GCS

Geometric Constraint Systems & Solvers

Ch Arber – September 9, 2021 - Barcelona

Summary

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  - The general problem.
  - Back to ruler and compass.
  - Modern geometry.
  - Algebraic geometry.
  - Back to 3D geometry.
- Conclusion
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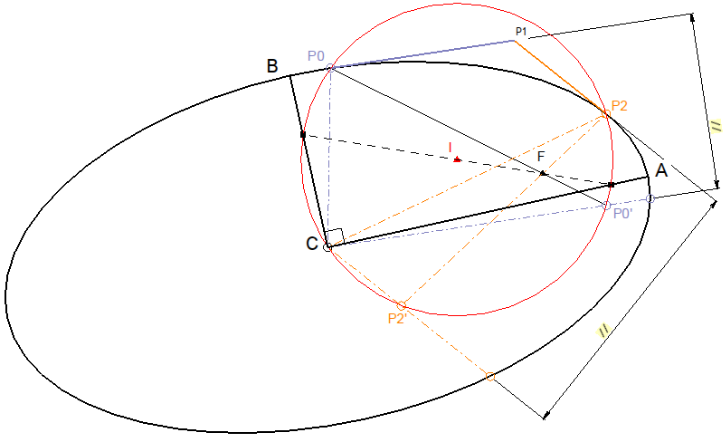
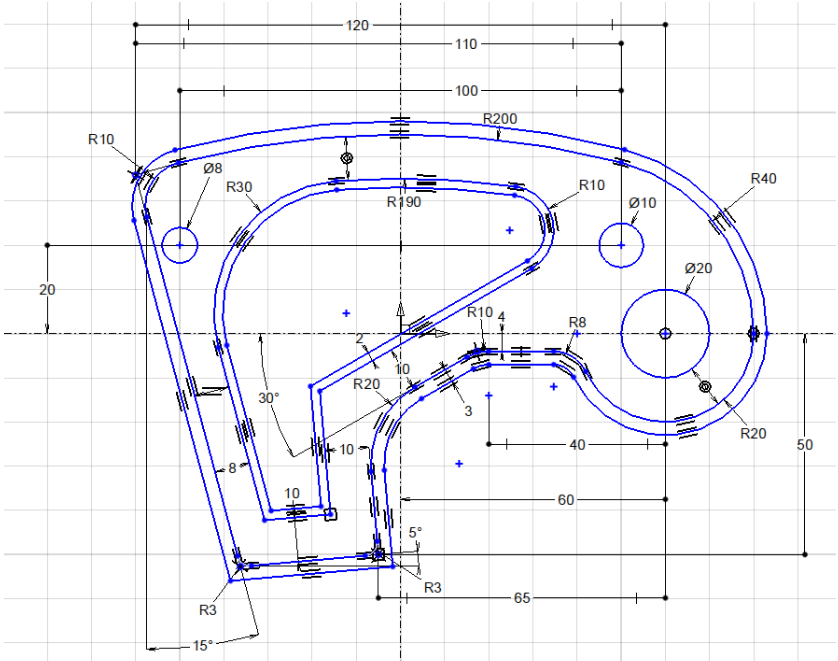


The GCS problem

The GCS problem: CAD

- Sketch figure (point, line, circle, BSpline, …) with implicit and explicit constraints (distance, angle, coincidence, …) in 2D and 3D.
- Solves the geometries with respect to constraints:
  - Tells the under and over-constrained geometries.
  - Tells the linked constraints.



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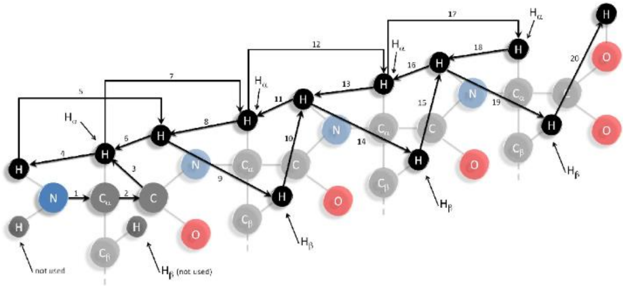
The GCS problem: others



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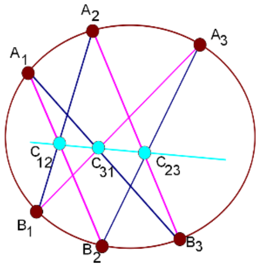
- Sketch a figure:
  - Molecular distance geometry.
  - Tensegrities.
- Solve a position:
  - Robotics.
- Automatic theorem proving



The general problem

The general problem

A graph $G = (V, E)$

V are geometries

E are constraints

An embedding map of V into a dimensional vector, affine, projective space, … : $\mathbb{R}^d$, $\mathbb{P}^d$, $\mathbb{C}^d$

