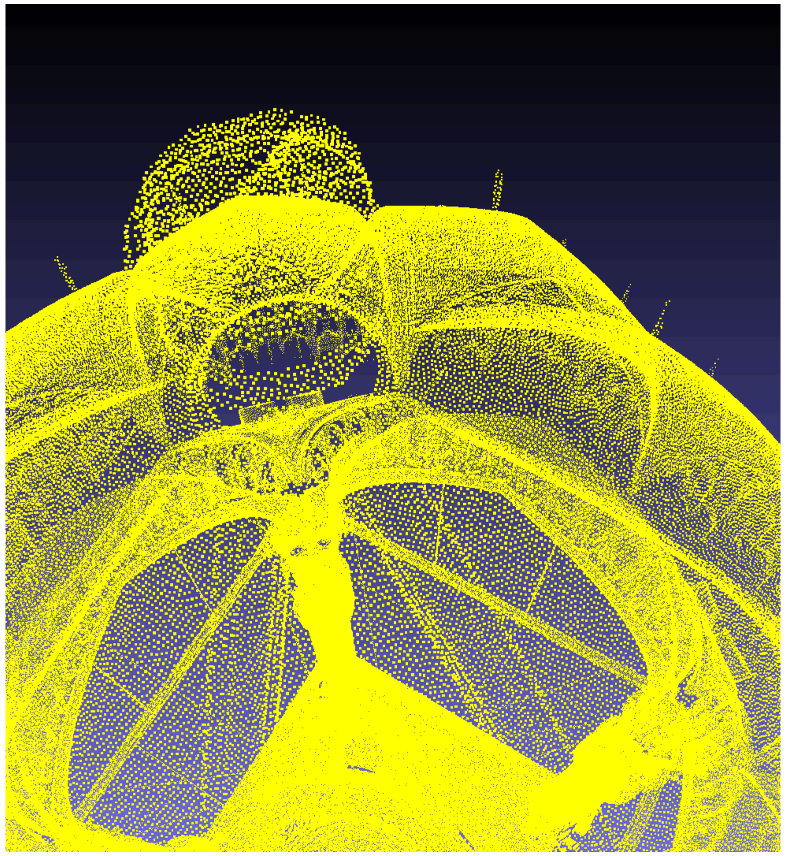
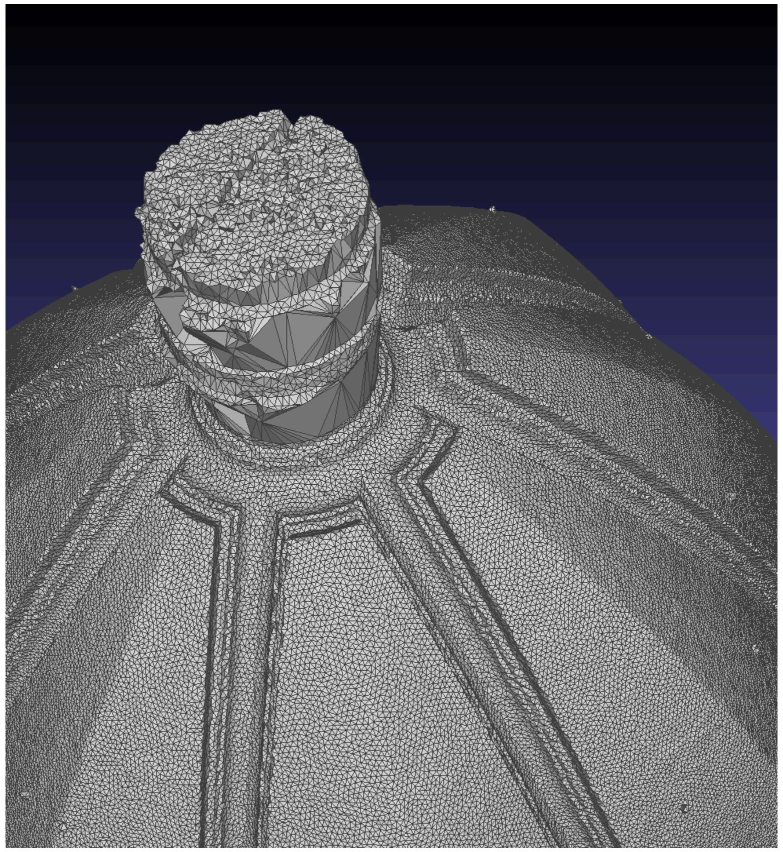
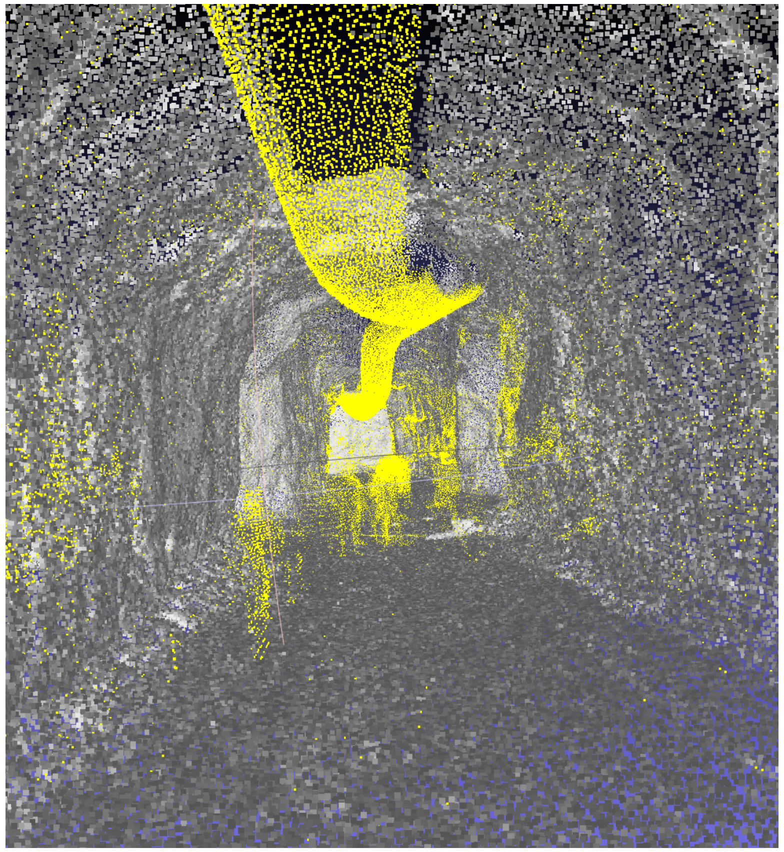
From topological inference to meshing algorithms

André Lieutier- Geometry Engineering







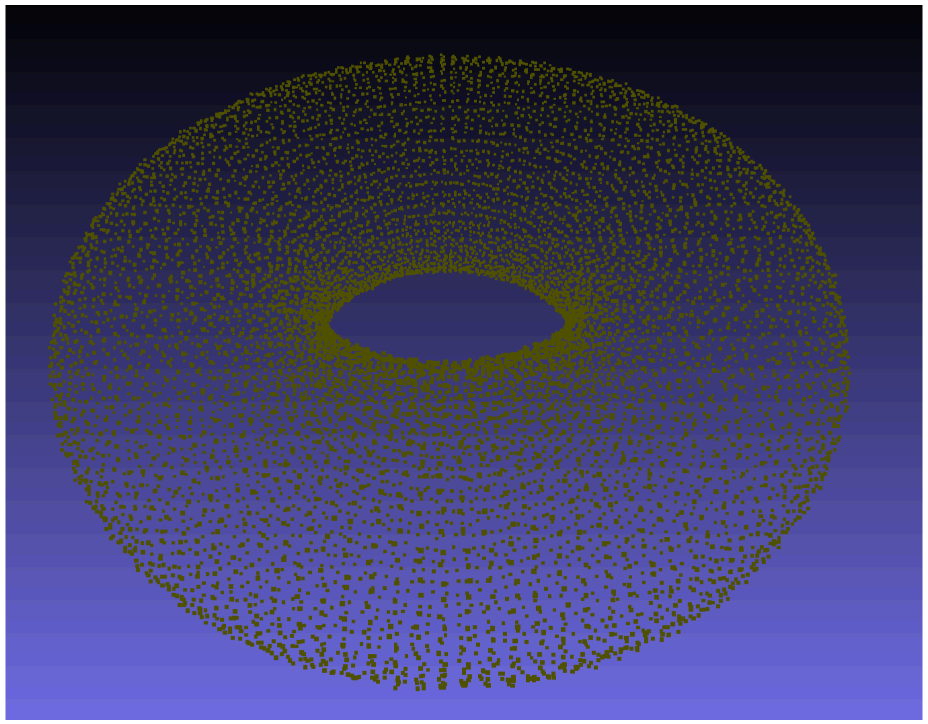
GRAPES-Software & Industrial workshop (Athens Feb. 2023)

Inferring topology from data

Part 1 focuses on the computation of a **simplicial complex** which reproduces the **homotopy type**.

In **part 2** we consider the computation of **homeomorphic** simplicial complexes, in other words **Triangulations**

Topological faithful reconstruction and topological inference



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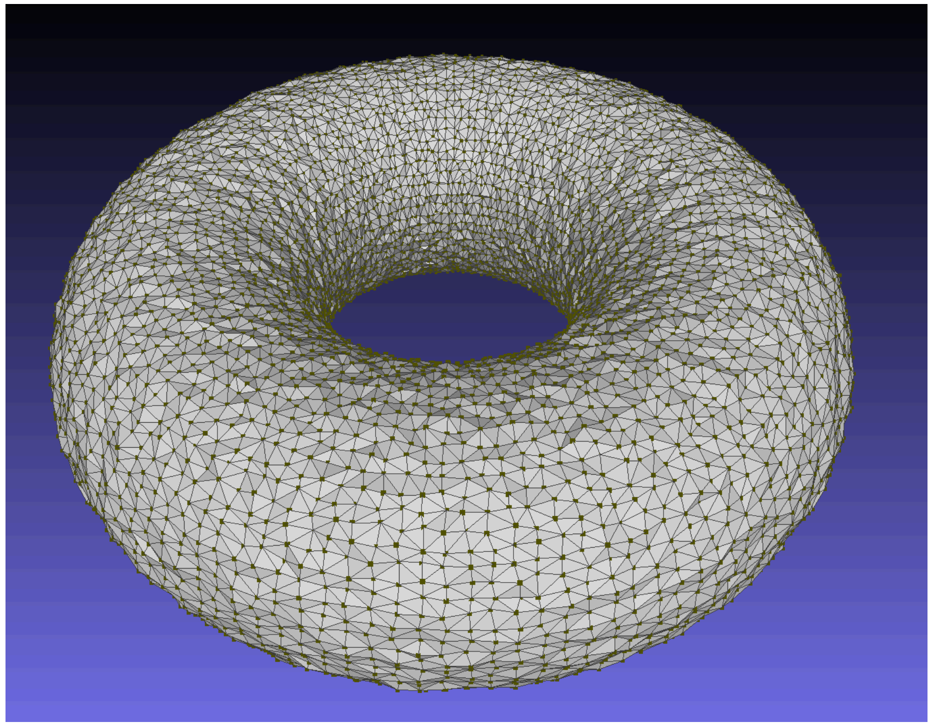

