

# New shape control tools for rational Bézier curve design

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## Abstract

Bézier curves are indispensable for geometric modelling and computer graphics. They have numerous favourable properties and provide the user with intuitive tools for editing the shape of a parametric polynomial curve. Even more control and flexibility can be achieved by associating a shape parameter with each control point and considering rational Bézier curves, which comes with the additional advantage of being able to represent all conic sections exactly. In this paper, we explore the editing possibilities that arise from expressing a rational Bézier curve in barycentric form. In particular, we show how to convert back and forth between the Bézier and the barycentric form, we discuss the effects of modifying the constituents (nodes, interpolation points, weights) of the barycentric form, and we study the connection between point insertion in the barycentric form with degree elevation of the Bézier form. Moreover, we analyse the favourable performance of the barycentric form for evaluating the curve.

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## 1 Introduction

A planar *rational Bézier curve*  $P: [0, 1] \rightarrow \mathbb{R}^2$  of degree  $n \in \mathbb{N}$  is defined by a set of *control points*  $P_0, \dots, P_n \in \mathbb{R}^2$  and a set of *weights*  $\alpha_0, \dots, \alpha_n \in \mathbb{R}$  as

$$P(t) = (x(t), y(t)) = \frac{\sum_{i=0}^n \alpha_i B_i^n(t) P_i}{\sum_{i=0}^n \alpha_i B_i^n(t)}, \quad (1)$$

where  $B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$  are the Bernstein polynomials. If  $\alpha_0 = \alpha_n = 1$ , then the curve is said to be in *standard form*, which can always be achieved, if the given weights  $\alpha_0$  and  $\alpha_n$  are non-zero and have the same sign, by uniformly scaling all weights and applying a linear rational parameter transformation [13, 9]. The curve  $P$  can also be written in homogeneous form and understood as the (central) projection of the spatial polynomial Bézier curve  $\hat{P}: [0, 1] \rightarrow \mathbb{R}^3$ ,

$$\hat{P}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t)) = \sum_{i=0}^n B_i^n(t) \hat{P}_i, \quad (2)$$

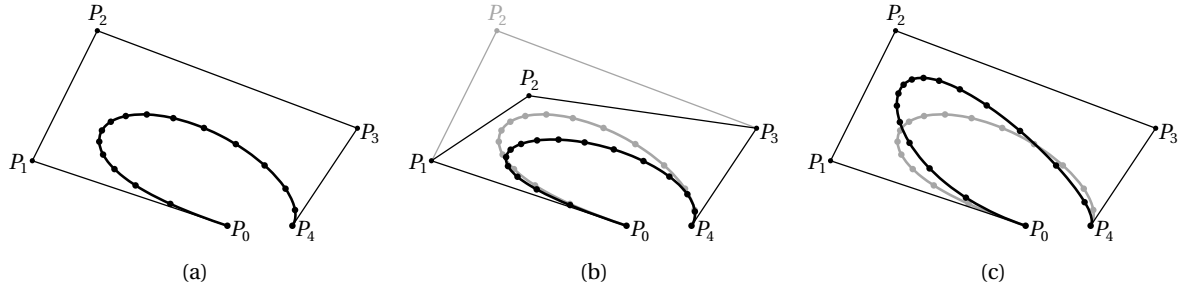
with homogeneous control points  $\hat{P}_i = (\alpha_i P_i, \alpha_i) \in \mathbb{R}^3$  into the  $\hat{z} = 1$  plane, because  $x(t) = \hat{x}(t)/\hat{z}(t)$  and  $y(t) = \hat{y}(t)/\hat{z}(t)$ . If all weights  $\alpha_i$  are equal, then  $P$  reduces to a planar polynomial Bézier curve.

Among the key properties that justify the popularity of rational Bézier curves for shape design, we recall that such a curve can be translated, scaled, or rotated by simply translating, scaling, or rotating its control polygon, and the same holds more generally for projective transformations [8]. Moreover, the shape of the curve can be controlled intuitively by modifying the control points  $P_i$  and the weights  $\alpha_i$  (see Figure 1), or by changing the *Farin points*  $F_i = (\alpha_i P_i + \alpha_{i+1} P_{i+1}) / (\alpha_i + \alpha_{i+1}) \in \mathbb{R}^2$  that can be associated with the  $i$ -th edge  $[P_i, P_{i+1}]$  of the control polygon for  $i = 0, \dots, n-1$  [7]. However, except at the endpoints, this control is *indirect* in the sense that it is difficult for the user to let the curve pass *exactly* through a specific point  $Q \in \mathbb{R}^2$ .

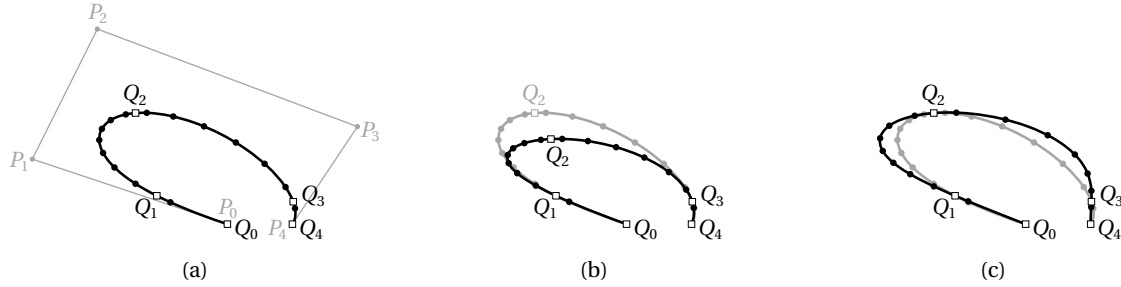
To overcome this limitation, we propose to convert the curve  $P$  to *barycentric form* and express it as

$$P(t) = \frac{\sum_{i=0}^n (-1)^i \frac{\beta_i}{t-t_i} Q_i}{\sum_{i=0}^n (-1)^i \frac{\beta_i}{t-t_i}}, \quad (3)$$

for certain distinct *nodes*  $t_0, \dots, t_n \in \mathbb{R}$  with corresponding *interpolation points*  $Q_0, \dots, Q_n \in \mathbb{R}^2$  and non-zero *weights*  $\beta_0, \dots, \beta_n \in \mathbb{R}$ . Since  $P(t_i) = Q_i$ , by construction [16], this representation allows for *direct* control, as we can force the curve to pass through some  $Q \in \mathbb{R}^2$  by simply moving one of the  $Q_i$  to  $Q$ . Moreover, the “flatness” of the curve at  $Q_i$  can be controlled by modifying the weight  $\beta_i$  (see Figure 2).



**Figure 1:** (a) a rational Bézier curve of degree  $n = 4$  with weights  $\alpha = (1, 3/2, 1, 1/2, 1)$ ; (b) the effect of moving the control point  $P_2$ ; (c) the effect of increasing the weight  $\alpha_2$  from 1 to 3. The dots visualize the curve points  $P(i/16)$  for  $i = 0, \dots, 16$ .



**Figure 2:** (a) converting a rational Bézier curve in (1) to the barycentric form in (3); (b) the effect of moving the interpolation point  $Q_2$ ; (c) the effect of decreasing the weight  $\beta_2$  by 50%. The dots visualize the curve points  $P(i/16)$  for  $i = 0, \dots, 16$ .