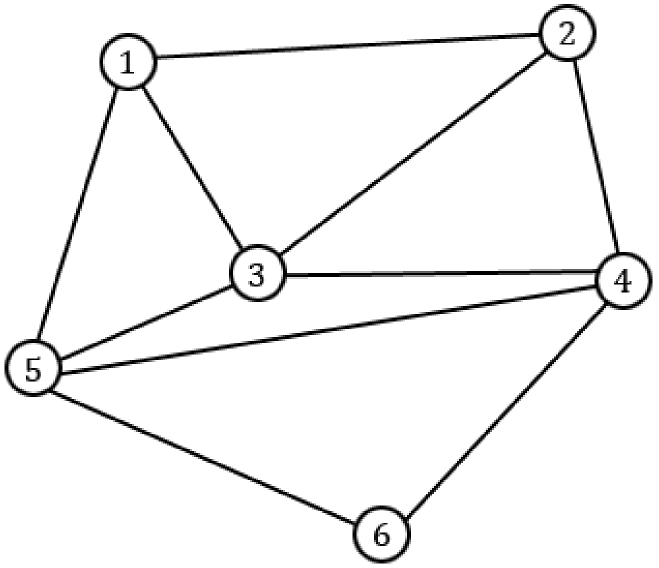
$$V \longrightarrow S$$

And elementary constraint functions:

$$S \times S \longrightarrow \mathbb{R} \text{ or } \mathbb{C}$$

$$c(v_i, v_j) = c_{ij}$$

One is looking for a realization, where the graph vertices are satisfying the elementary constraints, … or a function on the elementary constraints (in such a case, all geometries involved in the function are connected with an edge).



The general problem

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More precisely, consider the general problem of M constraints F, (C^1) fonctions on the S^n space, each geometry belonging to one S:

- Each copy S is a manifold with base vector space $\mathbb{R}^d$ and coordinates maps $x^{i\in\{1,..,d\}}$.
- The $x^{i\in\{1,..,N=n*d\}}$ are the DOF (degrees of freedom), and the $F_{j\in\{1,..,M\}}$are the constraints linking the geometries $v_{I\in\{1,..,n\}}$.
- To each geometry v_I, one can associate the set of constraints F_I, that are dependant of a DOF relative to a geometry I ($\frac{\partial F_j}{\partial x^i}$ not identically null). One can form the "Jacobian": $\frac{\partial F_j}{\partial x^i}$.
- The underlying graph, $G = (V, E)$ is given by the n d-dimensional geometries as vertices, and two vertices I_1, I_2 are linked by an edge if there is a common constraint that varies with a geometry move, i.e. $F_{I_1} \cap F_{I_2} \neq \emptyset$.
- Roughly, two geometries are linked by an edge, if there is a constraint involving the two geometries.

The general problem

Setting :

$X = (x^i)_{i \in \{1,\ldots,N=n*d\}}$

$Y = \left(F_j\left(x^i\right) - c_j\right) = \mathrm{F(X)} \in \mathbb{R}^M$

- We now have to solve $F(X) = 0$ or a classical optimization problem $Min_{X \in \mathbb{R}^N} \|F(X)\|$
- $\{X : F(X) = 0\}$ is itself a sub-manifold of $\mathbb{R}^N$, except in a singular locus, where Jacobian degenerates.

Indeed a classical engineer problem. Isn't it?

However, we have to use this precious information:

- We are talking about geometries, each belonging to a d-dimensional space.
- The underlying graph $G = (V, E)$ is a key information too!
- N and M are large, and the system is sparse.

The general problem

It is certainly worth taking into account that:

- Most of the sketches are constructible by ruler and compass, … even some are not.
- Geometry belongs to a metric space, and constraints are mainly distances and angles, the metric space invariants.
- Most of the sketches are sub $(M < N)$ or over-constrained $(M > N)$:
- One is then looking for a natural close solution from an initial one proposed by the user. The solution should be independent of a coordinates choice (frame).

Back to ruler and compass

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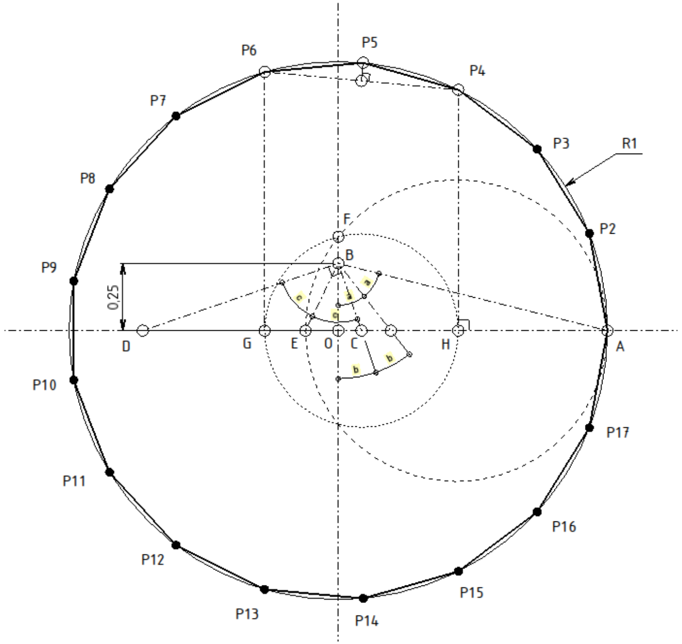
Back to ruler and compass

Solvers have existed for ages:

Most sketches can be solved using the classical “triangle and circle geometry”. It was the old way used by the draftsman, in front of his drawing board. Quite challenging sometimes, and the draftsman’s brain was the solver!