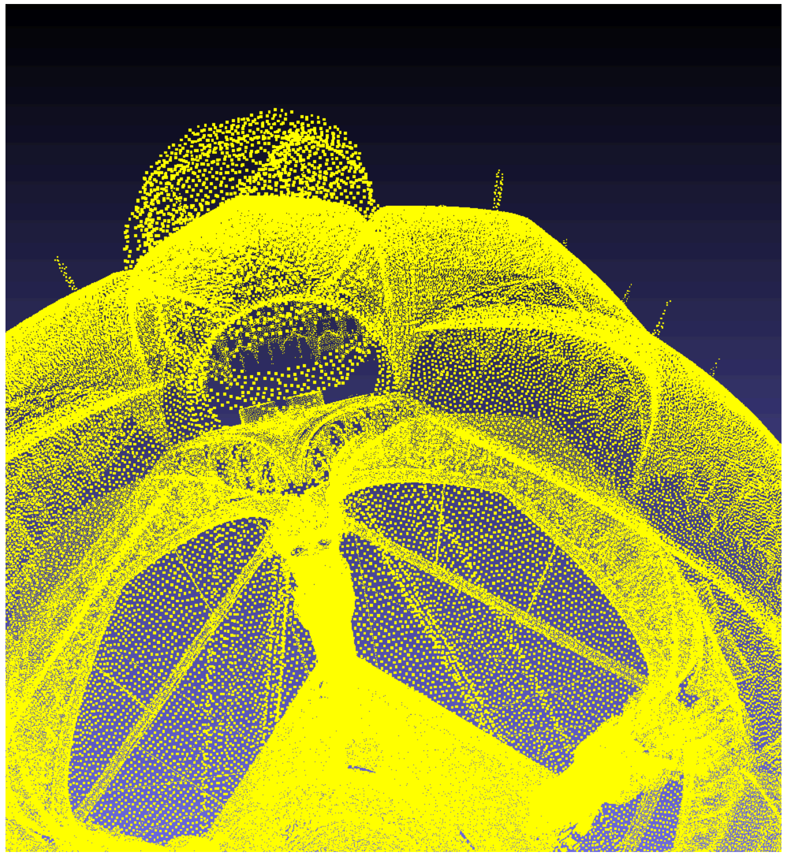

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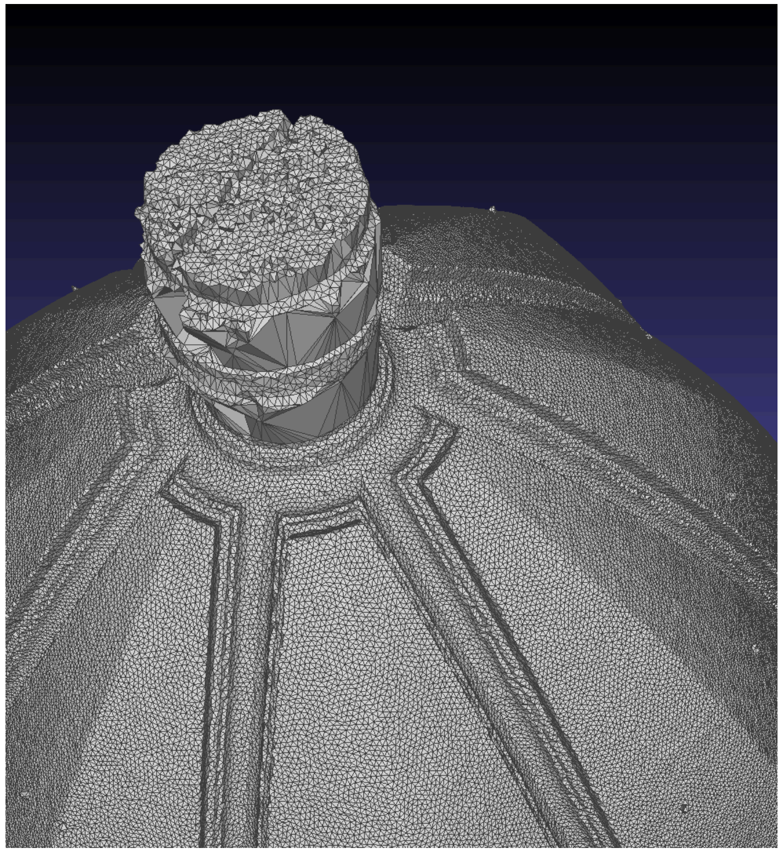
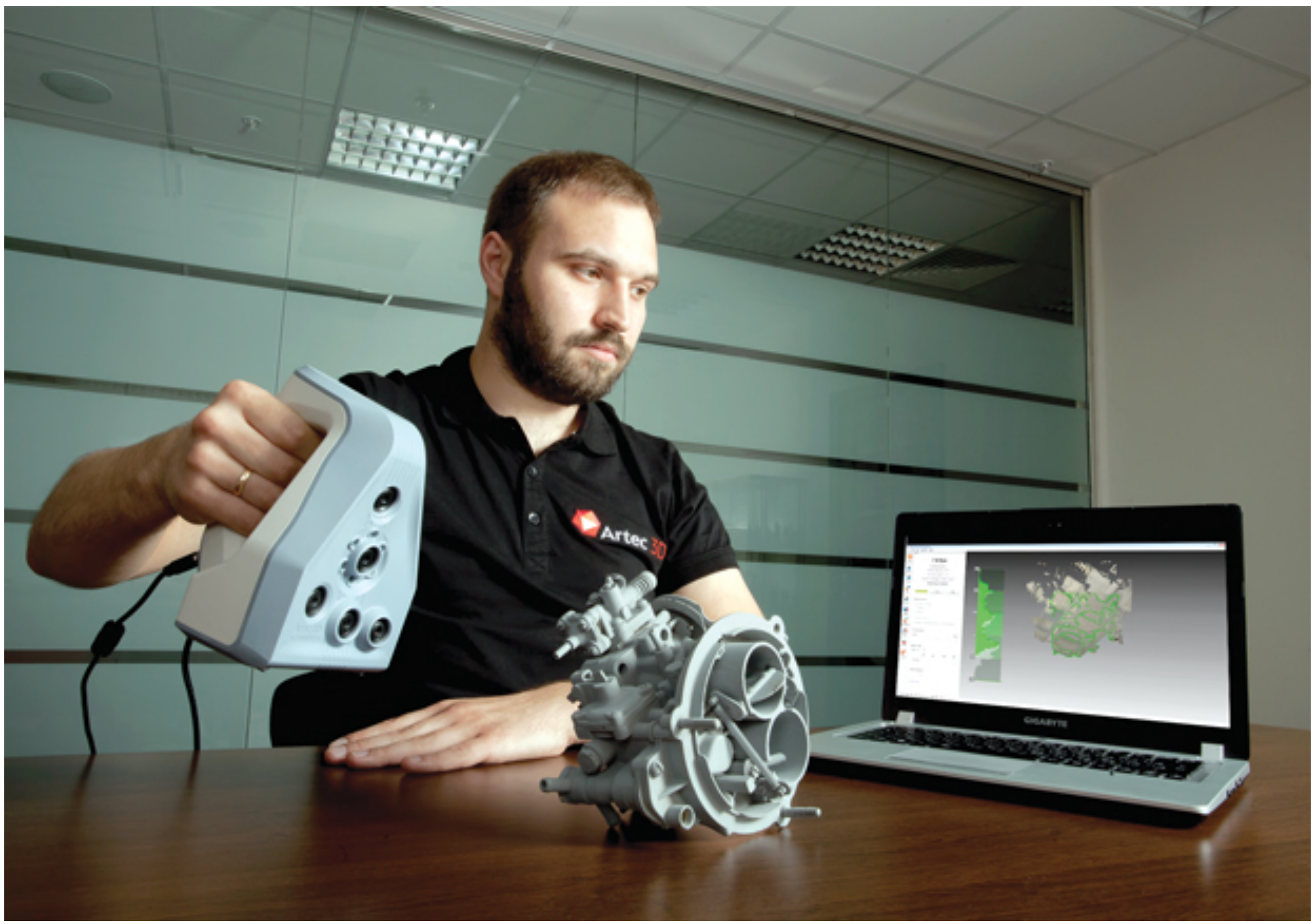


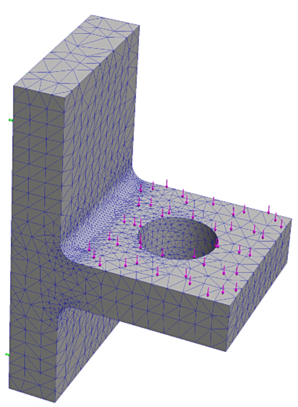
Topological faithful reconstruction and topological inference

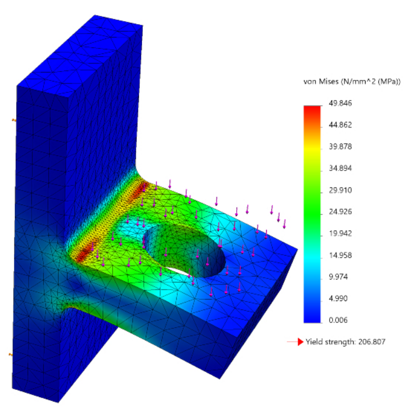


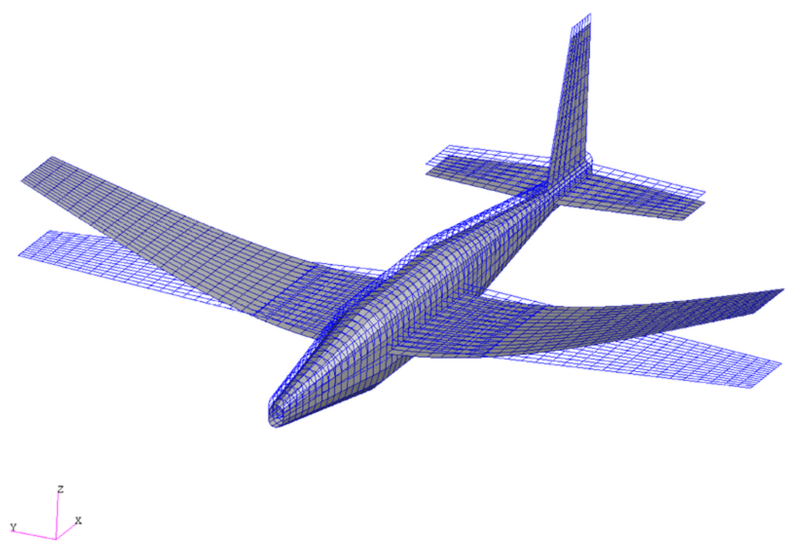
Topological faithful reconstruction and topological inference

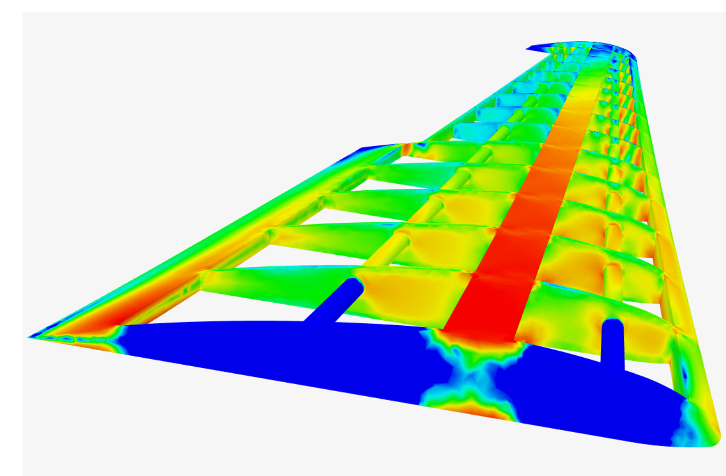
Reconstruction beyond visual realism: understanding the **topology**

Topological faithful reconstruction and topological inference





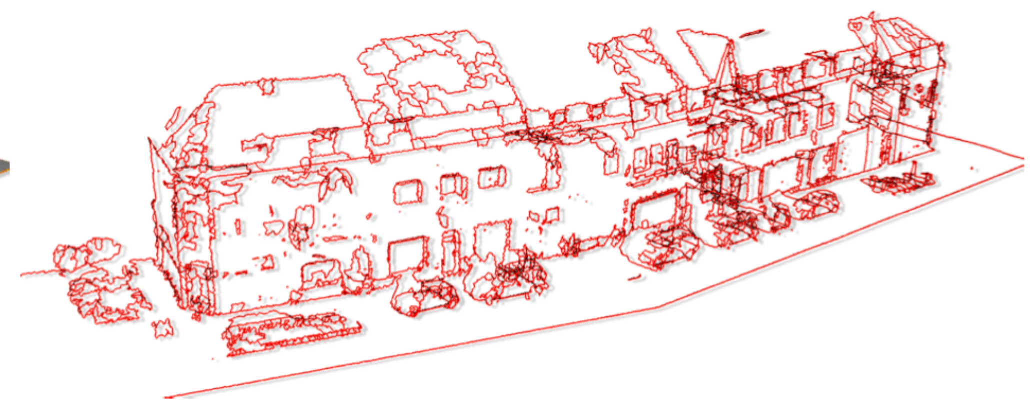




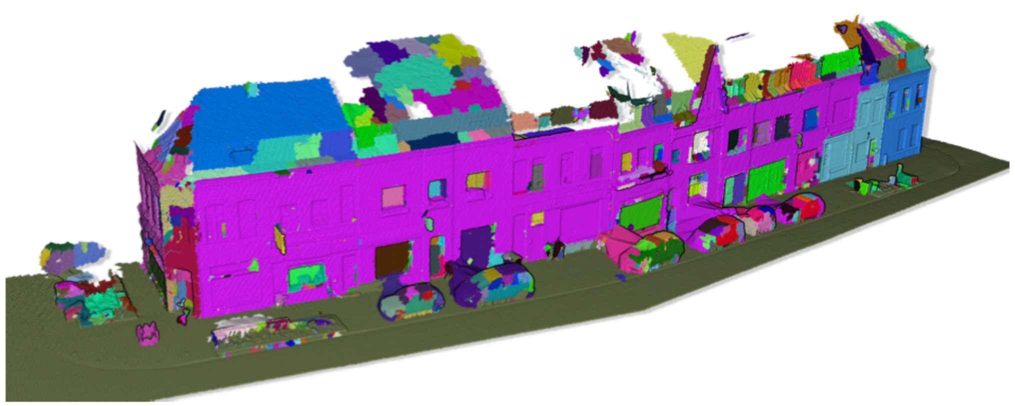
Reconstruction beyond visual realism:
understanding the **topology**

Topological faithful reconstruction and topological inference

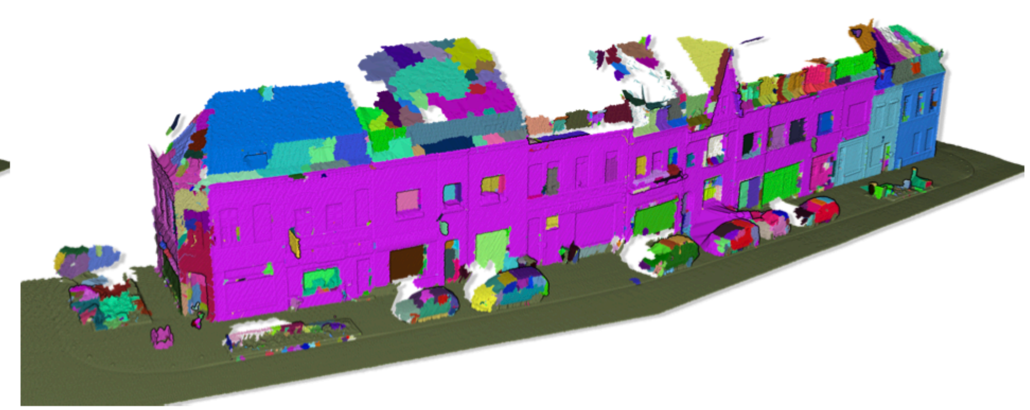
(a) Sharp feature estimation



(b) Cycle basis of $\mathbf{Z}_1^{\min}(B)$



(c) Critical basis of $\mathbf{Z}_2^{\min}(K, B)$



(d) Removing the first few chains

Reconstruction beyond visual realism:
Topology driven segmentation

Motivation: TDA
(**T**opological **D**ata **A**nalysis)