To the best of our knowledge, the barycentric form has been studied only in the functional setting, so far. For polynomials, it can be traced back to Taylor [17] and Dupuy [6], and Berrut and Trefethen [4] provide a detailed summary of its favourable properties. For rational functions, Salzer [15] and Schneider and Werner [16] were the first to identify the advantages of the barycentric form, and Berrut and Mittelmann [3] show that every rational interpolant can be expressed in barycentric form for a suitable choice of weights. For very specific weights, barycentric rational interpolants are guaranteed to have no poles and a high approximation order [2, 10], with slow-growing Lebesgue constants, in particular for equidistant nodes [5]. The barycentric form is also a key ingredient of the AAA algorithm [12], which extends the work of Antoulas and Anderson [1] and uses an adaptive node selection scheme for efficiently computing robust rational approximations of real and complex functions.

## 1.1 Contributions

In contrast to this previous work, the aim of this paper is to explore the use of the barycentric form in the context of curve design. We first show that rational Bézier curves (1) and barycentric rational curves (3) are essentially equivalent (Section 2), in the sense that any rational Bézier curve can be expressed in barycentric form and vice versa. We then discuss the shape editing possibilities offered by the barycentric form (Section 3), we show how to raise the degree from  $n$  to  $n+1$  without changing the curve (Section 4), and provide numerical evidence that the barycentric form is advantageous for curve evaluation (Section 5).

## 2 Equivalence of Bézier and barycentric form

Let us first recall how to derive the barycentric form for a polynomial  $p: \mathbb{R} \rightarrow \mathbb{R}$  of degree  $n$  [4]. Clearly, the Lagrange form of  $p$  is

$$p(t) = \sum_{i=0}^n \prod_{j=0, j \neq i}^n \frac{t - t_j}{t_i - t_j} p_i$$

where  $p_i = p(t_i)$ ,  $i = 0, \dots, n$ . Factoring out the polynomial  $\ell(t) = \prod_{j=0}^n (t - t_j)$ , we get the *first barycentric form*

$$p(t) = \ell(t) \sum_{i=0}^n \frac{w_i}{t - t_i} p_i,$$

with the *Lagrange weights*  $w_i$  defined as

$$w_i = \prod_{j=0, j \neq i}^n \frac{1}{t_i - t_j}, \quad i = 0, \dots, n. \quad (4)$$

Further dividing by the constant function 1, expressed in first barycentric form as  $1 = \ell(t) \sum_{i=0}^n \frac{w_i}{t - t_i}$ , and cancelling the common factor  $\ell(t)$ , then yields the *second barycentric form*

$$p(t) = \frac{\sum_{i=0}^n \frac{w_i}{t - t_i} p_i}{\sum_{i=0}^n \frac{w_i}{t - t_i}}.$$

To convert the rational Bézier curve in (1) to the barycentric form in (3), it remains to express the two components  $\hat{x}(t)$ ,  $\hat{y}(t)$  in the numerator of  $P(t)$  and its denominator  $\hat{z}(t)$  in the first barycentric form.

**Proposition 1.** *For any nodes  $0 \leq t_0 < t_1 < \dots < t_n \leq 1$ , we can express the rational Bézier curve (1) with control points  $P_i$  and weights  $\alpha_i$  in barycentric form (3) with interpolation points  $Q_i = P(t_i)$  and weights  $\beta_i = (-1)^{n+i} w_i z_i$ , where  $z_i = \hat{z}(t_i)$  and  $w_i$  as in (4).*

*Proof.* Let us first write the denominator of  $P(t)$  in first barycentric form as

$$\hat{z}(t) = \sum_{i=0}^n \alpha_i B_i^n(t) = \ell(t) \sum_{i=0}^n \frac{w_i}{t - t_i} z_i. \quad (5)$$

Likewise, the numerator of  $P(t)$  can be expressed in first barycentric form as

$$\sum_{i=0}^n \alpha_i B_i^n(t) P_i = \hat{z}(t) P(t) = \ell(t) \sum_{i=0}^n \frac{w_i}{t - t_i} z_i P(t_i). \quad (6)$$

The statement then follows after dividing (6) by (5), substituting  $P(t_i) = Q_i$  and  $w_i z_i = (-1)^{n+i} \beta_i$ , and cancelling the common factor  $(-1)^n \ell(t)$ .  $\square$

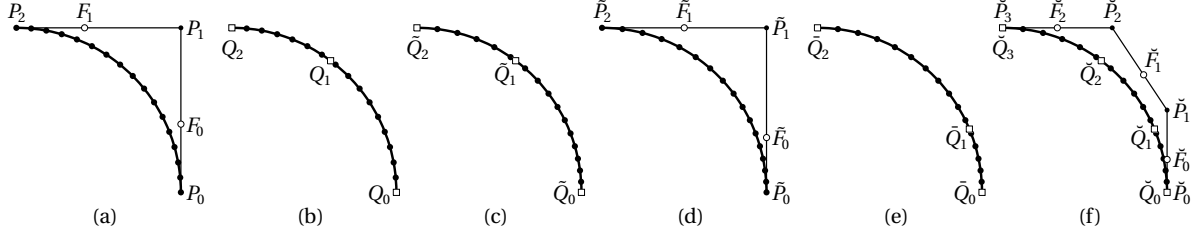
In principle, the nodes  $t_i$  do not have to be ordered or restricted to the interval  $[0, 1]$ , as long as they are distinct; however, in the context of interactive curve design, it seems natural to make these assumptions. Likewise, it is reasonable to set  $t_0 = 0$  and  $t_n = 1$ , so that  $Q_0 = P_0$  and  $Q_n = P_n$  mark the endpoints of the curve.

Under the usual assumption of positive weights  $\alpha_i$ , which guarantees that  $\hat{z}(t) > 0$  for all  $t \in [0, 1]$ , so that the curve is non-singular<sup>1</sup>, we conclude that the weights  $\beta_i$  are positive, too, because  $\text{sign}(w_i) = (-1)^{n-i}$ , which follows from the observation that the factors of  $w_i$  in (4) are negative, if and only if  $i < j \leq n$ . This is in line with a result by Schneider and Werner [16, Proposition 8], which implies that the positivity of the  $\beta_i$  is a necessary condition for the non-singularity of the barycentric rational curve in (3).

**Example 1.** Consider the quadratic rational curve  $P$  in Bézier form (1) with control points  $P_0 = (1, 0)$ ,  $P_1 = (1, 1)$ ,  $P_2 = (0, 1)$  and weights  $\alpha_0 = 1$ ,  $\alpha_1 = 1/\sqrt{2}$ ,  $\alpha_2 = 1$ , which describes a quarter circle (see Figure 3.a). Sampling this curve at the nodes  $t_0 = 0$ ,  $t_1 = 2 - \sqrt{2}$ ,  $t_2 = 1$  yields the interpolation points  $Q_0 = (1, 0)$ ,  $Q_1 = (3/5, 4/5)$ ,  $Q_2 = (0, 1)$ , and the weights of the barycentric form (3) turn out to be  $\beta_0 = 1 + 1/\sqrt{2}$ ,  $\beta_1 = 5/\sqrt{2}$ ,  $\beta_2 = 1 + \sqrt{2}$  (see Figure 3.b).

Once a rational curve is represented in barycentric form, we can use a linear rational reparameterization to bring it into *standard barycentric form* with  $\beta_0 = \beta_n = 1$ , very similarly to how the Bézier representation can be brought into standard form [13].