Back to ruler and compass

Euclidian geometry and graph theory:

- Knowing 3 distances, two distances and an angle, one distance and two angles, the shape of a triangle is completely solved (the sketch is rigid).
- A 2D sub-graph is presumably rigid if :$|E| = 2|V| - 3$.
- A dD sub-graph is presumably rigid if : $|E| = 2|V| - \frac{d(d+1)}{2}$.
- Geiringer-Laman theorem (1927, 1970): a 2D bar-joint $G = (V, E)$ is generically rigid, if and only if $|E| = 2|V| - 3$ and for any sub-graph $G' = (V', E')$, where $V' > 1$, $|E'| < 2|V'| - 3$.

Back to ruler and compass

This gives birth to a first strategy for Geometric Constraint Solvers:

- Analyse the graph.
- Decomposition and recomposition plan (DR plan):
  - Recognize, extract solvable sub-graph detecting dimensional coherence $N' - \frac{d(d+1)}{2} = M'$ (isostatic) : patterns as new nodes for a new graph to be analysed and further decomposed (recursive process).
  - Finally, recompose the patterns, in a constructive plan (assembly tree).
- Once the DR plan obtained, most of the irreducible components can be solved by ruler and compass method (quadratic equations). The non quadratic components are solved by a numerical method (back to the general problem), Newton or gradient like.

Back to ruler and compass

Benefits:

- A quadratic resolution scheme (QRS):

$$(N_1, M_1) \Rightarrow (N_2, M_2) \Rightarrow \cdots \Rightarrow (N_k, M_k)\ with\ N_i \ll N\ .$$

- Quadratic schemes support the chirality management: one can choose the solution chirality and display the possibly 2^k solutions.
- Fast solving process.

Drawbacks:

- Not a continuous and fluid process. Analyse of two closed sketches can lead to two different DR plans.

Back to ruler and compass

Not all sketches admit quadratic resolution schemes (solvable by ruler and compass):

- Gauss-Wantzel theorem (1796 – 1837): a n-regular polygon is constructible by ruler and compass if and only if n is a product of 2 power and Fermat numbers: p = 2^{2^k}+1.

- Wantzel theorem (1837): every ruler and compass constructible number is algebraic over field $\mathbb{Q}$, degree being 2^K(necessary condition).

A necessary and sufficient condition requires the Galois theory:

- Theorem: let $\alpha \in \mathbb{R}$, $P(X)$ its minimal polynom over $\mathbb{Q}$, and K the P dislocation field. α is constructible if and only if $[K{:}\ \mathbb{Q}]$ is a 2 power.

Modern geometry

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Modern geometry

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Using the modern definition of a geometry:

The unknown geometries to be solved are belonging to a geometric space. Following Klein, and the Erlangen program, a geometry is characterized by points in an abstract space and an associated group of transformations. Geometries are equivalent iff the underlying groups are the same.

- In our industrial case, the space is an affine space, and the transformation group is the similarities group, an extension of the classical affine group (translation, rotation) with the dilatations.
- Distance, up to a scaling factor, and angles are the invariants of the transformation group.

Felix Klein's Erlangen Program

"The remarkable power of this program is revealed in its applicability to situations which Klein himself had not yet envisaged."
H. S. M. Coxeter

Modern geometry

9.5. Similarities

Consider $f \in \mathrm{GA}(X)$ such that $\vec{f} \in \mathrm{GO}(\vec{X})$ (cf. 8.8.2). Writing μ for the ratio of $\vec{f}$, we have $f(x)f(y) = \mu xy$, and $f(x')f(y') = \mu x'y'$, whence

$$\frac{f(x')f(y')}{f(x)f(y)} = \frac{x'y'}{xy}$$

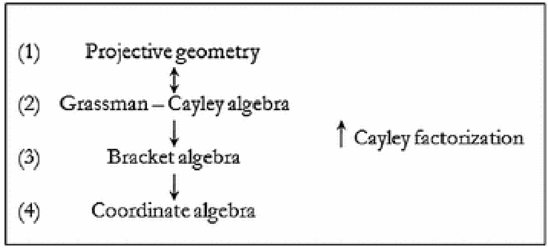
whenever $x \neq y$. In other words, f preserves ratios between distances. The converse also holds:

9.5.1. PROPOSITION AND DEFINITION. *Let $f : X \to X$ be a non-constant set-theoretical map such that $\frac{f(x')f(y')}{f(x)f(y)} = \frac{x'y'}{xy}$ whenever $x \neq y$ and $f(x) \neq f(y)$. Then $f \in \mathrm{GA}(X)$ and $\vec{f} \in \mathrm{GO}(\vec{X})$. Such maps are called similarities of X, and they are orientation-preserving or reversing according to whether $\vec{f} \in \mathrm{GO}^+(X)$ or $\vec{f} \in \mathrm{GO}^-(X)$. The set of similarities (resp. orientation-preserving, orientation-reversing) similarities of X is denoted by $\mathrm{Sim}(X)$ (resp. $\mathrm{Sim}^+(X)$, $\mathrm{Sim}^-(X)$). The ratio of f is defined as the ratio of $\vec{f}$.*