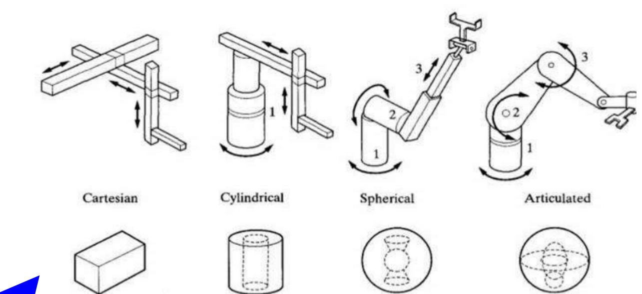
MANIFOLD LEARNING

input:

Data= movie/pictures

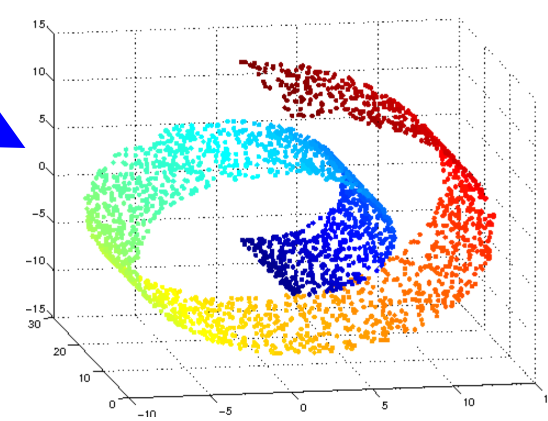
Topological inference
(topology learning)

output:



Robot configuration space

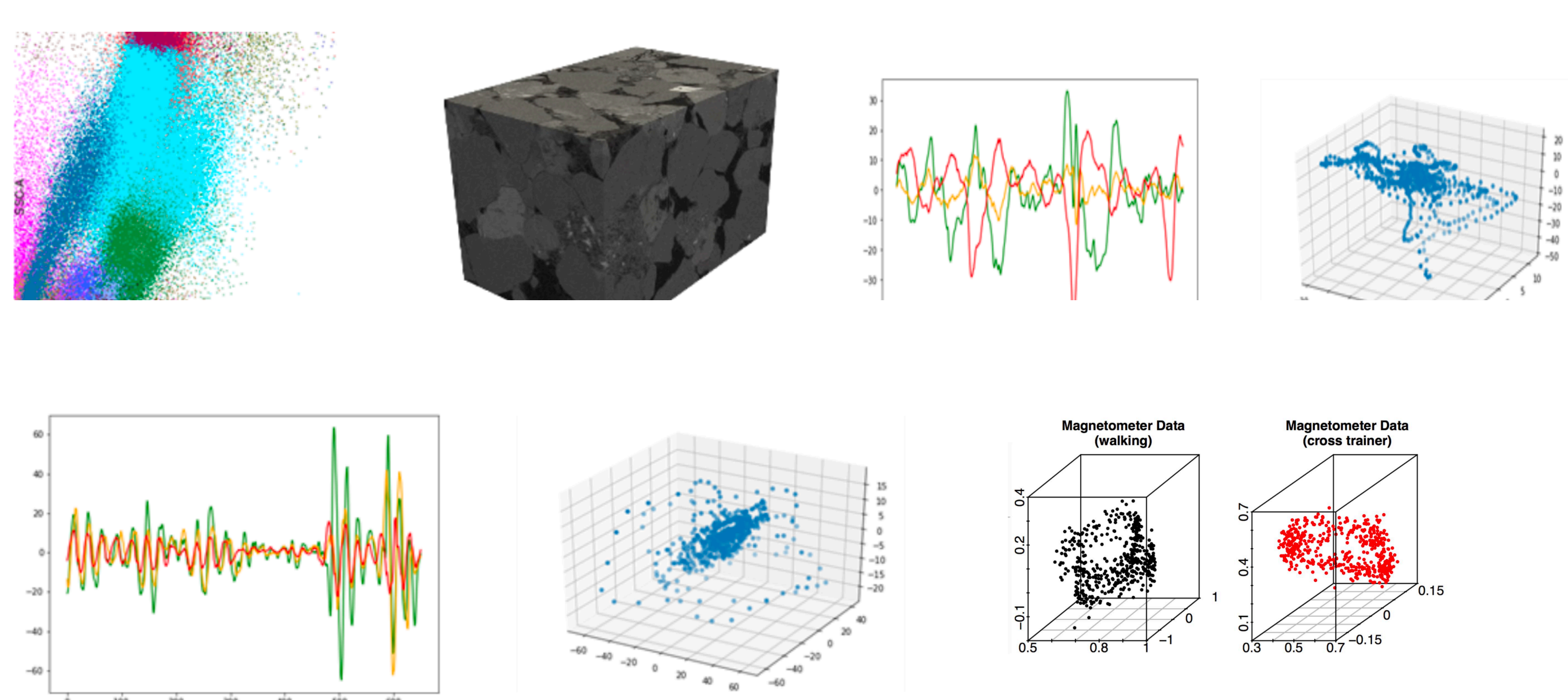
Manifold
reconstruction



topology
computation

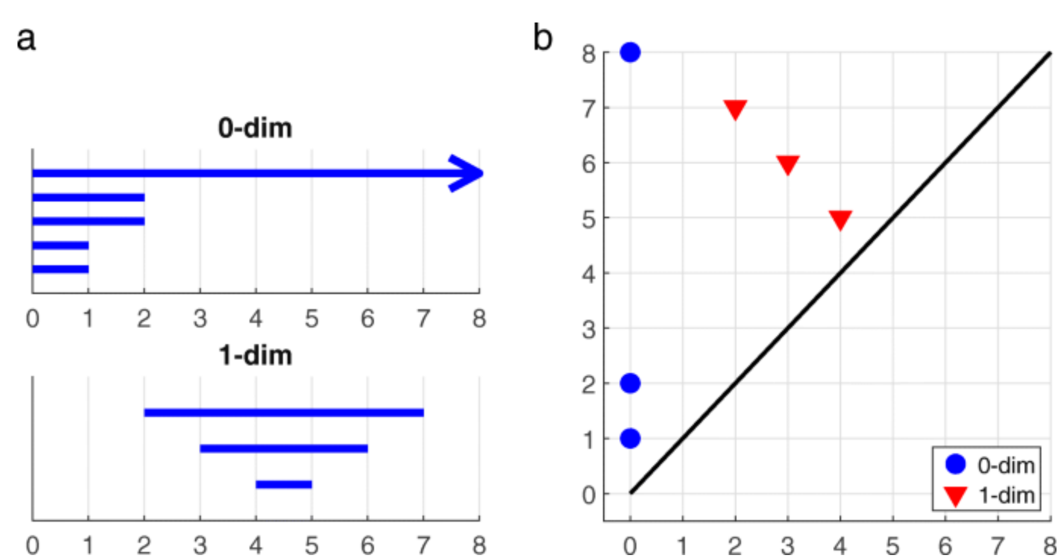
For example as
simplicial complex
= triangulation

Motivation: TDA
(**T**opological **D**ata **A**nalysis)



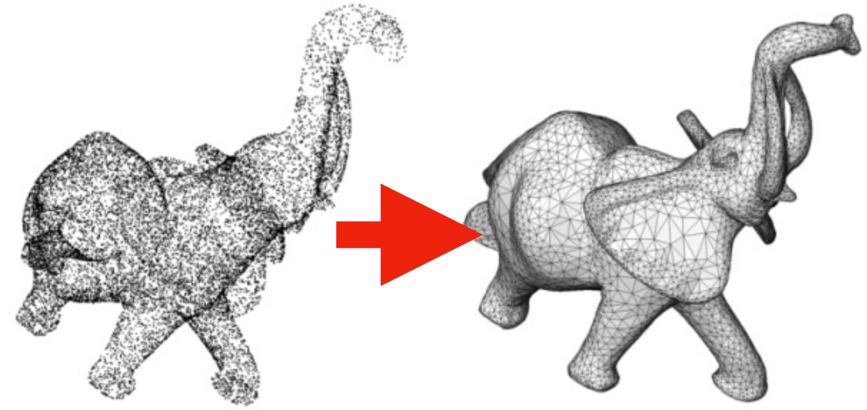
from raw data

To persistent homology
(barcode/diagram)

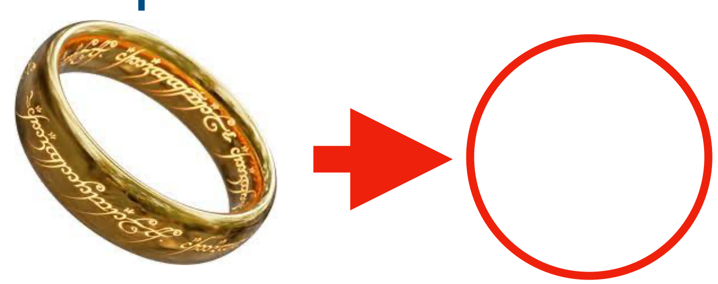


What does it mean to recover the topology ?
(of subsets of euclidean space)?

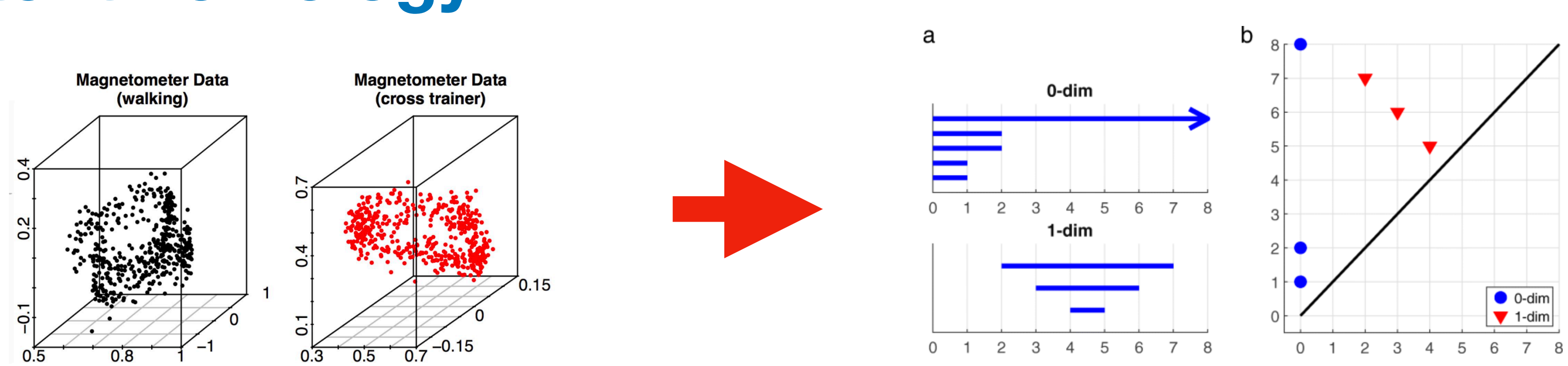
- Computing a finite representation, typically a **simplicial complex** which is **homeomorphic** = **triangulation** (or meshing)



- Computing a finite representation that shares the **homotopy type**



- Computing some **topological invariants, homology** and **persistent homology**



Inferring topology from data

Part 1 focuses on the computation of a simplicial complex which reproduces the **homotopy type**.

What does it mean to recover the topology ?

(of subset of euclidean space)

- Computing a finite representation, typically a **simplicial complex** which is **homeomorphic** = **triangulation** (or meshing)

WIKIPEDIA
The Free Encyclopedia

A function $f : X \to Y$ between two topological spaces is a **homeomorphism** if it has the following properties:

- f is a bijection (one-to-one and onto),
- f is continuous,
- the inverse function f^{-1} is continuous (f is an open mapping).

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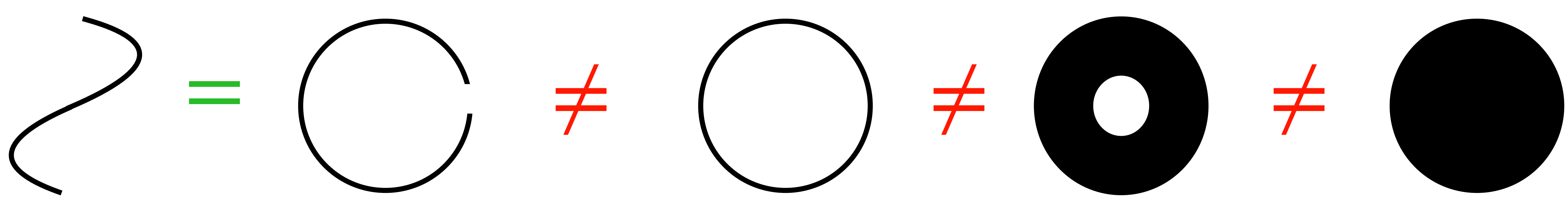

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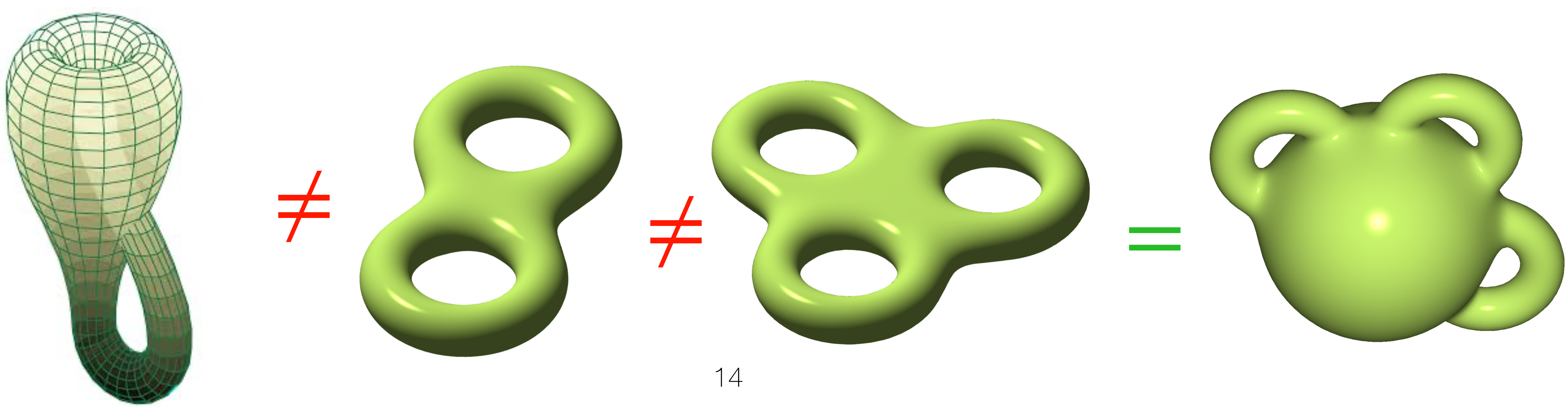