<sup>1</sup>It is actually sufficient to assume  $\alpha_0, \alpha_n > 0$  and  $\alpha_i \geq 0$  for  $i = 1, \dots, n-1$ .



**Figure 3:** Different representations of the quarter circle as a quadratic rational curve: (a) standard Bézier form with  $\alpha_0 = \alpha_2 = 1$  (instead of the weights  $\alpha_i$  we show the Farin points  $F_i$  to visualize the ratios  $\alpha_i/\alpha_{i+1}$ ); (b) corresponding barycentric form (cf. Proposition 1) for  $t_0 = 0, t_1 = 2 - \sqrt{2}, t_2 = 1$ ; (c) standard barycentric form with  $\tilde{\beta}_0 = \tilde{\beta}_2 = 1$  after reparameterization (cf. Proposition 3); (d) corresponding (non-symmetric) Bézier form (cf. Proposition 4); (e) after sliding the interpolation point  $\tilde{Q}_1$  to the new position  $\tilde{Q}_1$  (cf. Proposition 5); (f) as a cubic rational curve after inserting an additional interpolation point (cf. Proposition 7), in barycentric and Bézier form. The dots visualize the curve points  $P(i/16)$  for  $i = 0, \dots, 16$ .

**Lemma 2.** For any  $\lambda \in (0, 1)$ , consider the linear rational reparameterization  $\varphi: [0, 1] \rightarrow [0, 1]$ ,

$$\varphi(t) = \frac{(1-\lambda)t}{\lambda(1-t) + (1-\lambda)t}. \quad (7)$$

Let  $P$  be the barycentric rational curve (3) with nodes  $t_i$ , interpolation points  $Q_i$ , and weights  $\beta_i$  and let  $\tilde{P}$  be the barycentric rational curve with nodes  $\tilde{t}_i = \varphi(t_i)$ , the same interpolation points  $\tilde{Q}_i = Q_i$ , and weights  $\tilde{\beta}_i = \beta_i \lambda \tilde{t}_i / t_i$ . Then,  $P = \tilde{P} \circ \varphi$ .

*Proof.* Denoting the denominator of  $\varphi(t)$  by  $\delta(t) = \lambda(1-t) + (1-\lambda)t$ , we first observe that

$$\frac{\tilde{t}_i}{\varphi(t) - \tilde{t}_i} = \frac{\varphi(t_i)}{\varphi(t) - \varphi(t_i)} = \frac{\frac{(1-\lambda)t_i}{\delta(t_i)}}{\frac{(1-\lambda)t}{\delta(t)} - \frac{(1-\lambda)t_i}{\delta(t_i)}} = \frac{t_i \delta(t)}{t \delta(t_i) - t_i \delta(t)} = \frac{t_i \delta(t)}{\lambda(t - t_i)}$$

for any  $i = 0, \dots, n$ . Therefore,

$$\frac{\tilde{\beta}_i}{\varphi(t) - \tilde{t}_i} = \frac{\beta_i \lambda}{t_i} \cdot \frac{\tilde{t}_i}{\varphi(t) - \tilde{t}_i} = \frac{\beta_i}{t - t_i} \delta(t).$$

After substituting this into the numerator and the denominator of  $(\tilde{P} \circ \varphi)(t)$  and cancelling the common factor  $\delta(t)$  we get  $(\tilde{P} \circ \varphi)(t) = P(t)$ .  $\square$

Note that  $\tilde{\beta}_0$  in Lemma 2 is well-defined, even if  $t_0 = 0$ , because

$$\lim_{t \rightarrow 0} \frac{\varphi(t)}{t} = \lim_{t \rightarrow 0} \varphi'(t) = \frac{1-\lambda}{\lambda},$$

so that  $\tilde{\beta}_0 = \beta_0(1-\lambda)$  in that case. Moreover, we observe that  $\text{sign}(\tilde{\beta}_i) = \text{sign}(\beta_i)$  for  $i = 0, \dots, n$ . Therefore, if the  $\beta_i$  are all positive, then so are the new weights  $\tilde{\beta}_i$ .

**Proposition 3.** The barycentric rational curve (3) with nodes  $t_i$ , interpolation points  $Q_i$ , and weights  $\beta_i$  can be expressed in standard form by first reparameterizing it with  $\varphi$  in (7) for

$$\lambda = \frac{\beta_0 t_n - \beta_n t_0}{\beta_0(2t_n - 1) - \beta_n(2t_0 - 1)} \quad (8)$$

and then dividing all weights  $\tilde{\beta}_i$  by  $\tilde{\beta}_0$ , as long as  $\lambda \in (0, 1)$ .

*Proof.* According to Lemma 2, the first and the last weight of the reparameterized curve are

$$\tilde{\beta}_0 = \beta_0 \lambda \frac{\varphi(t_0)}{t_0} = \frac{\beta_0}{\delta(t_0)} \lambda(1-\lambda) \quad \text{and} \quad \tilde{\beta}_n = \beta_n \lambda \frac{\varphi(t_n)}{t_n} = \frac{\beta_n}{\delta(t_n)} \lambda(1-\lambda), \quad (9)$$

where  $\delta(t)$  is again the denominator of  $\varphi(t)$ . It remains for us to show that the choice of  $\lambda$  in (8) guarantees  $\tilde{\beta}_0 = \tilde{\beta}_n$ , which is equivalent to  $\beta_0\delta(t_n) = \beta_n\delta(t_0)$  by (9), so that both weights are 1 after dividing them by  $\tilde{\beta}_0$ . But as (8) implies

$$\lambda\beta_0(2t_n - 1) - \lambda\beta_n(2t_0 - 1) = \beta_0t_n - \beta_nt_0$$

and further

$$\beta_nt_0 - 2\lambda\beta_nt_0 + \lambda\beta_n = \beta_0t_n - 2\lambda\beta_0t_n + \lambda\beta_0,$$

we get the desired identity after noting that  $\delta(t_0) = \lambda - 2\lambda t_0 + t_0$  and  $\delta(t_n) = \lambda - 2\lambda t_n + t_n$ .  $\square$

The curve cannot be brought into standard form, if  $\lambda$  in (8) is outside the open interval  $(0, 1)$ , because  $\varphi$  is singular and no longer a monotonic reparameterization of  $[0, 1]$  in that case. However, if  $t_0 = 0$ ,  $t_n = 1$ , and all  $\beta_i$  are positive, then  $\lambda$  simplifies to  $\lambda = \beta_0/(\beta_0 + \beta_n) \in (0, 1)$  and  $\tilde{\beta}_0 = \tilde{\beta}_n = \beta_0\beta_n/(\beta_0 + \beta_n) > 0$ .

**Example 2.** Applying Proposition 3 to the barycentric rational curve  $P$  from Example 1 (see Figure 3.b), it turns out that the quarter circle can be described in standard barycentric form with nodes  $\tilde{t}_0 = 0$ ,  $\tilde{t}_1 = 2/3$ ,  $\tilde{t}_2 = 1$ , interpolation points  $\tilde{Q}_0 = (1, 0)$ ,  $\tilde{Q}_1 = (3/5, 4/5)$ ,  $\tilde{Q}_2 = (0, 1)$ , and weights  $\tilde{\beta}_0 = 1$ ,  $\tilde{\beta}_1 = 5/3$ ,  $\tilde{\beta}_2 = 1$  (see Figure 3.c).