Modern geometry

Calling the theory of invariants and the Cayley bracket algebra:

- Instead of working with coordinates charts on varieties, one can use invariant quantities associated to the geometry group transformations.
- Typically the volume functions on vectors.
- From the affine space embedded within a projective space, with the structural group being $GL_n(\mathbb{R})$, one can construct the Cayley bracket algebra, where product is invariant with respect to linear changes.
- Moreover, with an additional null quadric involved, one can derive all the metric spaces.
- This is much used in the field of automatic theorem proving.



Modern geometry

An automatic proof of Pascal theorem:

Theorem 3.0.1 (Pascal, Braikenridge-Maclaurin). *Six points, $P_1, \ldots, P_6$, lie on a conic if and only if the points $P_7 = \overline{P_1P_2} \cap \overline{P_4P_5}$, $P_8 = \overline{P_2P_3} \cap \overline{P_5P_6}$, and $P_9 = \overline{P_3P_4} \cap \overline{P_6P_1}$ are collinear, as depicted in Figure 2. Equivalently, the six points lie on a conic if and only if $(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$. In the coordinate ring of the Grassmannian, the condition for six points to lie on a conic reduces to the vanishing of the binomial*

$$f = [123][145][246][356] - [124][135][236][456]. \tag{2}$$

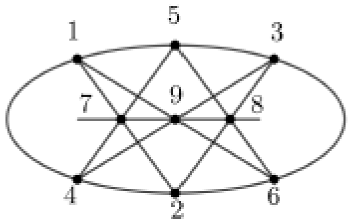


FIGURE 2. The six points $P_1, \ldots, P_6$ lie on a conic precisely when points P_7, P_8 and P_9 are collinear.

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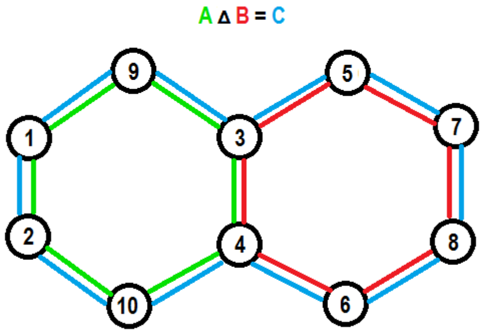
Modern geometry

Calling the theory of matroid (Whitney):

- The central algebra tool used is the matroid.
- It allows to model linear dependencies of rows and columns in a matrix.
- And can be strongly associated to graph theory and incidence matrix.
- One can use this too, to establish correspondences between the graph $G = (V, E)$ and the local differential structure given by the "Jacobian": $\frac{\partial F_j}{\partial x^i}$.



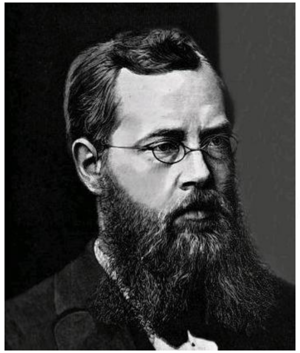
Modern geometry

Calling to Lie groups and Lie theory:

- One can then reformulate the general optimization problem:
  - Starting form an initial position $(v_1(0), \dots, v_n(0))$, one is looking for the transformations $(g_1, \dots, g_n)$ such:

$\big(v_1(t), \dots, v_n(t)\big) = (g_1, \dots, g_n).\,(v_1(0), \dots, v_n(0))$ will satisfy the constraints.

  - In this setting, the unknown are no longer the v_i, but the transformations acting on the v_i. These transformations belong to the variety ${}^{G}/_{Is(v)}$, as the quotient of the structural group considered (similarities) by the isotropy group of v.
  - Because the constraints are formulated as invariants of the transformation group, the "diagonal" transformation $(g, \dots, g)$ is a trivial solution too.



Modern geometry

Calling to Riemannian geometry:

- We are solving elements belonging to a product of Lie groups, modulo isotropy groups, and the diagonal group.
- The gradient flow is over a quotient of a Lie group.
- The Lie group is equipped with a bi-invariant metric (Killing metric).