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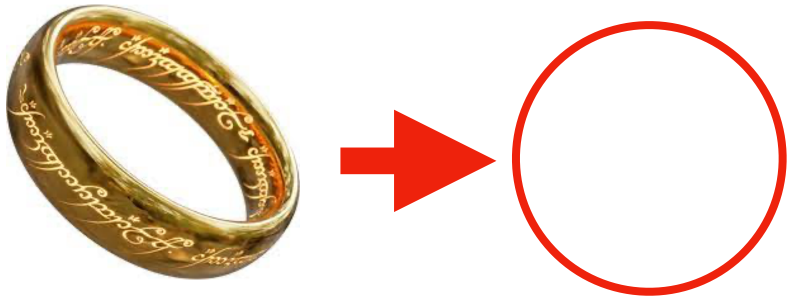
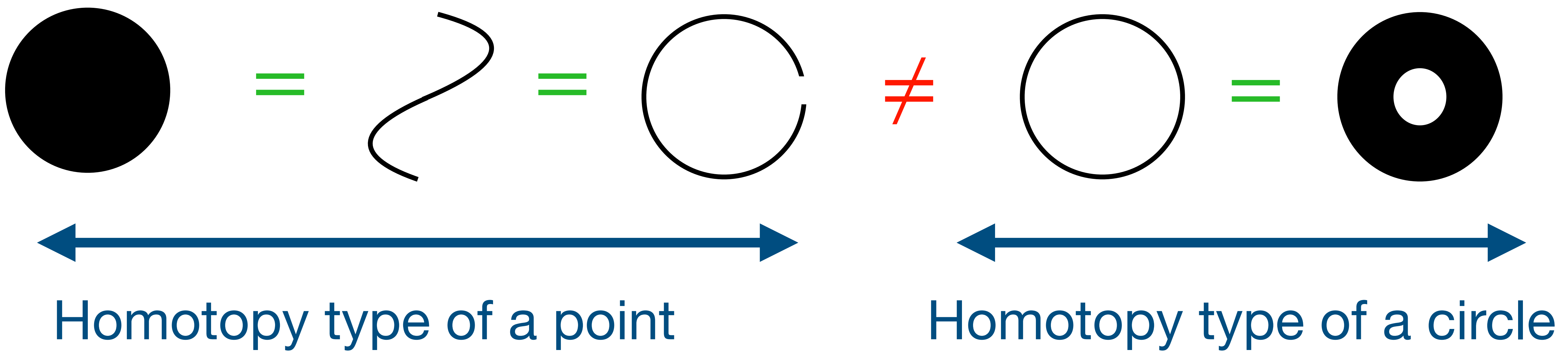
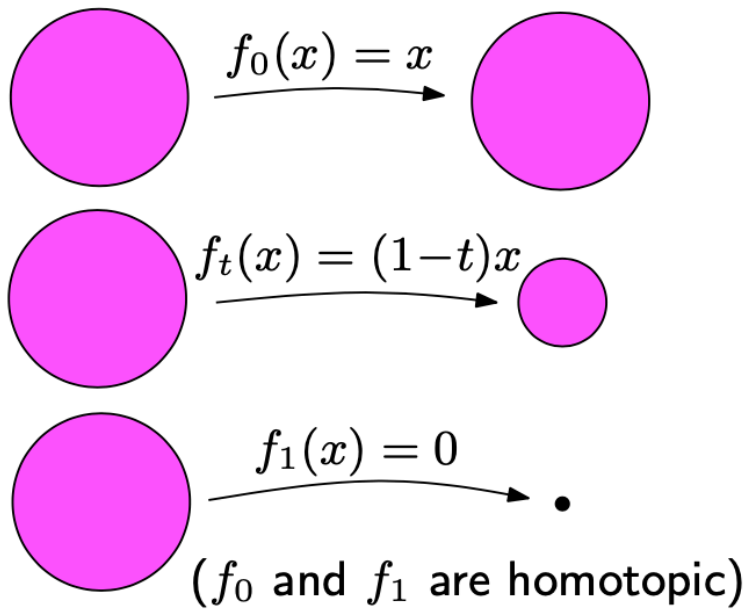
- Computing a finite representation that shares the **homotopy type**





Homotopy type

(thanks to Frederic Chazal)

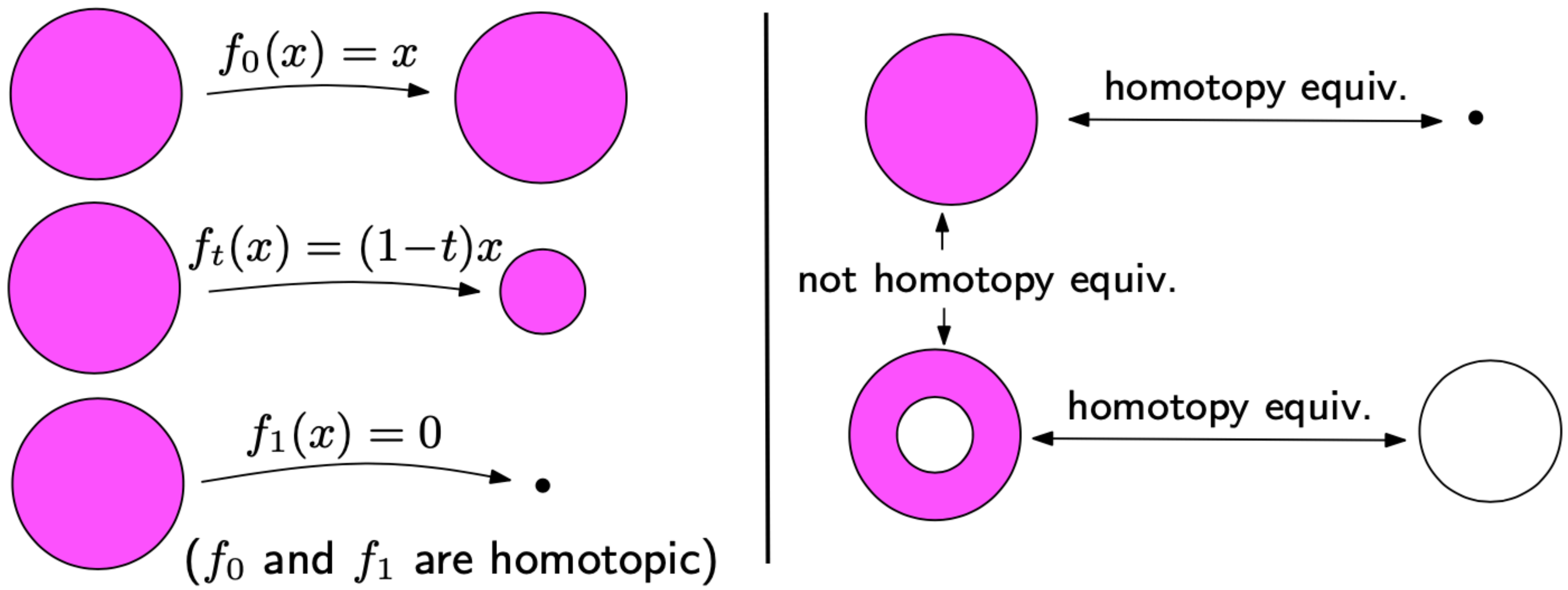


- Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic (denoted $f_0 \simeq f_1$) if there exists a continuous map $H : [0,1] \times X \to Y$ s. t. $\forall x \in X$, $H(0,x) = f_0(x)$ and $H_1(1,x) = f_1(x)$.

$$H(t,x) := (1-t)x$$

Homotopy type

(thanks to Frederic Chazal)



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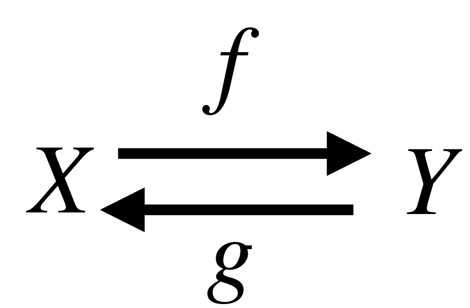
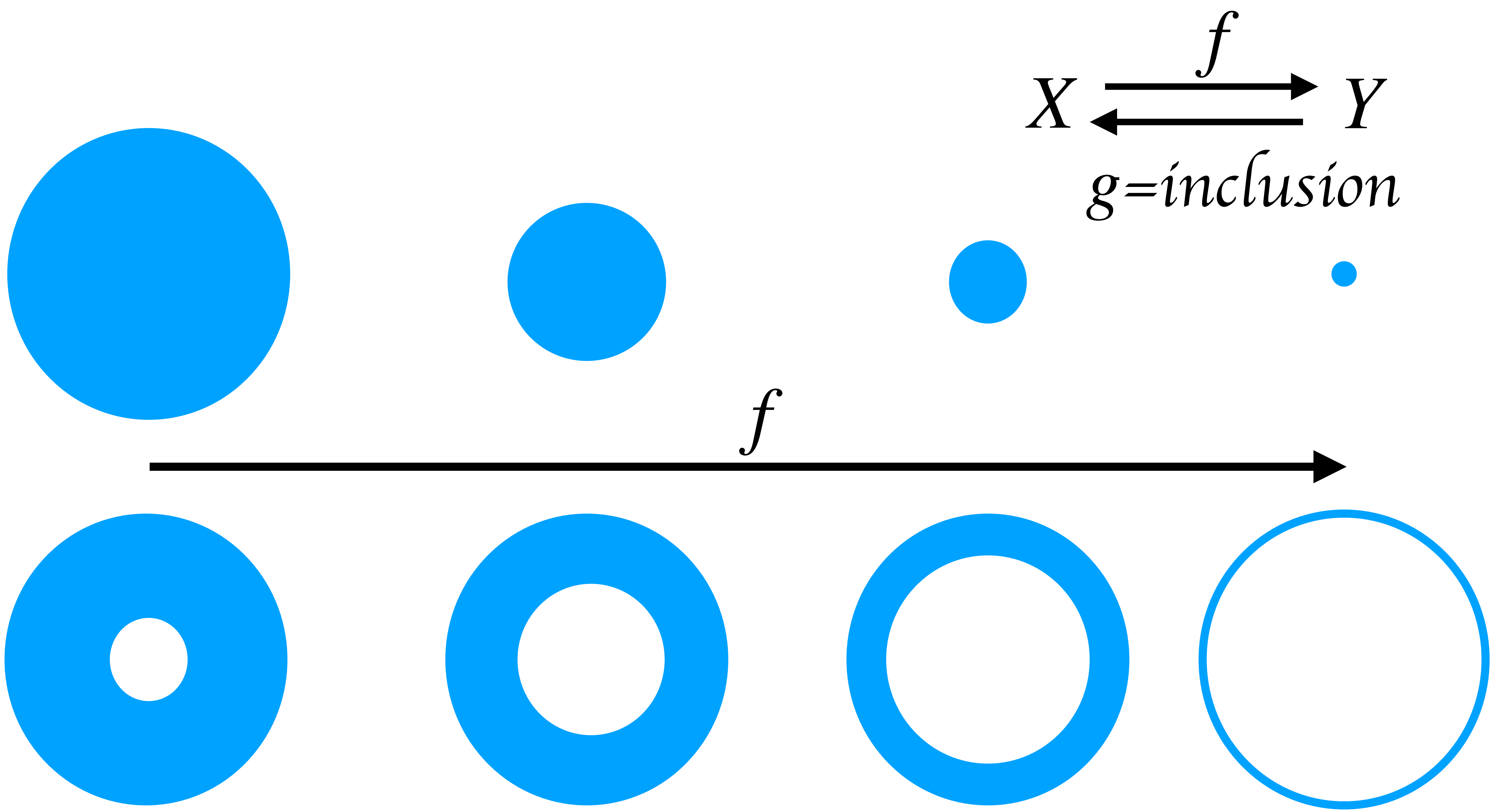
- X and Y have the same homotopy type (or are homotopy equivalent) if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ s. t. $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.