A natural question to ask at this point is: how does one get back from barycentric to Bézier form? To this end, it helps to recall that  $\hat{P}_i = (\alpha_i P_i, \alpha_i)$  and to let  $\hat{Q}_i = (z_i Q_i, z_i)$ , so that the assignments  $Q_i = P(t_i)$  and  $z_i = \hat{z}(t_i)$  for  $i = 0, \dots, n$  in the statement of Proposition 1 can be written compactly as  $\hat{Q} = \mathbf{B}\hat{P}$ , where<sup>2</sup>

$$\mathbf{B} = \begin{pmatrix} B_0^n(t_0) & \cdots & B_n^n(t_0) \\ \vdots & \ddots & \vdots \\ B_0^n(t_n) & \cdots & B_n^n(t_n) \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} \hat{P}_0 \\ \vdots \\ \hat{P}_n \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} \hat{Q}_0 \\ \vdots \\ \hat{Q}_n \end{pmatrix}. \quad (10)$$

**Proposition 4.** The barycentric rational curve (3) with nodes  $t_i$ , interpolation points  $Q_i$ , and weights  $\beta_i$  can be expressed in Bézier form (1) with control points  $P_i = (\hat{x}_i, \hat{y}_i)/\hat{z}_i$  and weights  $\alpha_i = \hat{z}_i$ , where the vector  $\hat{P}$  of points  $\hat{P}_i = (\alpha_i P_i, \alpha_i) = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$  is defined as  $\hat{P} = \mathbf{B}^{-1}\hat{Q}$  and  $\hat{Q}$  is the vector of points  $\hat{Q}_i = (z_i Q_i, z_i)$  with  $z_i = (-1)^{n+i} \beta_i/w_i$  and  $w_i$  as in (4).

*Proof.* First recall that the Bernstein–Vandermonde matrix  $\mathbf{B}$  in (10) is non-singular, because the Bernstein basis is a Chebyshev system. The assertion then follows immediately by applying Proposition 1 to the rational Bézier curve with the stated control points  $P_i$  and weights  $\alpha_i$  and verifying that it gives back the interpolation points  $Q_i$  and weights  $\beta_i$ .  $\square$

Note that  $\hat{P} = \mathbf{B}^{-1}\hat{Q}$  can be computed fast and accurately with  $O(n^2)$  time complexity [11]. If the last coordinate  $\alpha_i = \hat{z}_i$  of the homogeneous control point  $\hat{P}_i$  happens to vanish for some  $i$ , it means that the given barycentric rational curve cannot be written as a classical rational Bézier curve with control points in  $\mathbb{R}^2$ . Instead, the control point  $P_i$  needs to be replaced by the control vector  $\alpha_i P_i = (\hat{x}_i, \hat{y}_i) \in \mathbb{R}^2$ , representing an infinite control point in this case [14, 8].

**Example 3.** Using Proposition 4 to convert the barycentric rational curve  $\tilde{P}$  in standard form from Example 2 (see Figure 3.c) back to Bézier form, we find that the quadratic rational Bézier curve with control points  $\tilde{P}_0 = (1, 0)$ ,  $\tilde{P}_1 = (1, 1)$ ,  $\tilde{P}_2 = (0, 1)$  and weights  $\tilde{\alpha}_0 = 2/3$ ,  $\tilde{\alpha}_1 = 1/3$ ,  $\tilde{\alpha}_2 = 1/3$  also describes a quarter circle (see Figure 3.d). Bringing these weights into standard form, we return to the rational Bézier curve  $P$  that we started with in Example 1 (see Figure 3.a).

### 3 Shape editing using the barycentric form

Once a rational curve is given in barycentric form (3), several new options arise for manipulating the curve by modifying the different parameters of the barycentric form: the nodes  $t_i$ , the interpolating points  $Q_i$ , and the weights  $\beta_i$ .

Changing one of the nodes, say  $t_k$ , in isolation, while keeping the  $Q_i$  and the  $\beta_i$  fixed, has a rather unpredictable effect. However, it is possible to preserve the shape (and the parameterization) of the curve by simultaneously adapting the corresponding  $Q_k$  and all  $\beta_i$ , so that the effect amounts to “sliding”  $Q_k$  along

<sup>2</sup>The observant reader may have already noticed that throughout this paper we write points (in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ ) as row vectors, so as to avoid excessive use of the transposition operator and to be able to conveniently stack them into matrices, like  $\hat{P}\hat{Q} \in \mathbb{R}^{(n+1) \times 3}$ .

the curve. This can be achieved by first using Proposition 4 to express the curve in Bézier form and then applying Proposition 1 with the modified nodes to get back to the barycentric form. However, it turns out that we do not have to carry out these conversions explicitly, as the new interpolation points and weights can be expressed directly in terms of the given parameters of the barycentric form.

**Proposition 5.** *Suppose we change the node  $t_k$  for some  $k \in \{0, \dots, n\}$  to some new value  $\bar{t}_k \notin \{t_0, \dots, t_n\}$  and keep the other nodes fixed, that is, we let  $\bar{t}_i = t_i$  for  $i \neq k$ . The barycentric rational curve (3) with nodes  $t_i$ , interpolation points  $Q_i$ , and weights  $\beta_i$  can then be expressed alternatively in terms of the nodes  $\bar{t}_i$ , the interpolation points  $\bar{Q}_k = P(\bar{t}_k)$  and  $\bar{Q}_i = Q_i$  for  $i \neq k$ , and the weights*

$$\bar{\beta}_k = \sum_{i=0}^n (-1)^{n+k+i} \frac{\bar{t}_k - t_k}{\bar{t}_k - t_i} \beta_i, \quad \bar{\beta}_i = \frac{t_i - t_k}{t_i - \bar{t}_k} \beta_i, \quad i \neq k. \quad (11)$$

*Proof.* To prove this statement, we stick to the idea sketched out above. After converting the given curve to Bézier form, it follows directly from Proposition 1 that the new interpolation points are  $\bar{Q}_i = P(\bar{t}_i)$ , which simplifies to  $\bar{Q}_i = P(t_i) = Q_i$  for  $i \neq k$ . Moreover, we know that the given weights satisfy  $\beta_i = (-1)^{n+i} w_i \hat{z}(t_i)$ , where  $w_i$  is defined in (4) and the denominator polynomial  $\hat{z}$  can be written, independently of the Bézier form, in first barycentric form as  $\hat{z}(t) = \ell(t) \sum_{i=0}^n (-1)^i \frac{\beta_i}{t - t_i}$ . Likewise, the new weights satisfy  $\bar{\beta}_i = (-1)^{n+i} \bar{w}_i \hat{z}(\bar{t}_i)$ , where  $\bar{w}_i = \prod_{j=0, j \neq i}^n \frac{1}{\bar{t}_i - \bar{t}_j}$ . If  $i \neq k$ , then this expression simplifies to

$$\bar{\beta}_i = (-1)^{n+i} \frac{1}{\bar{t}_i - \bar{t}_k} \prod_{j=0, j \neq i, k}^n \frac{1}{\bar{t}_i - \bar{t}_j} \hat{z}(\bar{t}_i) = (-1)^{n+i} \frac{1}{t_i - \bar{t}_k} \prod_{j=0, j \neq i, k}^n \frac{1}{t_i - t_j} \hat{z}(t_i) = \frac{t_i - t_k}{t_i - \bar{t}_k} \beta_i,$$

because  $\bar{t}_i = t_i$  for  $i \neq k$ . For the remaining weight  $\bar{\beta}_k$ , note that  $\ell(\bar{t}_k) = \prod_{j=0}^n (\bar{t}_k - t_j) = (\bar{t}_k - t_k) / \bar{w}_k$ , hence

$$\bar{\beta}_k = (-1)^{n+k} \bar{w}_k \ell(\bar{t}_k) \sum_{i=0}^n (-1)^i \frac{\beta_i}{\bar{t}_k - t_i} = \sum_{i=0}^n (-1)^{n+k+i} \frac{\bar{t}_k - t_k}{\bar{t}_k - t_i} \beta_i.$$