Modern geometry

Pure distances GCS:

In the GCS theory related to Math, the problem intensively studied is the distance problem, where the v are points embedded in an affine space, and constraints are distances between these points.
The central tool is the Cayley-Menger determinant, shaping the form. It captures some interesting results about these kind of GCS:

$$\Gamma(x_0, x_1, \ldots, x_k) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{01}^2 & \cdots & d_{0k}^2 \\ 1 & d_{10}^2 & 0 & \cdots & d_{1k}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{k0}^2 & d_{k1}^2 & \cdots & 0 \end{vmatrix}$$

Modern geometry

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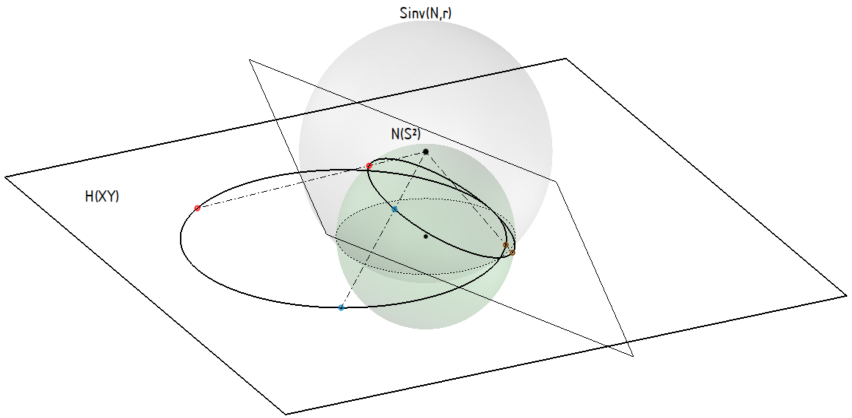
9.7.3.4. Theorem. *Let $(x_i)_{i=0,1,\ldots,n}$ be arbitrary points in an $(n-1)$-dimensional Euclidean affine space X. Then $\Gamma(x_0, x_1, \ldots, x_n) = 0$. A necessary and sufficient condition for $(x_i)_{i=0,1,\ldots,n-1}$ to be a simplex of X is that $\Gamma(x_0, x_1, \ldots, x_{n-1}) \neq 0$. Given $k(k+1)/2$ real numbers d_{ij} $(i, j = 0, 1, \ldots, k)$, a necessary and sufficient condition for the existence of a simplex $(x_i)_{i=0,1,\ldots,n}$ satisfying $d_{ij} = x_i x_j$ is that for every $h = 2, \ldots, k$ and every h-element subset of $\{0, 1, \ldots, k\}$ the corresponding Cayley–Menger determinant be non-zero and its sign be $(-1)^{k+1}$.*

From an industrial point of view, it is however too reductive, and we have at least to deal with points, lines, circles as geometries and distances, angles and radii as constraints.

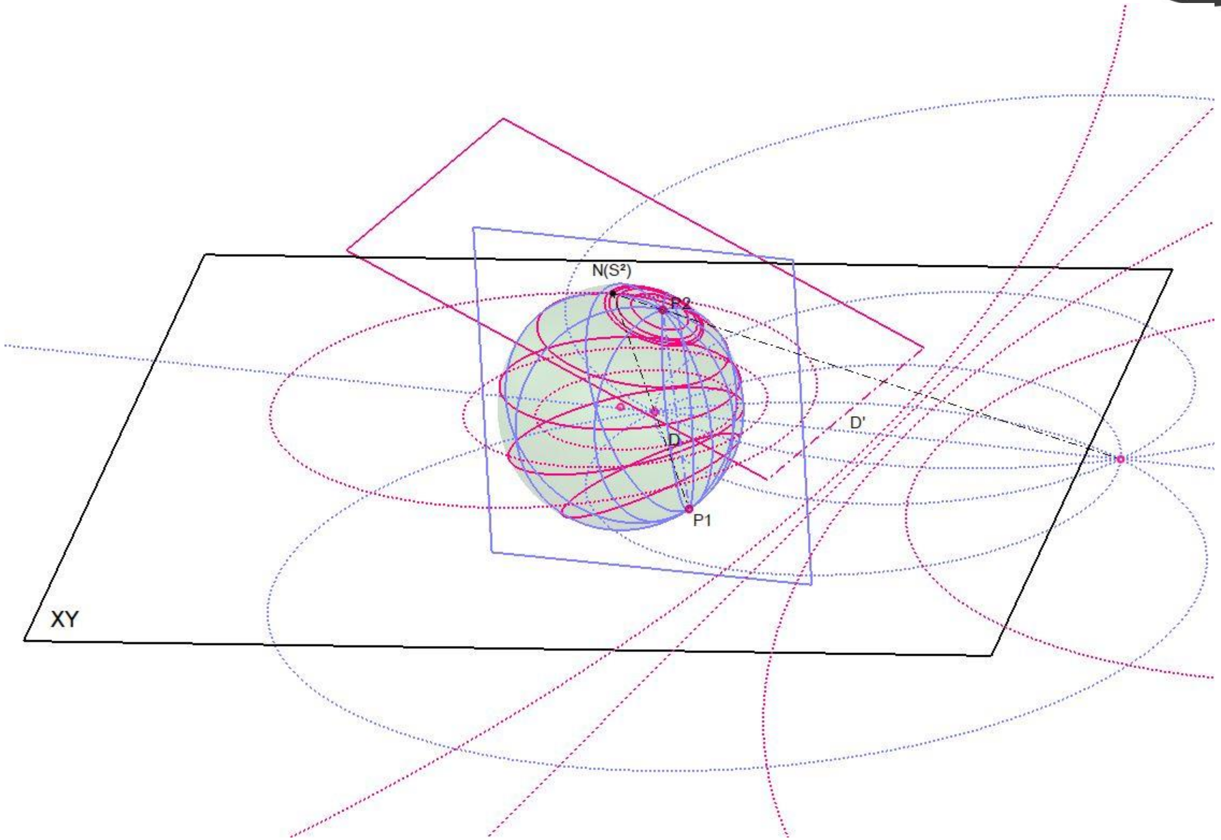
Modern geometry

Spheres space:

There is however a space, the space of spheres, as a sub space of the space of quadratic forms, that allows to extend the problem, combining a pure math formulation and a broader industrial scope.