Homotopy type

A particular case : **deformation retract**



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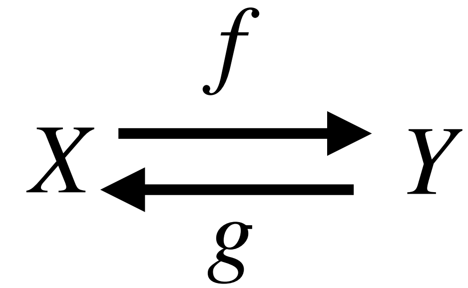
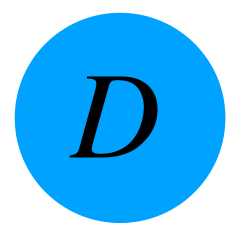
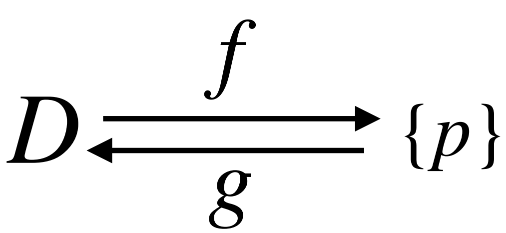
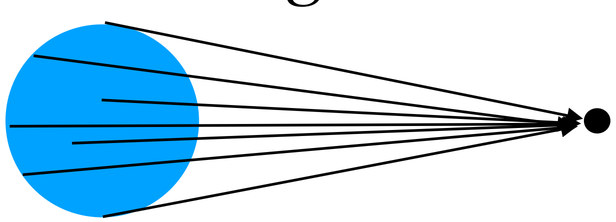


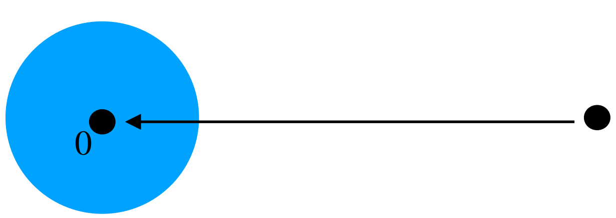
$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \right\}$$







$\forall x \in D, f(x) = p$



$g(p) = 0 \in D$

Homotopy type

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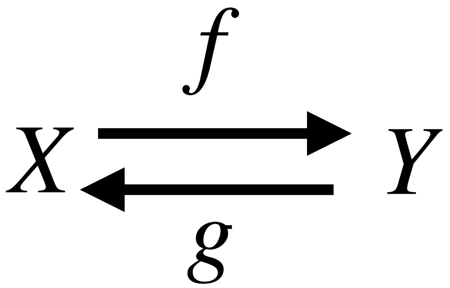


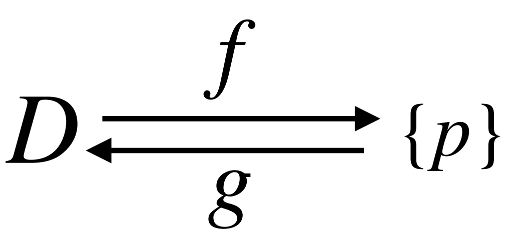
$$f \circ g = 1_{\{p\}}$$

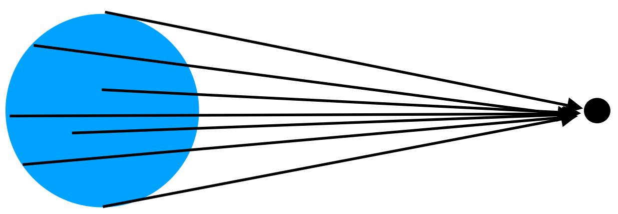
$$D = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\}$$

$$H(t, x) = (1 - t)x$$

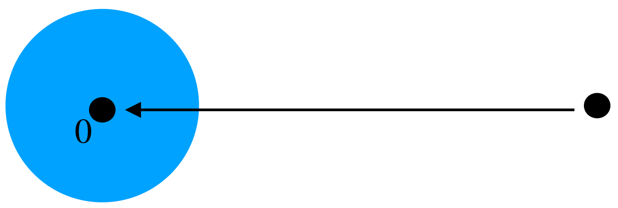
$$g \circ f = H(1, x)$$

$$H(0, x) = 1_D$$





$\forall x \in D, f(x) = p$



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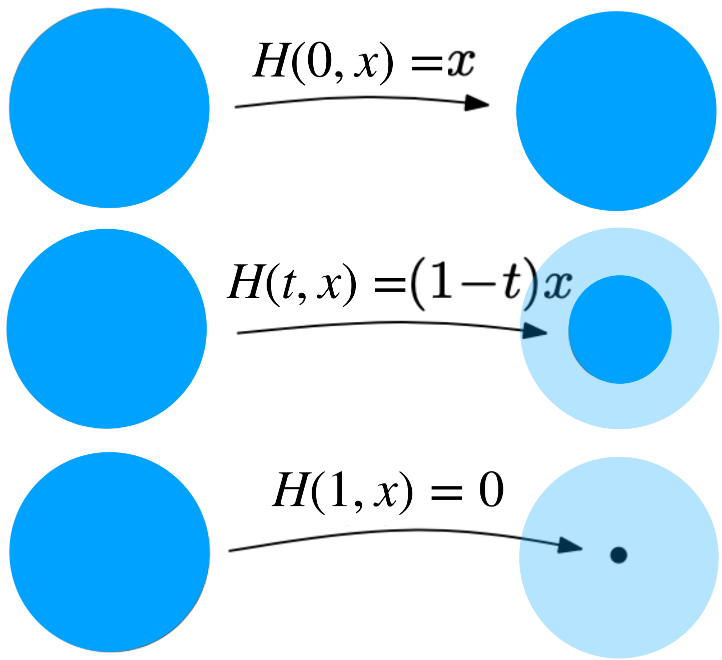
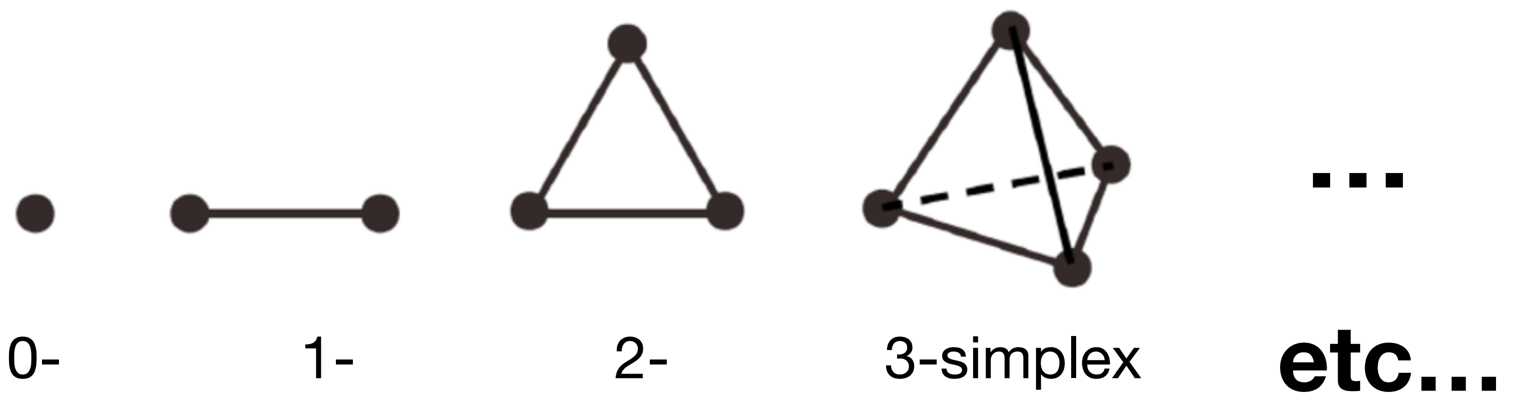
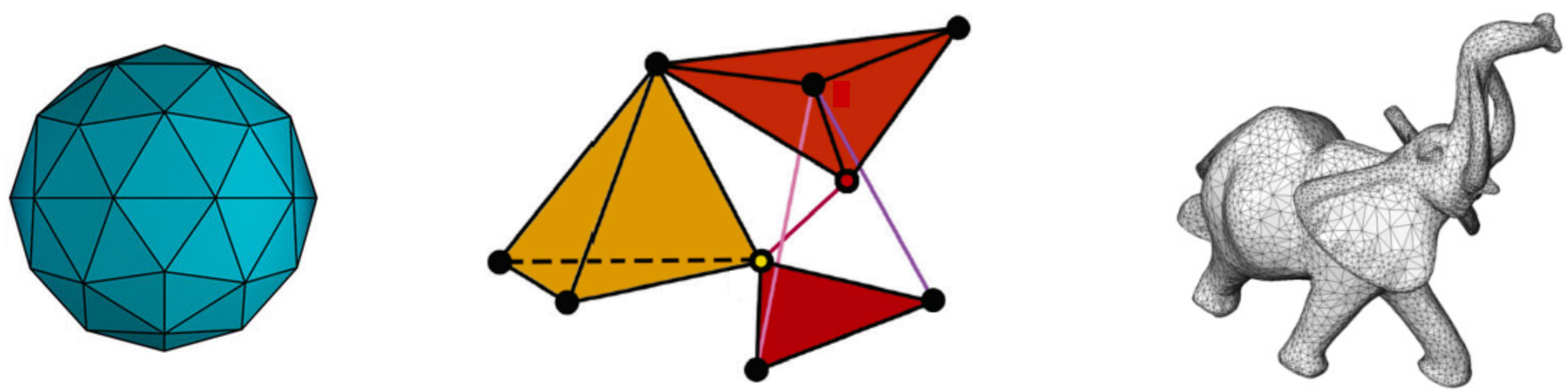


(Abstract) simplicial complexes

A k-**simplex** is a set of $k+1$ **vertices**



An **simplicial complex** is a collection of simplices glued along common faces:

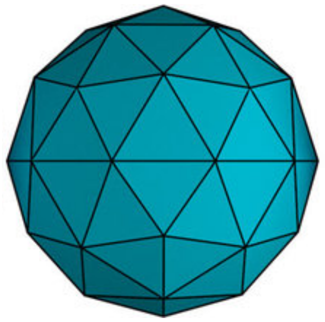


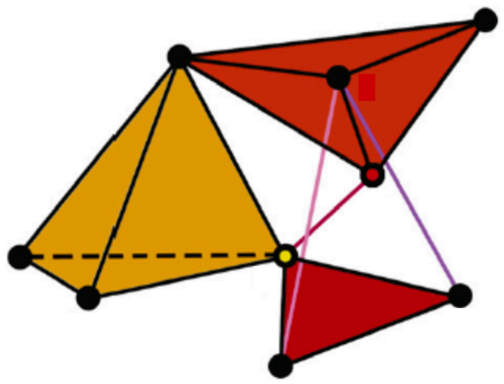
…such that if a simplex is in the complex, all its faces (i.e. its non empty subsets) are also in the complex.

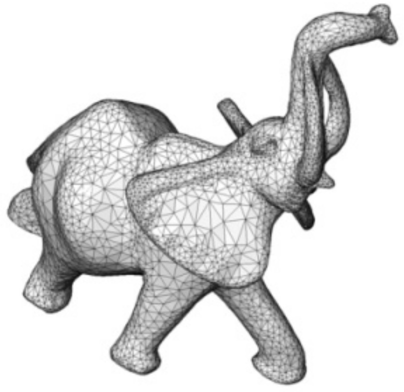
Simplicial complexes (geometric realization)

A k-**simplex** is a set of $k+1$ **vertices**

A **simplicial complex** defines a **topological space** by associating to each k-simplex the **convex hull** of $k+1$ points in general position in Euclidean space.
This topological space is called its **geometric realization**.





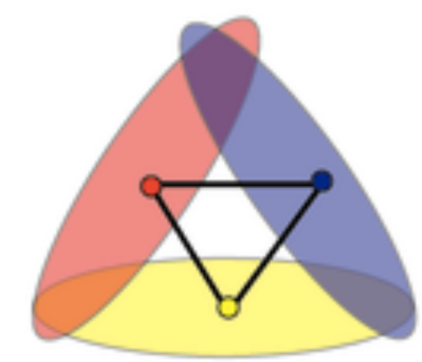


Nerve Theorem

(finite, convex case)

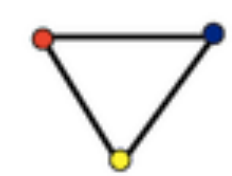
Definition 1. *A finite family of convex sets $\mathcal{F} = \{C_1, \ldots, C_n\}$ is a* **finite convex cover** *of a set X if:*

$$X = \bigcup_{i=1,n} C_i$$



Definition 2. *Given a finite cover $\mathcal{F}$, the* **nerve** *$\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex whose vertex set is $\mathcal{F}$ and with one simplex for each subset of $\mathcal{F}$ whose sets have a non empty common intersection:*

$$\mathcal{N}(\mathcal{F}) = \left\{ \sigma \subset \mathcal{F} \;\middle|\; \bigcap_{C_i \in \sigma} C_i \neq \emptyset \right\}$$

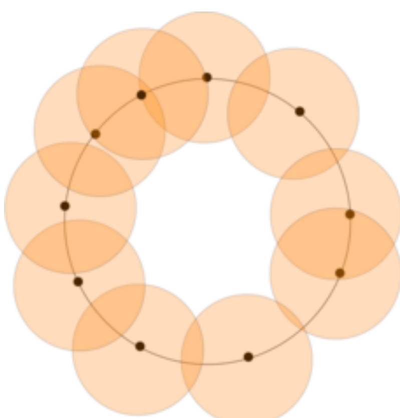


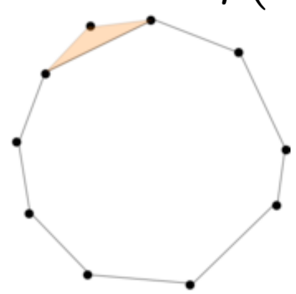
Theorem 1 (Nerve Theorem). *If $\mathcal{F}$ is a finite convex cover of X, then $\mathcal{N}(\mathcal{F})$* **and** *X* **have same homotopy type**.