□

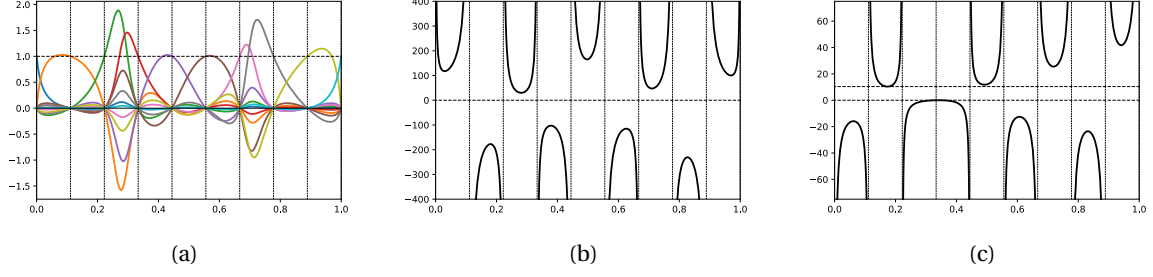
In an interactive application, this “sliding” of  $Q_k$  can be realized, for example, by letting the user click on the desired interpolation point, while holding the ‘shift’ key (to distinguish the action from a displacement of  $Q_k$ ; see below), and translating the subsequent mouse movement (left/right or up/down) into an increase or decrease of  $t_k$  until the mouse button is released. Note that the time complexity for updating  $\beta_k$  is  $O(n)$ ,  $O(1)$  for updating each of the other  $\beta_i$ , and  $O(n)$  for updating  $Q_k$  (see Section 5), hence  $O(n)$  overall, which is much more efficient than computing the conversion to Bézier form and back.

For the reasons pointed out in Section 2, it seems reasonable to prevent sliding the endpoints  $Q_0$  and  $Q_n$ , that is, to exclude the cases  $k = 0$  and  $k = n$  in Proposition 5, and to restrict  $\bar{t}_k$  to the open interval  $(t_{k-1}, t_{k+1})$ , so that  $\bar{Q}_k$  remains between its neighbours  $Q_{k-1}$  and  $Q_{k+1}$  along the curve. In this case, it follows immediately from (11) that  $\text{sign}(\bar{\beta}_i) = \text{sign}(\beta_i)$  for  $i \neq k$ , hence the positivity of the weights  $\beta_i$  carries over to the new weights  $\bar{\beta}_i$ . For  $\bar{\beta}_k$ , this is not obvious from (11), but implied by the fact that changing  $t_k$  does not change the curve, so that the non-singularity of the curve still guarantees that all  $\beta_i$  have the same weight [16].

**Example 4.** Applying Proposition 5 to the barycentric rational curve  $\bar{P}$  in standard form from Example 2 (see Figure 3.c) and changing  $\bar{t}_1 = 2/3$  to  $\bar{t}_1 = 1/3$ , thus sliding  $\bar{Q}_1 = (3/5, 4/5)$  to  $\bar{Q}_1 = (12/13, 5/13)$ , we find that the quarter circle can also be described in barycentric form with nodes  $\bar{t}_0 = 0$ ,  $\bar{t}_1 = 1/3$ ,  $\bar{t}_2 = 1$ , interpolation points  $\bar{Q}_0 = (1, 0)$ ,  $\bar{Q}_1 = (12/13, 5/13)$ ,  $\bar{Q}_2 = (0, 1)$ , and weights  $\bar{\beta}_0 = 2$ ,  $\bar{\beta}_1 = 13/6$ ,  $\bar{\beta}_2 = 1/2$  (see Figure 3.e).

The most direct control over the shape of the curve is given by displacing one of the interpolation points, say  $Q_k$ , while keeping all other parameters fixed. By the interpolation property of the barycentric form, this will force the curve to pass through the new position of  $Q_k$  at  $t_k$  (see Figure 2.b). Compared to moving a Bézier control point  $P_k$ , it should be noted that the basis function  $C_k(t) = (-1)^k \frac{\beta_k}{t - t_k} / \sum_{i=0}^n (-1)^i \frac{\beta_i}{t - t_i}$  that corresponds to  $Q_k$  (see Figure 4.a) is neither non-negative nor as nicely “bell-shaped” as the basis function  $\alpha_k B_k^n(t) / \sum_{i=0}^n \alpha_i B_i^n(t)$  that corresponds to  $P_k$ . Hence, for large displacements, the shape may change less intuitively as it does in the case of editing the control polygon. However, while the general shape of the curve is more easily controlled with the Bézier control points  $P_i$ , changing the interpolation points  $Q_i$ , combined with the “sliding” procedure outlined above and inserting points with ease (see Section 4) provides a useful tool for “micro-editing” the curve shape. For example, it can be used to “snap” the curve to some point  $Q$





**Figure 4:** Plots of (a) the basis functions  $C_k(t)$  for  $k = 0, \dots, n$  of a barycentric rational curve of degree  $n = 9$  with equidistant nodes  $t_i = i/n$  and weights  $(\beta_0, \dots, \beta_n) = (1, 8, 3, 2, 5, 6, 2, 5, 8, 1)$ , (b) the denominator  $D(t)$ , and (c) the function  $S_k(t)$  for  $k = 3$ . The horizontal lines represent  $M_*$  and  $M^*$ .

that must be interpolated exactly and the interpolation property guarantees that  $Q$  remains a point on the curve during subsequent editing operations, as long as  $Q$  is identical to one of the  $Q_i$ .

It remains for us to discuss what happens to the curve if we change one of the weights, say  $\beta_k$ , and it turns out that we can use this parameter to modify the “flatness” of the curve at  $Q_k$ . Indeed, Schneider and Werner [16] show that the derivative of  $P$  at  $Q_k = P(t_k)$  is

$$P'(t_k) = \frac{\sum_{i=0, i \neq k}^n (-1)^{k+i+1} \frac{\beta_i}{t_k - t_i} (Q_k - Q_i)}{\beta_k}.$$

As the numerator, which determines the direction of the tangent at  $Q_k$ , does not depend on  $\beta_k$ , which in turn appears only in the denominator, it follows that  $\beta_k$  controls the length of the tangent, but not its direction. Therefore, decreasing  $\beta_k$  “flattens” the curve locally at  $Q_k$ , while increasing  $\beta_k$  has the effect of letting the curve bend more tightly at  $Q_k$  (see Figure 2.c and Figure 5). However, some limits on the possible values of  $\beta_k$  need to be respected, if we want to guarantee that the curve remains non-singular.