Modern geometry



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Modern geometry : spheres space

Geometries:

$$S = n_x.H_x + n_y.H_y - n_z.S_1 + \text{v}.S_i$$

$$S = n_x.H_x + n_y.H_y + (-n_z + v).O + \left(\frac{-n_z - v}{2}\right).\infty$$

$H_x \; real\; y\; axis : x = 0$

$H_y \; real\; x\; axis : y = 0$

$S_1 \; real\; unit\; sphere: -\frac{1}{2}\left(x^2 + y^2 - 1\right) = 0 : S(O, 1)$

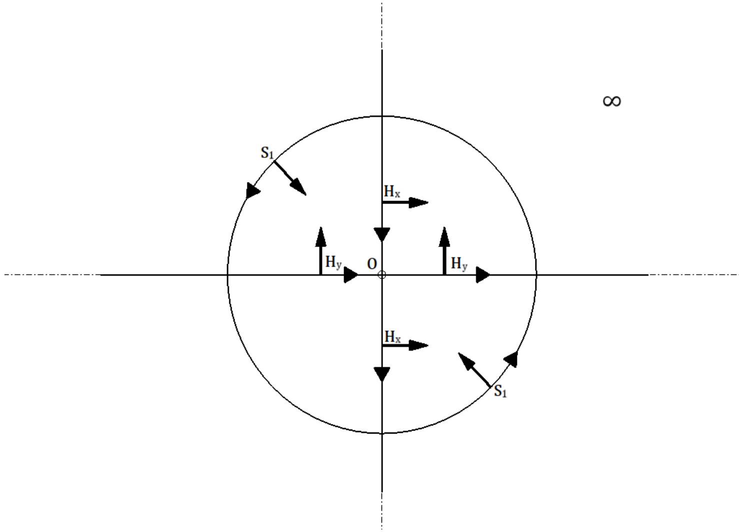
$S_i \; imaginary\; \text{"unit"}\; sphere: -\frac{1}{2}\left(x^2 + y^2 + 1\right) = 0: S(O, -1)$

$O\; origin\; point\; (south\; pole): -\frac{1}{2}\left(x^2 + y^2\right) = 0 : O = \frac{1}{2}(S_1 + S_i)$

$\infty\; infinite\; point\; (north\; pole): 1 = 0!!! : \; \infty = S_1 - S_i$

$$q(S) = n_x^2 + n_y^2 + n_y^2 - v^2 = \vec{n}.\vec{n} - v^2 : Lorentz\; quadratic\; form$$

$$b(S, S') = n_x.n_x' + n_y.n_y' + n_z.n_z' - v.v' = \vec{n}.\vec{n}' - v.v'$$



Modern geometry : spheres space

Constraints:

<u>Shape constraint :</u>

Point P : $q(P) = 0$

Line D: $q(D) = 1, b(D, \infty) = 0$

Sphere S: $q(S) = 1, b(S, \infty) = -\frac{1}{r}$

<u>Topological constraint:</u>

$$P \in S \iff b(P) = 0$$

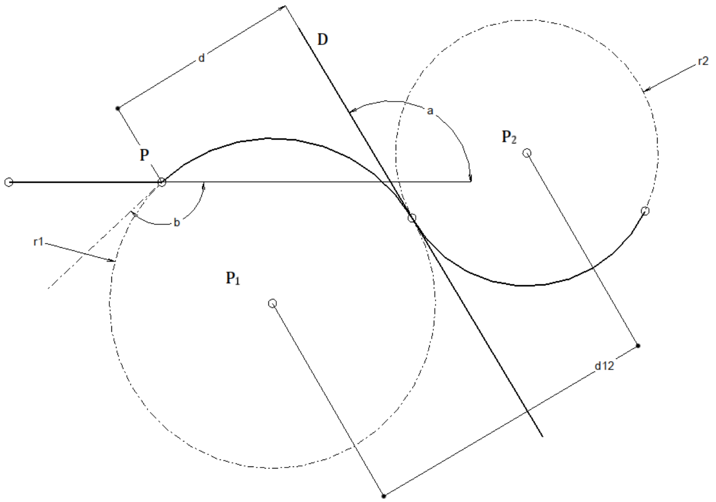
<u>Distance constraint:</u>

$$b(P, P') = -\frac{1}{2}d^2(P, P'), \qquad b(P, D) = d(P, D),$$

$$b(P, S) = -\frac{1}{2}\left(d^2(P, P') - r^2\right)$$

<u>Angle constraint :</u>

$$\cos(S, S') = b(S, S')$$



Modern geometry

One can combine these two “modern” geometrical points of view:

- Use a larger quadratic space of signature $(d+1,1)$, to deal with points, hyperplanes and spheres as vectors of a same vector space
- Solve transformations as member of a Lie group $SO_{(d+1,1)}$, instead of solving manifold coordinates

Finding a solution is a gradient flow descent on a Riemannian manifold as a product of Lie groups, the metric being given by the iso-constraints lines. The metric can degenerate in some points, as singular locus of the Riemannian metric. This is the solving method implemented inside TopSolid.GCS.