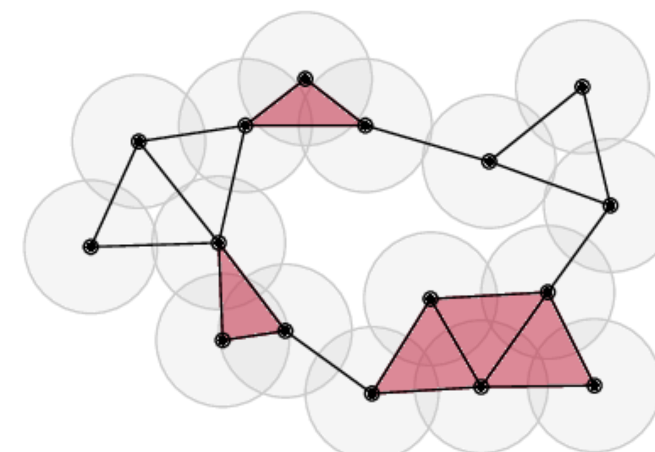
Čech complex and α-complex

Given a finite set $\mathcal{P}$ and a radius $r > 0$, the **Čech complex** $\mathrm{C}_r(\mathcal{P})$ is the **nerve** of the family made of the closed balls $\mathbb{B}(p, r)$ with radius r for each $p \in \mathcal{P}$:

$$\mathrm{C}_r(\mathcal{P}) = \mathcal{N}\left(\{\mathbb{B}(p, r) \mid p \in \mathcal{P}\}\right)$$







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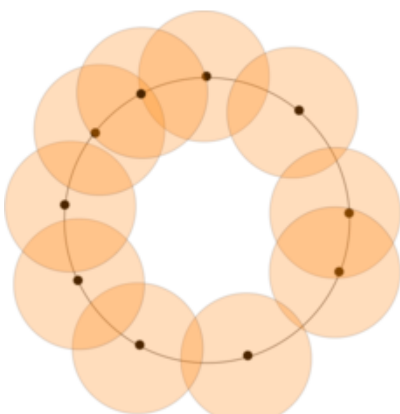

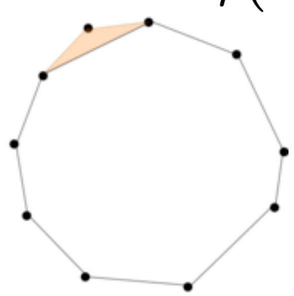

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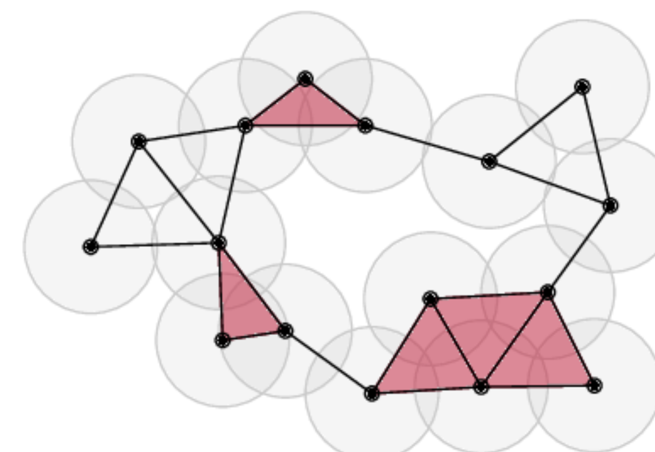
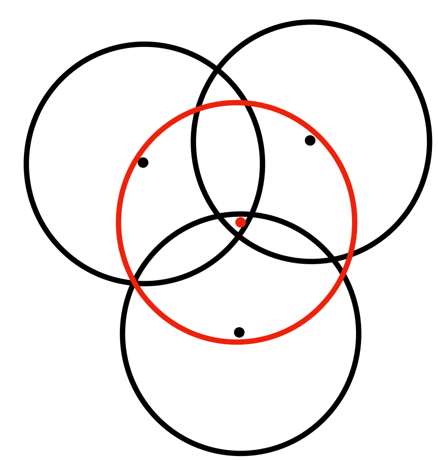


Equivalently:

Given a finite set $\mathcal{P}$ and a radius $r > 0$, the **Čech complex** $\mathrm{C}_r(\mathcal{P})$ is the set of simplices in $\mathcal{P}$ enclosed in ball of radius r.



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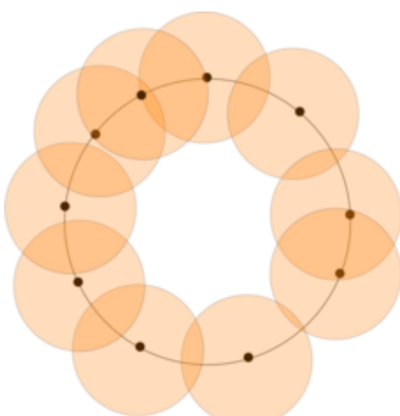
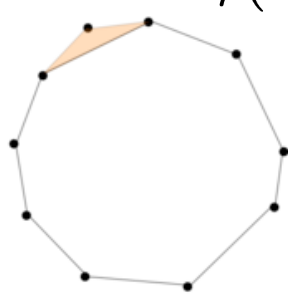
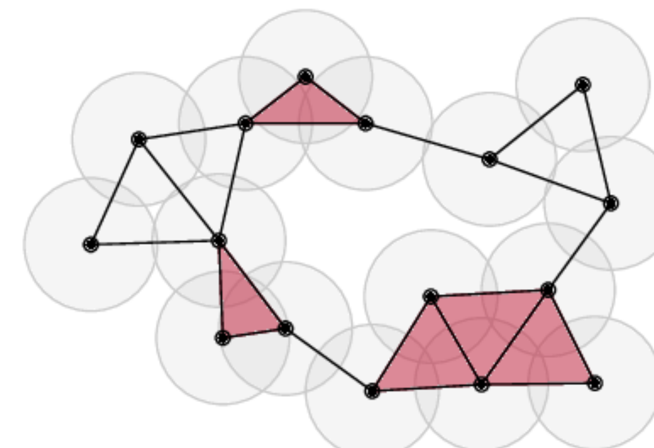


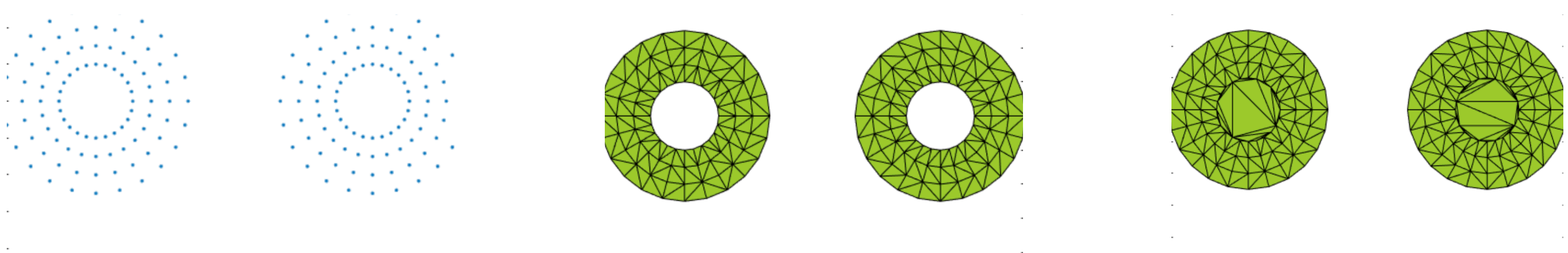


$$\mathrm{C}_r(\mathcal{P}) = \mathcal{N}\left(\{\mathbb{B}(p, r) \mid p \in \mathcal{P}\}\right)$$

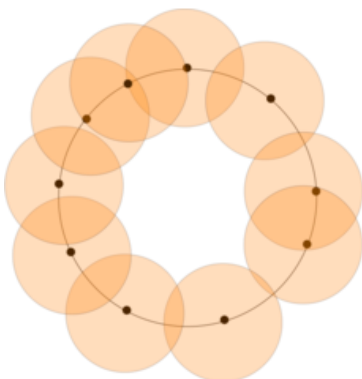
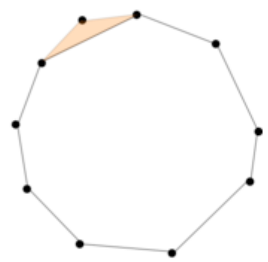
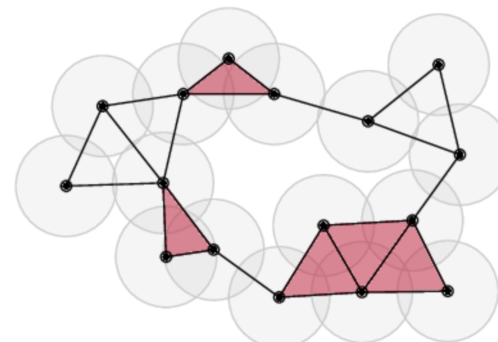
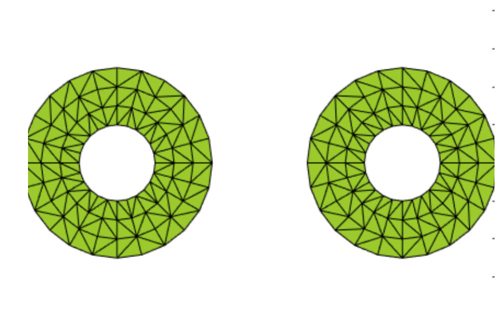


Given a finite set $\mathcal{P}$ and a radius $r > 0$, the α**-complex** $\mathrm{A}_r(\mathcal{P})$ is the **nerve** of the family made of the intersections of the (closed) Voronoi cell of p with the closed balls $\mathbb{B}(p, r)$ with radius r for each $p \in \mathcal{P}$:

$$\mathrm{A}_r(\mathcal{P}) = \mathcal{N}\left(\{\mathbb{B}(p, r) \cap \mathrm{Vor}_{\mathcal{P}}(p) \mid p \in \mathcal{P}\}\right)$$

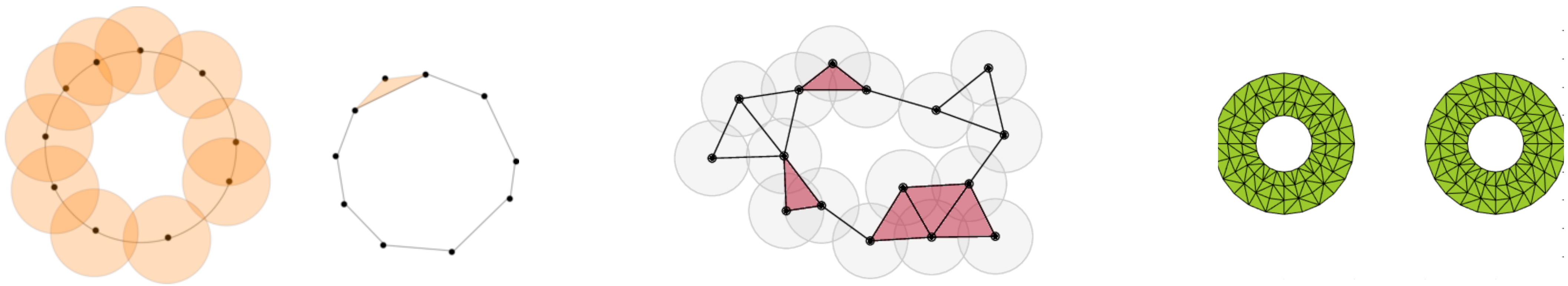


Čech complex and α-complex

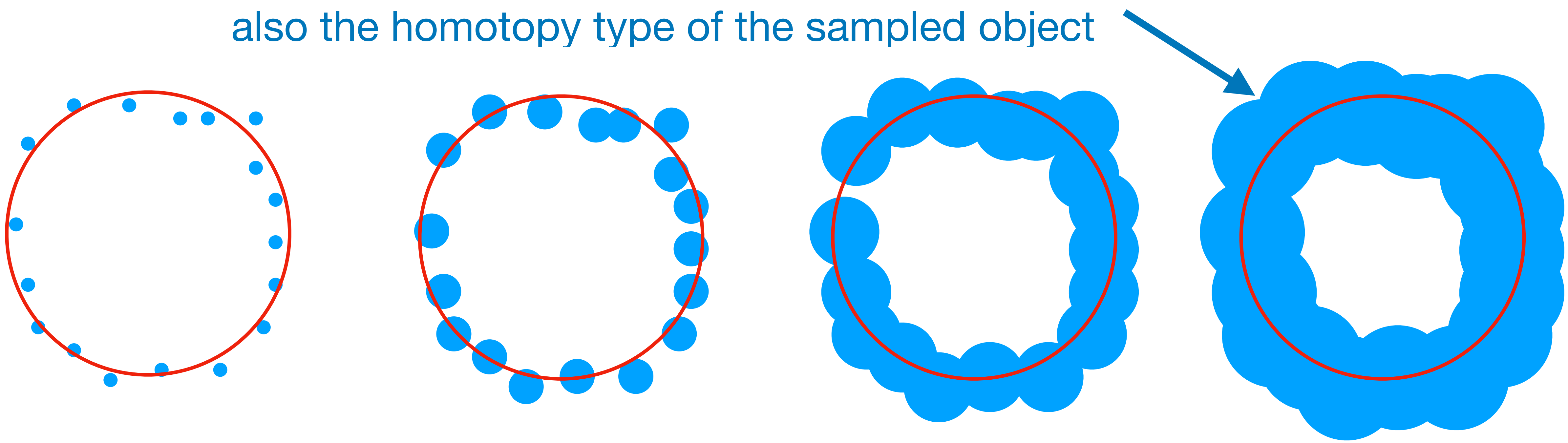
By the nerve Theorem, both Čech complex and α-complex have the homotopy type of the corresponding union of balls

Čech complex and α-complex



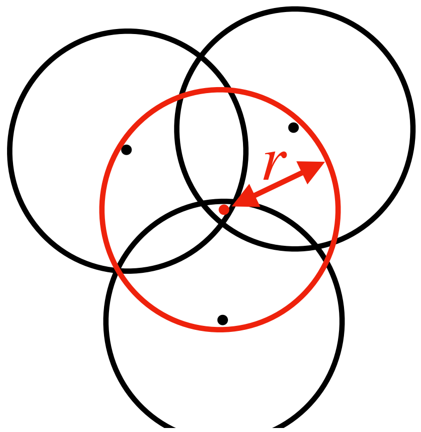
By the nerve Theorem, both Čech complex and α-complex have the homotopy type of the corresponding union of balls

Intuition: under some conditions, it may retrieve also the homotopy type of the sampled object