**Proposition 6.** Consider a non-singular barycentric rational curve (3) with nodes  $0 = t_0 < t_1 < \dots < t_n = 1$ , interpolation points  $Q_i$ , and weights  $\beta_i > 0$  and suppose we change the weight  $\beta_k$  for some  $k \in \{0, \dots, n\}$  to some new value  $\tilde{\beta}_k$ . Then the modified curve  $\tilde{P}$  remains non-singular as long as  $\tilde{\beta}_k \in (M_*, M^*)$ , where

$$M_* = \max\{\dots, M_{k-2}, M_k, M_{k+2}, \dots\}, \quad M^* = \min\{\dots, M_{k-3}, M_{k-1}, M_{k+1}, M_{k+3}, \dots\},$$

with

$$M_{k+i} = \begin{cases} \max\{S_k(t) : t \in (t_{k+i+i}, t_{k+i+i'})\}, & i \text{ even}, \\ \min\{S_k(t) : t \in (t_{k+i+i}, t_{k+i+i'})\}, & i \text{ odd}, \end{cases} \quad i_* = \begin{cases} -1, & i \leq 0, \\ 0, & i > 0, \end{cases} \quad i^* = \begin{cases} 0, & i < 0, \\ 1, & i \geq 0, \end{cases} \quad (12)$$

and

$$S_k(t) = \sum_{i=0, i \neq k}^n (-1)^{k+i+1} \frac{t - t_k}{t - t_i} \beta_i.$$

*Proof.* First recall from (5) that the denominator  $D(t) = \sum_{i=0}^n (-1)^i \frac{\beta_i}{t - t_i}$  of a non-singular barycentric rational curve with positive weights  $\beta_i$  (see Figure 4.b) satisfies

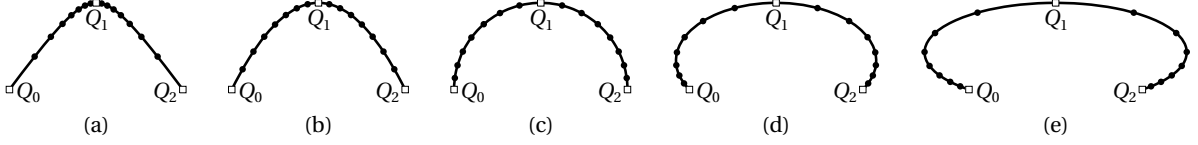
$$(-1)^n \ell(t) D(t) = \hat{z}(t) > 0, \quad t \in [0, 1], \quad (13)$$

where  $\ell(t) = \prod_{i=0}^n (t - t_i)$ . Next observe that the denominator

$$\bar{D}(t) = (-1)^k \frac{\tilde{\beta}_k}{t - t_k} + \sum_{i=0, i \neq k}^n (-1)^i \frac{\beta_i}{t - t_i}$$

of the modified curve  $\tilde{P}$  vanishes at  $\bar{t}$ , if and only if  $\tilde{\beta}_k = S_k(\bar{t})$ . By the interpolation property of barycentric rational curves, it is clear that  $\bar{t} \notin \{t_0, \dots, t_n\}$ . Therefore,  $\tilde{P}$  is non-singular for  $t \in [0, 1]$ , as long as  $\tilde{\beta}_k$  is not in the image of  $I = [0, 1] \setminus \{t_0, \dots, t_n\}$  under  $S_k$ . To better understand the behaviour of  $S_k$ , note that

$$S_k(t) = (-1)^{k+1} (t - t_k) D(t) + \beta_k$$



**Figure 5:** The effect of changing the central weight of a quadratic rational curve  $P$  in barycentric form (3): (a)  $\beta_1 = 4$ ; (b)  $\beta_1 = 2$ ; (c)  $\beta_1 = 1$ ; (d)  $\beta_1 = 1/2$ ; (e)  $\beta_1 = 1/4$ . The direction of the derivative  $P'(t_1)$  at  $Q_1$  is fixed, but its length is inverse proportional to  $\beta_1$ . The dots visualize the curve points  $P(i/16)$  for  $i = 0, \dots, 16$ .

and assume that  $t \in (t_j, t_{j+1})$  for some  $j \in \{0, \dots, n-1\}$ . Since  $(-1)^{n-j} \ell(t)$  is clearly positive, it follows from (13) that  $(-1)^j D(t)$  is positive too, with  $\lim_{t \rightarrow t_j} (-1)^j D(t) = \lim_{t \rightarrow t_{j+1}} (-1)^j D(t) = +\infty$ . Therefore,  $S_k(t) - \beta_k$  is positive, if  $j \geq k$  and  $j+k$  is odd, or if  $j < k$  and  $j+k$  is even, and negative otherwise, converging to  $+\infty$  or  $-\infty$  as  $t$  approaches  $t_j$  or  $t_{j+1}$ , except at  $t_k$ , because  $S_k(t_k) = 0$ . Consequently, the image of  $(t_j, t_{j+1})$  under  $S_k$  is

$$S_k[(t_j, t_{j+1})] = \begin{cases} [M_{j+1}, +\infty), \\ (-\infty, M_{j+1}], \end{cases} \quad \text{if } j < k-1 \text{ and } j+k \text{ is } \begin{cases} \text{even,} \\ \text{odd,} \end{cases}$$

$$S_k[(t_j, t_{j+1})] = \begin{cases} [M_j, +\infty), \\ (-\infty, M_j], \end{cases} \quad \text{if } j \geq k+1 \text{ and } j+k \text{ is } \begin{cases} \text{odd,} \\ \text{even,} \end{cases}$$

and

$$S_k[(t_{k-1}, t_{k+1})] = (-\infty, M_k],$$

for the  $M_j$  in (12) (see Figure 4.c). Combining these images, we find that  $S_k[I] = (-\infty, M_*) \cup [M^*, +\infty)$ , which, together with the considerations above, shows that the stated condition for  $\tilde{\beta}_k$  guarantees  $\tilde{P}$  to be non-singular.  $\square$

Note that  $M_*$  in Proposition 6 is always non-negative, because  $S_k(t_k) = 0$  and thus  $M_* \geq M_k \geq 0$ , which is in line with our expectation that  $\tilde{\beta}_k$  should be positive, just like  $\beta_k$ , in order for  $\tilde{P}$  to be non-singular. In general, it does not seem feasible to determine the bounds  $M_*$  and  $M^*$  analytically, but they can be computed numerically by first finding the roots of  $S'_k$  over the relevant intervals with Newton's method, using, for example, the midpoint of the interval as initial value, and then evaluating  $S_k$  at these roots to get the  $M_j$  in (12).

**Example 5.** The derivative of the quadratic barycentric rational curve  $P$  with nodes  $t_0 = 0$ ,  $t_1 = 1/2$ ,  $t_2 = 1$ , interpolation points  $Q_0 = (-1, 0)$ ,  $Q_1 = (0, 1)$ ,  $Q_2 = (1, 0)$ , and weights  $\beta_0 = \beta_1 = \beta_2 = 1$  at  $t_1$  is  $P'(t_1) = (4, 0)$  (see Figure 5.c). Increasing the weight  $\beta_1$  to 2 or 4 shortens the derivative by a factor of 1/2 or 1/4, respectively (see Figure 5.a,b), while decreasing  $\beta_1$  to 1/2 or 1/4 extends the derivative by a factor of 2 or 4, respectively (see Figure 5.d,e). In this example, the curve is well-defined for all  $\beta_1 > 0$ . Instead, if we want to modify  $\beta_0$ , then we must ensure that  $\beta_0 \in (0, 9)$ .