Algebraic geometry now!

Algebraic geometry now!

Always using the spheres space formulation, we can reformulate the question, with enough "industrial" perspective, as:

Find $(v_1, \dots, v_n)$, with some constraints $b(v_i, v_j) = c_{ij}$
Using a standard 2D basis, (H_x, H_y, S_1, S_i), and coordinates (x, y, z, t) we have to solve a set of quadratic + linear equations of the special "Lorentz-Minkowski" kind:

$$x_i x_j + y_i y_j + z_i\, z_j - v_i v_j = c_{ij}$$
$$x_k^2 + y_k^2 + z_k^2 - v_k^2 = c_{kk}$$
$$a_l x_l + b_l y_l + c_l z_l - d_l v_l = c_{ll}$$

Algebraic geometry now!

$$x_i x_j + y_i y_j + z_i\, z_j - v_i v_j = c_{ij}$$
$$x_k^2 + y_k^2 + z_k^2 - v_k^2 = c_{kk}$$
$$a_l x_l + b_l y_l + c_l z_l - d_l v_l = c_{ll}$$

An upper-bound 2^K for the number of solutions should be given by Bezout. A more precise bound should be given by the mixed volume of the associated Newton polytope, which should have strong relationship with the underlying $G = (V, E)$ graph.

And maybe there is, or will be a good (fast, robust, complete) solver among the sparse polynomial solvers available or in the mathematician's pocket !

Algebraic geometry now!

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Bernstein theorem (1975):

Our object of study is a system $F = (f_1, \ldots, f_n)$ of polynomial equations of the form

$$f_i = \sum_{\mathbf{q} \in \mathcal{A}_i} c_{i,\mathbf{q}} \cdot \mathbf{x}^{\mathbf{q}}, \qquad \text{where} \quad c_{i,\mathbf{q}} \in \mathbf{C}^* \quad \text{and} \quad \mathbf{x}^{\mathbf{q}} = x_1^{\mathbf{q}_1} \cdots x_n^{\mathbf{q}_n}. \tag{1}$$

Here $\mathcal{A}_i$ is a finite subset of $\mathbf{N}^n$, called the *support* of f_i, and $Q_i = \operatorname{conv}(\mathcal{A}_i)$ is the *Newton polytope* of f_i. The *mixed volume* $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is the coefficient of $l_1 l_2 \cdots l_n$ in the homogeneous polynomial $\operatorname{Vol}(l_1 Q_1 + \cdots + l_n Q_n)$, where Vol is the Euclidean volume, and

$$Q_1 + \cdots + Q_n := \{x_1 + \cdots + x_n \in \mathbf{R}^n \colon x_i \in Q_i \text{ for } i = 1, \ldots, n\} \tag{2}$$

denotes the Minkowski sum of polytopes [2]. The following toric root count is well known.

Theorem 1 (Bernstein's Theorem [1]). *The number of isolated zeros of F in $(\mathbf{C}^*)^n$ is bounded above by $\mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. This bound is exact for generic choices of the coefficients $c_{i,\mathbf{q}}$.*

Algebraic geometry now!

Determinantal variety:

Another approach can be obtained considering the symmetric shape matrix:

$$MS = \begin{bmatrix} b(s_1, s_1) & \cdots & b(s_1, s_n) \\ \vdots & \ddots & \vdots \\ b(s_n, s_1) & \cdots & b(s_n, s_n) \end{bmatrix}$$

This is a symmetric matrix, all minors (m, m) with $m > d + 2$ have null determinant. It can be considered as a determinantal variety. Constraints are sections of the determinantal variety. The number of solutions is then limited, given by the algebraic degree of the variety.

Algebraic geometry now!

According to Harris and Tu (1984), we have:

(c) ***Application*: *The degrees of determinantal varieties***

As an application of the degeneracy locus formulas we will now compute the degrees of determinantal varieties in a projective space.