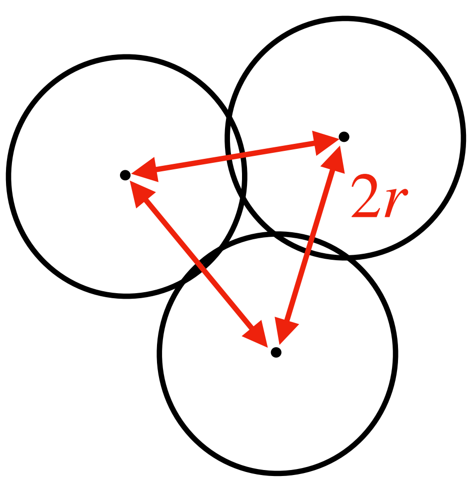


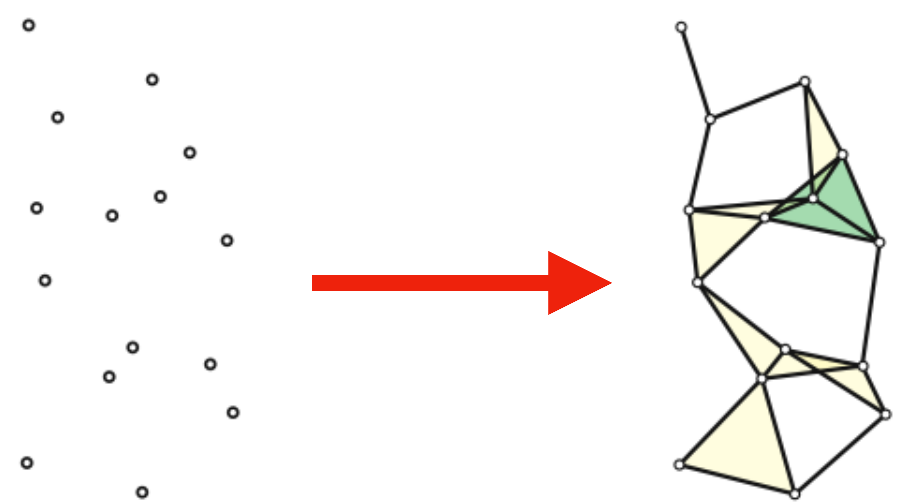
Vietoris-Rips complex

Given a finite set $\mathcal{P}$ and a radius $r > 0$, the **Čech complex** $\mathrm{C}_r(\mathcal{P})$ is the set of simplices in $\mathcal{P}$ enclosed in ball of radius r.

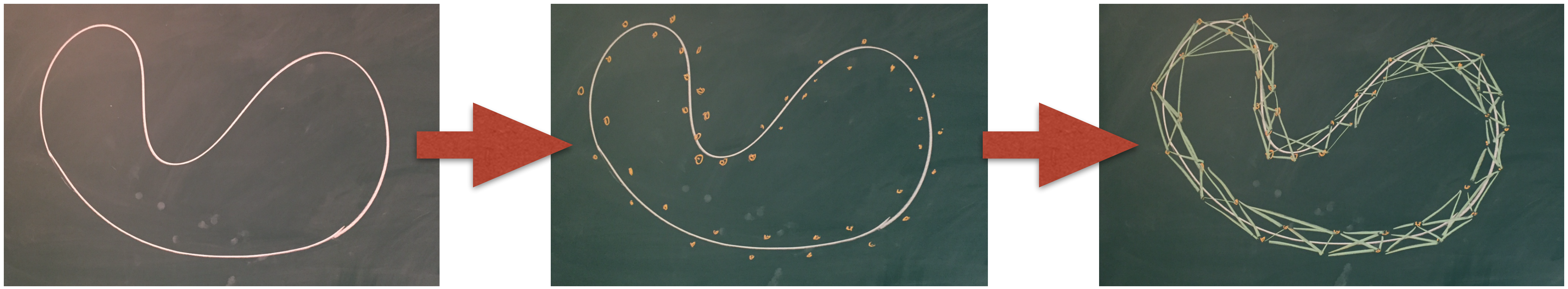


Given a finite set $\mathcal{P}$ and a parameter $r > 0$, the **Vietoris-Rips complex** $\mathrm{R}_r(\mathcal{P})$ is the set of simplices in $\mathcal{P}$ with diameter at most $2r$.





Is it possible to capture the **topology** of a ``shape” from a finite sampling ?



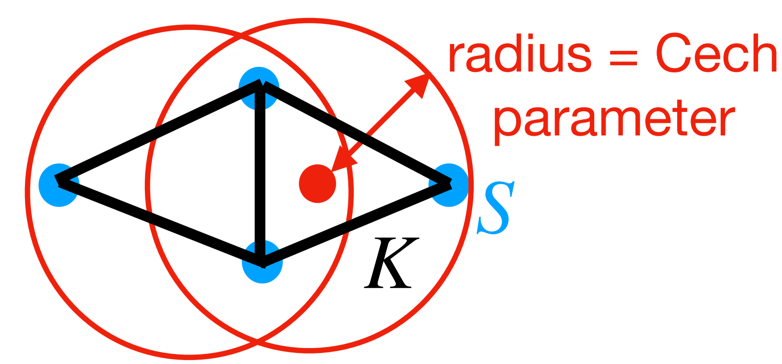
An embedded **manifold** M

A **Point cloud** S **sampling** M

A **simplicial complex** K built upon S, typically a **parametrized** Cech or Rips,

$$K \simeq M \,?$$

(i.e. is K **homotopy equivalent** to M ?)



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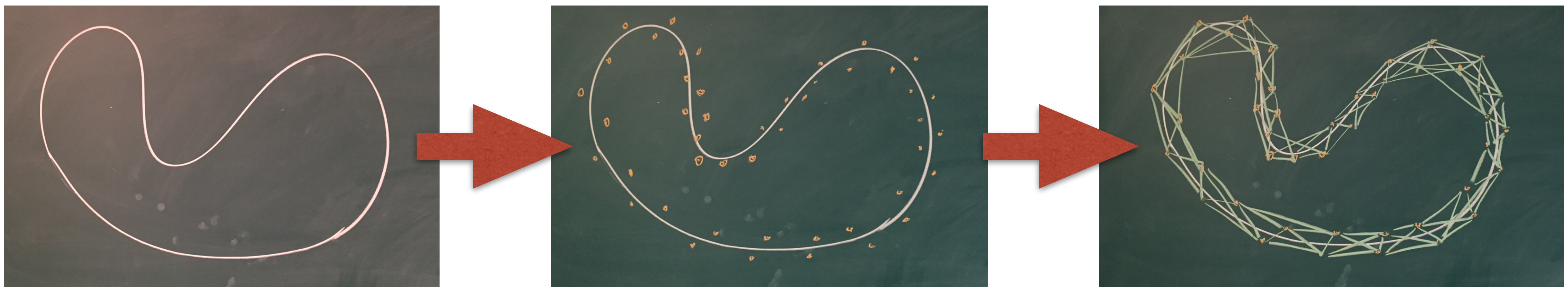


An embedded **shape** M

with some quantified regularity

A **Point cloud** S **sampling** M

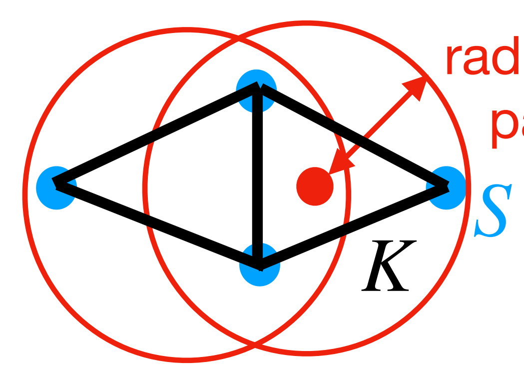
with a sampling density related to the shape regularity

A **simplicial complex** K built upon S, typically a **parametrized** Cech or Rips,

with a parameter related to the shape regularity and sampling density

$$K \simeq M\,?$$

(i.e. is K **homotopy equivalent** to M ?)



Regularity measures

Reach and medial axis

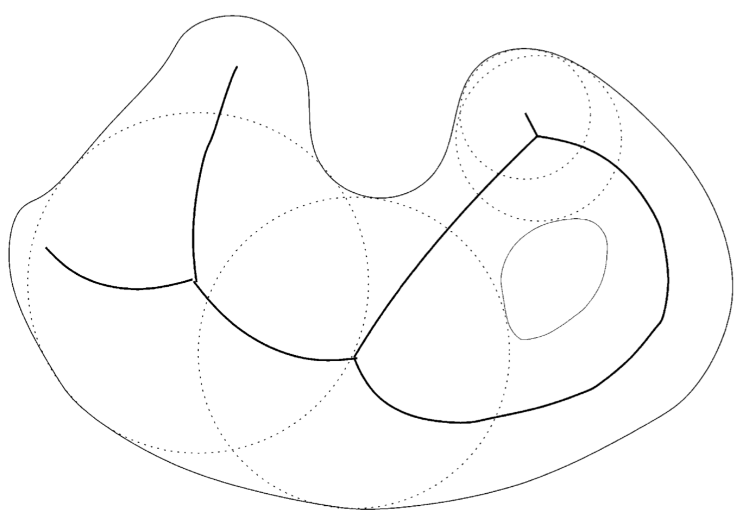
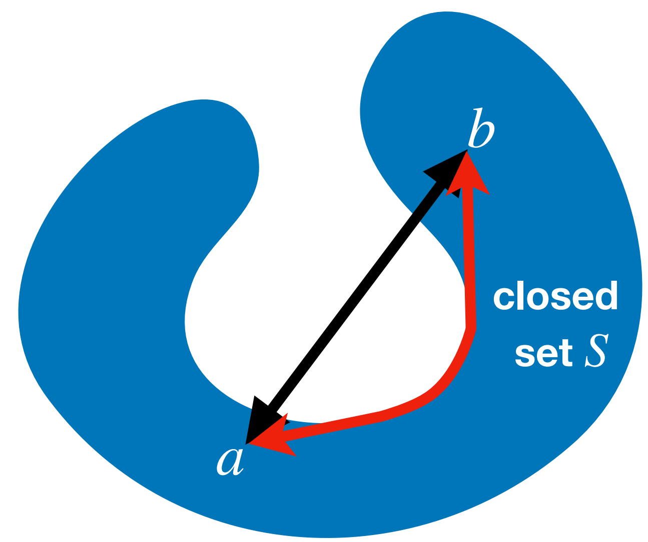


Fig. 1. A set and its medial axis.