## 4 Point insertion and degree elevation

A common tool for increasing the flexibility of a rational Bézier curve is degree elevation, which can be used to represent a given curve of degree  $n$  as a curve of degree  $n+1$  without changing its shape. This increases the number of control points and weights by one and hence gives the user more control to model the desired shape. The equivalent of degree elevation in the barycentric form simply amounts to adding an interpolation point  $Q_* = P(t_*)$  for some  $t_* \in [0, 1] \setminus \{t_0, \dots, t_n\}$ , adapting the weights  $\beta_i$ , and computing the appropriate new weight for  $Q_*$ .

**Proposition 7.** Let  $k \in \{0, \dots, n+1\}$  and  $t_* \in (t_{k-1}, t_k)$ , where  $t_{-1} = 0$  and  $t_{n+1} = 1$ . The barycentric rational curve  $P$  of degree  $n$  in (3) with nodes  $t_i$ , interpolation points  $Q_i$ , and weights  $\beta_i$  can then be expressed alternatively as a barycentric rational curve  $\tilde{P}$  of degree  $n+1$  with parameters

$$\tilde{t}_i = \begin{cases} t_i, \\ t_*, \\ t_{i-1}, \end{cases} \quad \tilde{Q}_i = \begin{cases} Q_i, \\ Q_* = P(t_*), \\ Q_{i-1}, \end{cases} \quad \tilde{\beta}_i = \begin{cases} \frac{\beta_i}{t_* - t_i}, \\ \sum_{i=0}^n (-1)^{n+k+i} \frac{\beta_i}{t_i - t_*}, \\ \frac{\beta_{i-1}}{t_{i-1} - t_*}, \end{cases} \quad \text{if } \begin{cases} i < k, \\ i = k, \\ i > k. \end{cases} \quad (14)$$



*Proof.* As in the proof of Proposition 5, we first use Proposition 4 to convert  $P$  to Bézier form and then conclude from Proposition 1 that the given weights satisfy  $\beta_i = (-1)^{n+i} w_i \hat{z}(t_i)$ , where  $w_i$  is defined in (4) and  $\hat{z}(t) = \ell(t) \sum_{i=0}^n (-1)^n \frac{\beta_i}{t-t_i}$ . Likewise, applying Proposition 1 to the nodes  $\check{t}_0, \dots, \check{t}_{n+1}$ , it follows that  $\check{Q}_i = P(\check{t}_i)$ , which simplifies to what is stated in (14), and that  $\check{\beta}_i = (-1)^{n+1+i} \check{w}_i \check{z}(\check{t}_i)$ , where  $\check{w}_i = \prod_{j=0, j \neq i}^{n+1} \frac{1}{\check{t}_i - \check{t}_j}$ . If  $i < k$ , then this expression simplifies to

$$\check{\beta}_i = (-1)^{n+1+i} \frac{1}{\check{t}_i - \check{t}_k} \prod_{j=0, j \neq i, k}^{n+1} \frac{1}{\check{t}_i - \check{t}_j} \check{z}(\check{t}_i) = (-1)^{n+1+i} \frac{1}{t_i - t_\star} \prod_{j=0, j \neq i}^n \frac{1}{t_i - t_j} \hat{z}(t_i) = \frac{\beta_i}{t_\star - t_i},$$

because  $\check{t}_i = t_i$  for  $i < k$  and  $\check{t}_i = t_{i-1}$  for  $i > k$ , and similarly to

$$\check{\beta}_i = (-1)^{n+1+i} \frac{1}{\check{t}_i - \check{t}_k} \prod_{j=0, j \neq i, k}^{n+1} \frac{1}{\check{t}_i - \check{t}_j} \check{z}(\check{t}_i) = (-1)^{n+1+i} \frac{1}{t_{i-1} - t_\star} \prod_{j=0, j \neq i}^n \frac{1}{t_{i-1} - t_j} \hat{z}(t_{i-1}) = \frac{\beta_{i-1}}{t_{i-1} - t_\star},$$

if  $i > k$ . For the remaining weight  $\check{\beta}_k$ , note that  $\ell(\check{t}_k) = \prod_{j=0}^n (\check{t}_k - t_j) = 1/\check{w}_k$ , hence

$$\check{\beta}_k = (-1)^{n+1+k} \check{w}_k \ell(\check{t}_k) \sum_{i=0}^n (-1)^i \frac{\beta_i}{\check{t}_k - t_i} = \sum_{i=0}^n (-1)^{n+k+i} \frac{\beta_i}{t_i - t_\star}.$$

□

At this point, one may ask: what happens if we convert the barycentric rational curve  $\check{P}$  of degree  $n+1$  with the parameters in (14) to Bézier form? But as  $\check{P}$  is just a different representation of the same curve  $P$ , its Bézier form must simply be the degree-elevated Bézier form of  $P$ . Indeed, since degree elevation does not change the denominator polynomial  $\hat{z}$ , this fact can also be observed by applying Proposition 1 to the degree-elevated Bézier form of  $P$  and noticing that this gives the parameters in (14).

**Example 6.** Adding the point  $Q_\star = P(1/3)$  to the quadratic barycentric rational curve  $\check{P}$  from Example 4 (see Figure 3.e), it follows from Proposition 7 that the quarter circle can be described as a cubic barycentric rational curve with nodes  $\check{t}_0 = 0$ ,  $\check{t}_1 = 1/3$ ,  $\check{t}_2 = 2/3$ ,  $\check{t}_3 = 1$ , interpolation points  $\check{Q}_0 = (1, 0)$ ,  $\check{Q}_1 = (12/13, 5/13)$ ,  $\check{Q}_2 = (3/5, 4/5)$ ,  $\check{Q}_3 = (0, 1)$ , and weights  $\check{\beta}_0 = 3$ ,  $\check{\beta}_1 = 13/2$ ,  $\check{\beta}_2 = 5$ ,  $\check{\beta}_3 = 3/2$  (see Figure 3.f). Applying Proposition 4 to this curve, we find that its Bézier form is given by the control points  $\check{P}_0 = (1, 0)$ ,  $\check{P}_1 = (1, 1/2)$ ,  $\check{P}_2 = (2/3, 1)$ ,  $\check{P}_3 = (0, 1)$  and the weights  $\check{\alpha}_0 = 2/3$ ,  $\check{\alpha}_1 = 4/9$ ,  $\check{\alpha}_2 = 1/3$ ,  $\check{\alpha}_3 = 1/3$  (see Figure 3.f), which is just the degree-elevated quadratic rational Bézier curve from Example 3 (see Figure 3.d).

## 5 Performance of the barycentric form for curve evaluation

The classical way to evaluate a rational Bézier curve (1) at some parameter  $t \in [0, 1]$  is by applying the *de Casteljau algorithm* to the numerator and the denominator of  $P(t)$  and dividing through, which has time complexity  $O(n^2)$ . The rational de Casteljau algorithm [7] provides a more robust, but less efficient alternative with the same time complexity. A recent paper by Woźny and Chudy [18] presents a novel evaluation procedure for rational Bézier curves with a nice geometric interpretation. Like the de Casteljau algorithm, it is based on robust convex combinations, but it has a favourable linear time complexity. Yet another option is to use Proposition 1 to convert the curve into the barycentric form in (3) and to evaluate the latter. This can clearly be done in linear time, too, by first computing the sums in the numerator and the denominator and then dividing through (see Algorithm 1).