PROPOSITION 12. (a) *Let V be the space of all $m \times n$ matrices and V_k those of rank at most k. Then in the projective space $\mathbb{P}(V)$,*

$$\deg \mathbb{P}(V_k) = \prod_{\alpha=0}^{m-k-1} \frac{\binom{n+\alpha}{m-1-\alpha}}{\binom{n-k+\alpha}{m-k+-\alpha}}.$$

(b) *Let W be the space of all $n \times n$ symmetric matrices and S_r those of corank at least r. Then in $\mathbb{P}(W)$*

$$\deg \mathbb{P}(S_r) = \prod_{\alpha=0}^{r-1} \frac{\binom{n+\alpha}{r-\alpha}}{\binom{2\alpha+1}{\alpha}}$$

Back to 3D geometry

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Back to 3D geometry

A very open problem:

- Most of the publications are dealing with the 2D problem (plan sketch). CAD modelling is indeed a 3D question, and the same problem has to be solved in 3D.
- The previous formulations are working in 3D too.
- According to the literature, very few is known considering $G = (V, E)$ embedded in a 3D space and more.
- The spheres space is a quadratic space, which orthogonal group is $SO_{(d+1,1)}$.
  - In 2D this is $SO_{(3,1)}$ with strong isomorphisms. The transformations are the 2D Möbius transformations: $z \mapsto \frac{az+b}{cz+d}$.
  - In 3D this is $SO_{(4,1)}$, not so common.

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Back to 3D geometry : rigidity

Cauchy theorem (1813):

12.8.1. THEOREM (CAUCHY). *Let P, P' be two convex polytopes in X and $f : \operatorname{Fr} P \to \operatorname{Fr} P'$ a bijection taking vertices into vertices, edges into edges and faces into faces. Assume that, for each face F of P, the restriction $f|_F : F \to f(F)$ is an isometry. Then there exists an isometry $\overline{f}$ of X such that $\overline{f}(P) = P'$ and $\overline{f}|_{\operatorname{Fr} P} = f$; in particular, P and P' are isometric.*

12.8.2. COROLLARY. *A convex polytope P is not flexible, that is, if $P(t)$, $t \in [0,1]$ is a family of (not necessarily convex) polyhedra with $P(0) = P$ and $f_t : \operatorname{Fr}(P(0)) \to \operatorname{Fr}(P(t))$ $(t \in [0,1])$ is a family of bijections such that $f_t|_F$ is an isometry between F and $f_t(F)$ for every face F of $P(0)$, that $f_0 = \operatorname{Id}_P$ and that*

$$f : \operatorname{Fr} P \times [0,1] \ni (x,t) \mapsto f_t(x) \in X$$

is continuous, then there exists a family $\overline{f}_t \in \operatorname{Isom}(X)$ $(t \in [0,1])$, such that $\overline{f}_t|_{\operatorname{Fr}(P(0))} = f_t$ and $\overline{f}_t(P(0)) = P(t)$ for any $t \in [0,1]$.

Back to 3D geometry

One had to wait until recently :

- Connelly (1977!) was first to exhibit a non convex polytope with rigid faces that was not rigid in the 3D space.
- This is only in 1974, that Gluck proved that the convex requirement is a non generic requirement:
- A non rigid polyhedral has singular dimensions, i.e. formulated as a GCS, it has a singular Jacobian. So difficult to manufacture a flexible polyhedron!

If the question of rigidity deals with order 0 and order 1, it has to be studied exploring the further orders:

- this is the question of tensegrities, which is strongly connected to the one of structural analysis (FEA).

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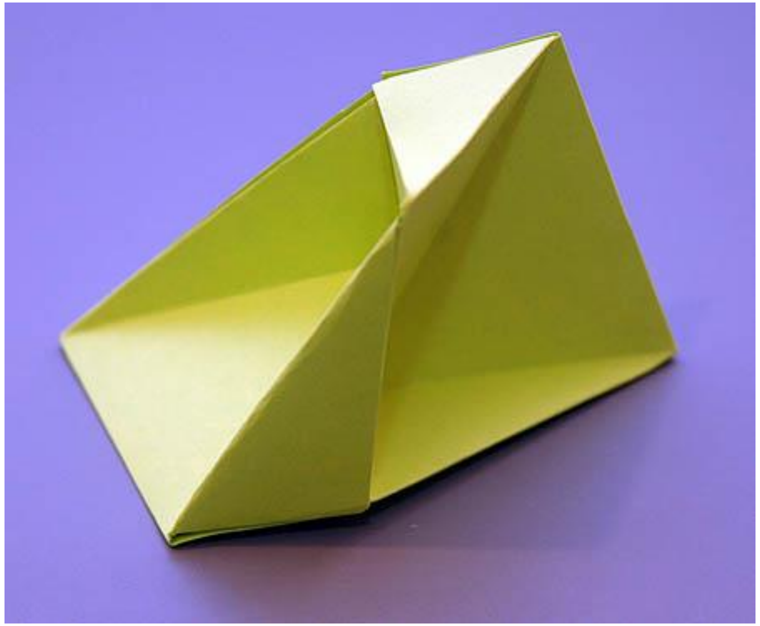
Back to 3D geometry: rigidity

Flexible polyhedron:

- Steffen example
- Flexible and not generically rigid:
  - special relationships between distances are required
  - tensegrity existence



Conclusion

Conclusion

GCS as Geometric Constraint Solver :

- An important and practical algorithm, used daily by millions CAD users (and gamers !).
- It is a very competitive question with important economic implications.
- An appealing problem for mathematicians : a lot still to be discovered !
- Could be a very good challenge for advancing Maths, and establishing better connexions between Maths community and advanced Industrial Software development.
- Old and new geometry is back, as always!

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Questions?

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