To compare the efficiency of these three algorithms (classical de Casteljau, linear-time geometric, barycentric), we implemented them in C++ on an Ubuntu 20.04.2 LTS laptop with 1.8 GHz Intel Core i7-10510U processor and 16 GB RAM. The time complexities of the algorithms are confirmed by the plots in Figure 6.a, which show the average running times for evaluating a rational Bézier curve of degree  $n$  with random control points  $P_i \in [-1, 1]^2$  and random weights  $\alpha_i \in [0.01, 10]$  at  $10^6$  random parameters  $t$  for  $n = 3, 5, 10, 20, \dots, 80$ . Regardless of  $n$ , the evaluation in barycentric form is the fastest, even though the running time includes the time for pre-computing the interpolation points  $Q_i$  (with Woźny and Chudy's linear-time algorithm) and the weights  $\beta_i$  (with the polynomial de Casteljau algorithm) for equidistant nodes  $t_i = i/n$  and Lagrange weights  $w_i = (-1)^{n-i} \binom{n}{i} n^n / n!$  [4]. Compared to the  $10^6$  evaluations, this pre-computation is negligible, but for fewer evaluations it is not. This effect can be seen in Figure 6.b, which reports the running times for evaluating

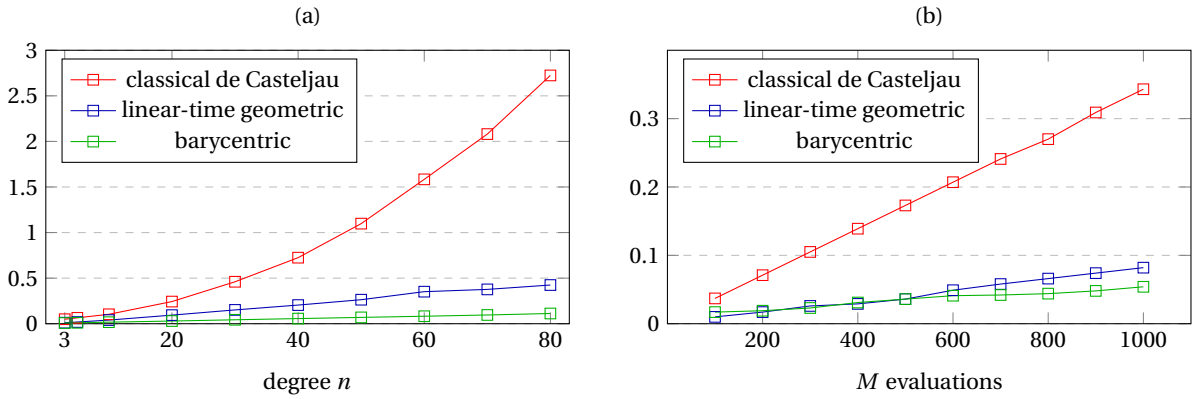
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**Algorithm 1** Evaluation of  $P$  using the barycentric rational form

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**Input:** nodes  $t_0, \dots, t_n$ , interpolation points  $Q_0, \dots, Q_n$ , and weights  $\beta_0, \dots, \beta_n$  of  $P$  and parameter  $t$ **Output:**  $P(t)$  $N := 0$  $D := 0$  $\sigma := 1$ **for**  $i$  from 0 to  $n$  **do**     $d := t - t_i$     **if**  $d = 0$  **then**        return  $Q_i$      $a := \sigma \beta_i / d$      $N := N + a Q_i$      $D := D + a$      $\sigma := -\sigma$ return  $N/D$ 

---



**Figure 6:** Comparison of different algorithms for evaluating a rational Bézier curve: (a) running times (in seconds) for  $10^6$  evaluations of curves of different degree  $n$ ; (b) running times (in milliseconds) for  $M$  evaluations of a curve of degree 3.

a cubic rational Bézier curve at  $M = 100, 200, \dots, 1000$  random parameters  $t \in [0, 1]$ . While the linear-time algorithm is the fastest for small  $M$ , the pre-computation needed for the barycentric evaluation pays off for large  $M$ , with the break-even at about  $M = 300$  evaluations.

While we use equidistant nodes in this comparison, we should point out that this may lead to numerical inaccuracies for  $n \geq 50$ , due to the large variance of the  $w_i$ , which carries over to the  $\beta_i$ . Instead, stable results can be obtained by using Chebyshev points of the second kind as nodes, that is,  $t_i = (1 - \cos \frac{i\pi}{n})/2$ , because the corresponding Lagrange weights satisfy  $|w_0| = |w_n| = |w_i|/2$  for  $i = 1, \dots, n-1$  [4].

## 6 Conclusion

In this paper, we explored the use of barycentric rational curves in the context of shape design, and we studied their properties. Converting a given rational Bézier curve to barycentric form is simple and comes with several advantages. On the one hand, the barycentric form offers new tools for controlling the shape of the curve that are complementary to the classical way of manipulating rational Bézier curves. As the barycentric form provides neither a convex hull, nor a vanishing diminishing property, these new tools may be less intuitive, but we believe that they are still useful, at least for “micro-editing”. After a modification in the barycentric form, the curve can easily be transformed back into Bézier form. On the other hand, the barycentric form is very efficient to evaluate in linear time.

Analogously to rational Bézier curves, the barycentric rational curve  $P$  in (3) can be seen as the projection of the polynomial curve  $\hat{P}$  that interpolates the homogeneous interpolation points  $\hat{Q}_i = (z_i Q_i, z_i)$  at the nodes  $t_i$ , where  $z_i = (-1)^{n+i} \beta_i / w_i$ , into the  $\hat{z} = 1$  plane. In this homogeneous setting, the proposed editing operations can be understood as follows: (1) “sliding”  $Q_k$  along  $P$  is equivalent to “sliding”  $\hat{Q}_k$  along  $\hat{P}$ , and

as the modification of  $t_k$  entails a change of the  $w_i$ , the  $\beta_i$  need to be updated for  $i \neq k$  as in Proposition 5, so that the corresponding  $z_i$  and  $\hat{Q}_i$  remain the same; (2) moving  $Q_k$  is like moving  $\hat{Q}_k$  in the  $\hat{z} = z_k$  plane; (3) modifying  $\beta_k$  is tantamount to displacing  $\hat{Q}_k$  along the line through  $\hat{Q}_k$  and the origin. The larger  $\beta_k$ , the further  $\hat{Q}_k$  is from the origin, and the more the projected curve  $P$  bends at  $Q_k$ , very similarly to how increasing the weight  $\alpha_i$  pulls  $P$  towards the control point  $P_k$  in the case of rational Bézier curves.

For the important class of non-singular rational curves, there is an interesting difference between the Bézier and the barycentric form. While the positivity of the  $\alpha_i$  is *sufficient*, the positivity of the  $\beta_i$  is only *necessary* for the non-singularity of the curve. Consequently, the set of all Bézier curves with positive  $\alpha_i$  does not contain all non-singular curves, while the set of all barycentric rational curves with positive  $\beta_i$  does, but also contains singular curves. In both cases, additional non-linear constraints are needed to fix this (cf. Proposition 6).

Another difference is that the barycentric form can describe curves that cannot be represented in Bézier form, at least not with the same degree. For example, a semi-circle can be modelled as a quadratic rational curve in barycentric form (see Figure 5.c), but not in Bézier form without using control vectors, and likewise for a full circle as a quartic rational curve.

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