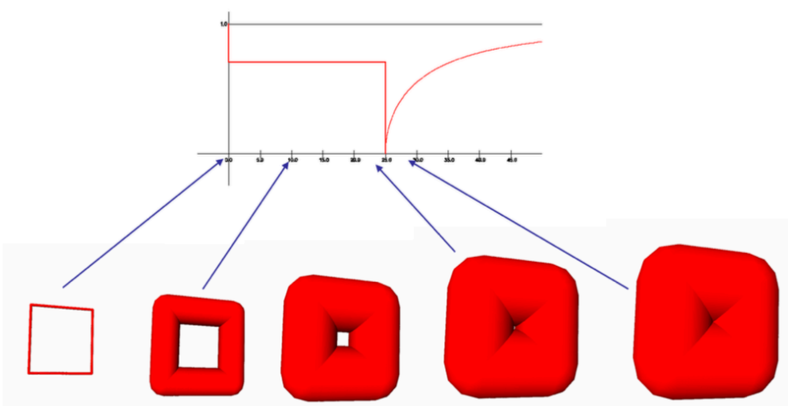
Metric distortion



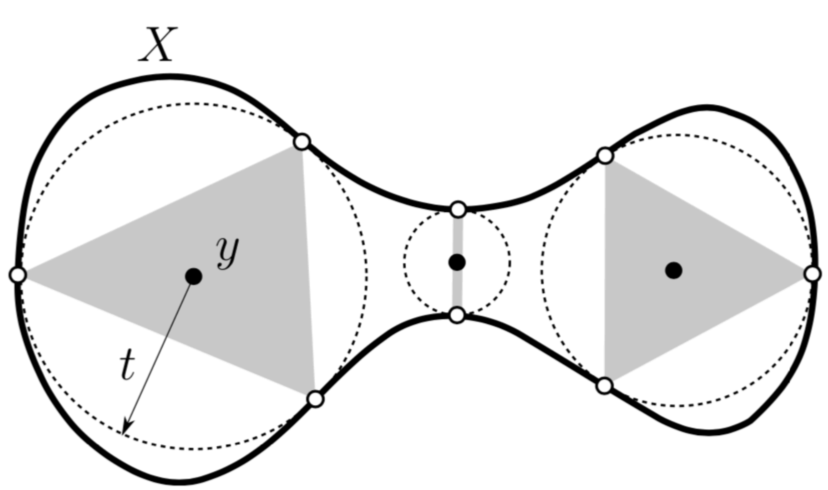
(smooth objects**)**

(not necessarily smooth**)**

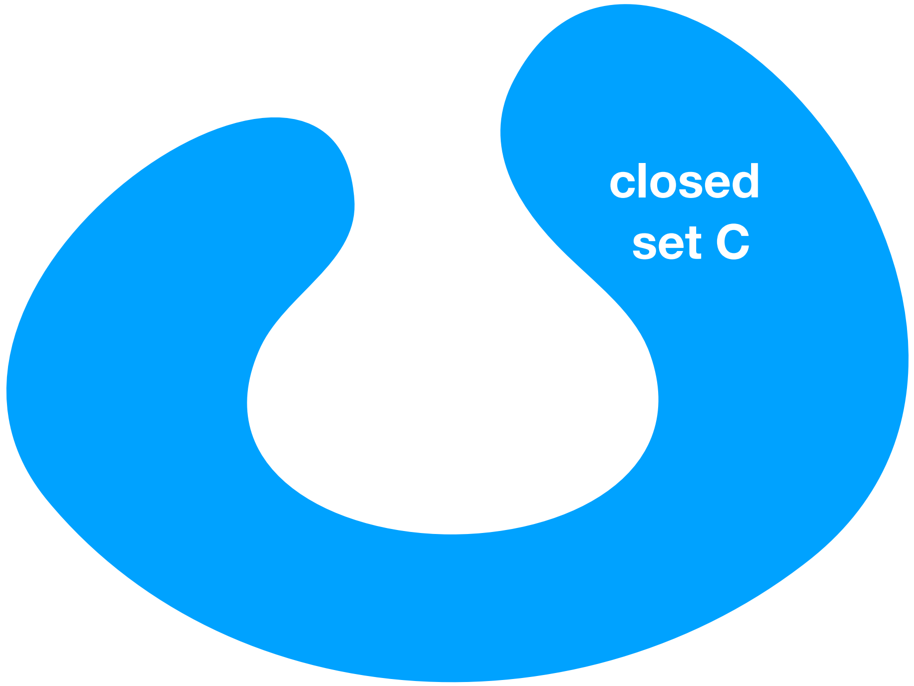
Critical function



Convexity deffect

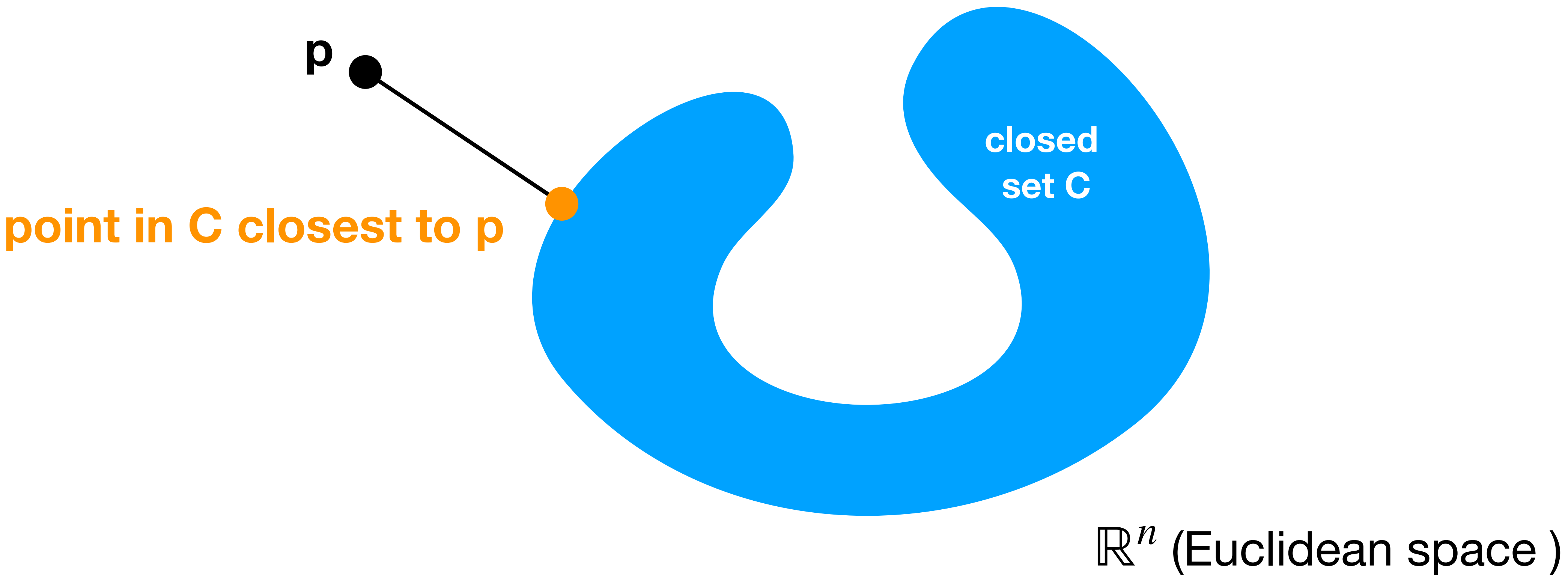


Medial Axis and Reach



$\mathbb{R}^n$ (Euclidean space)

Medial Axis and Reach



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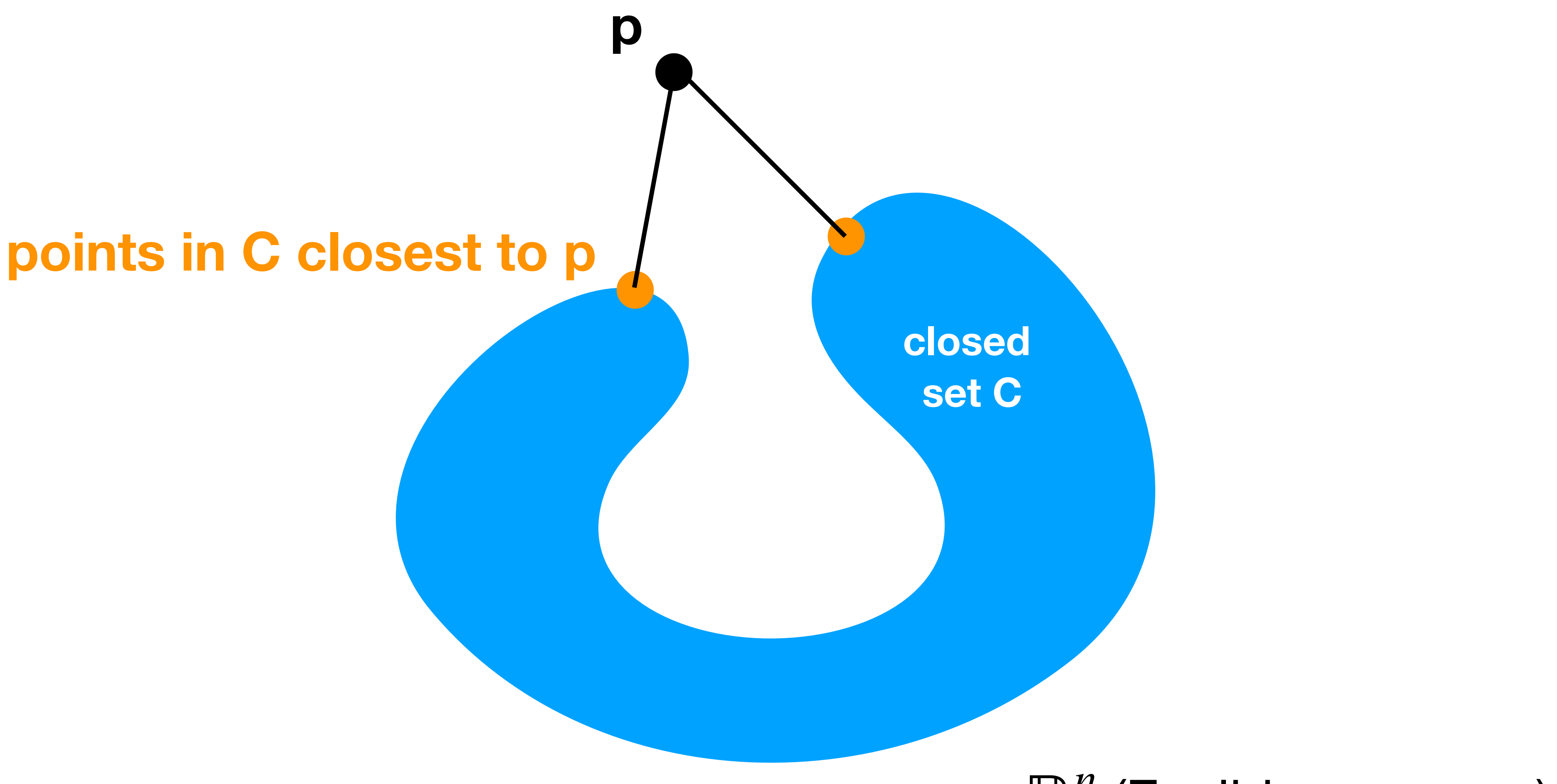

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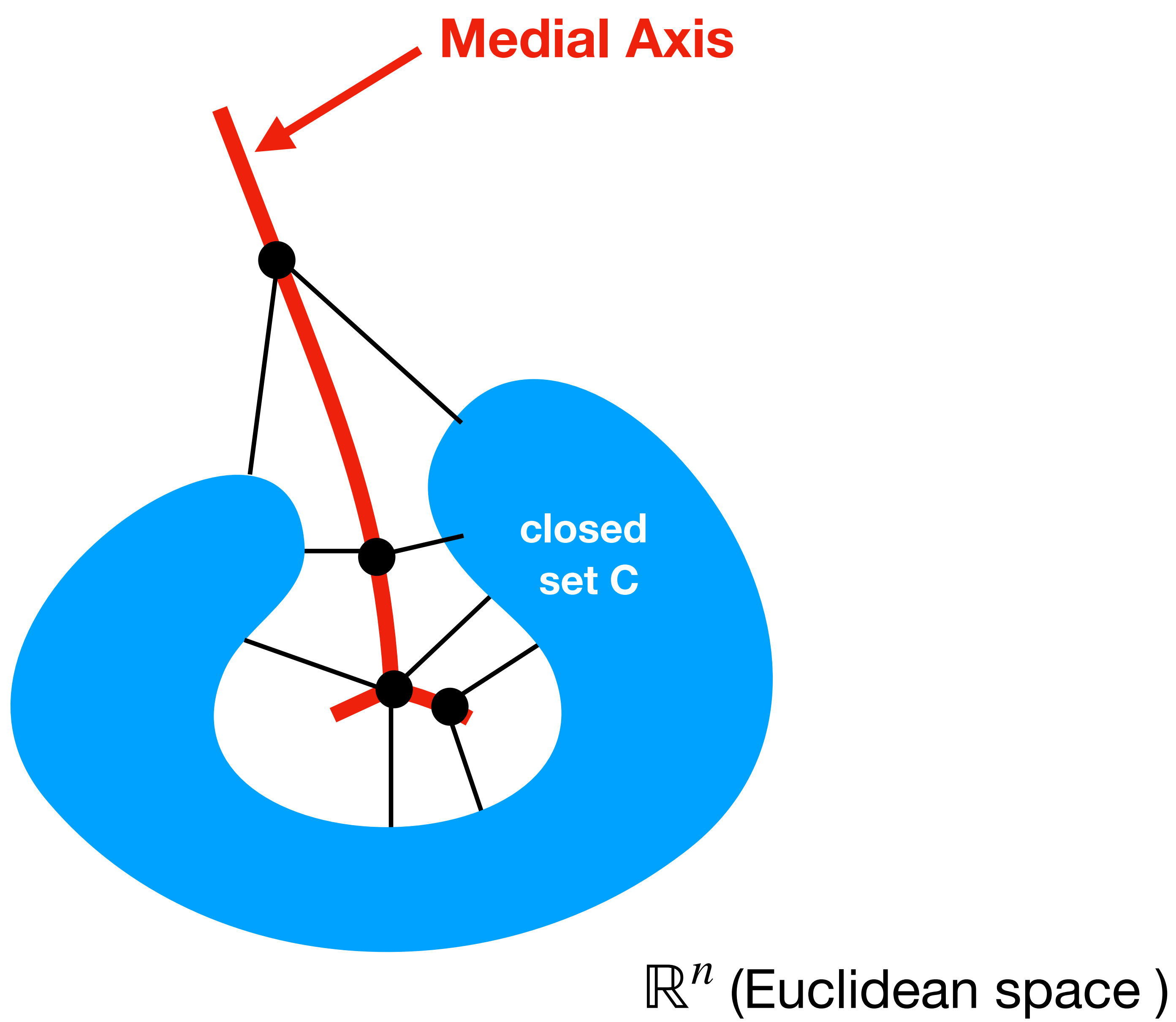
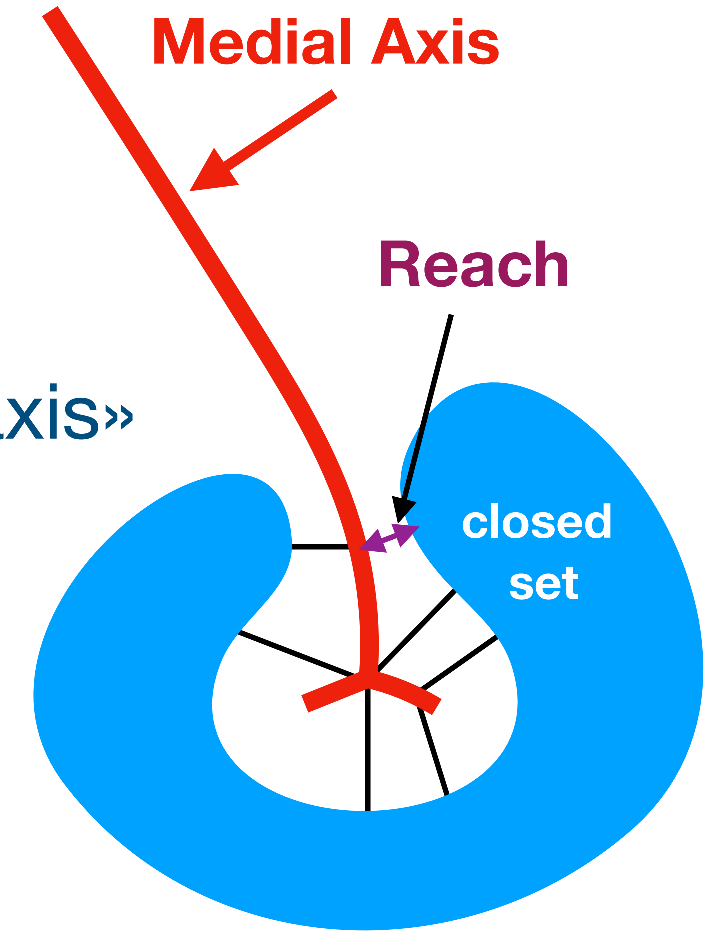


Medial Axis and Reach

Reach of a **closed set C**
« infimum of distances between C and its medial axis»



Medial Axis and Reach

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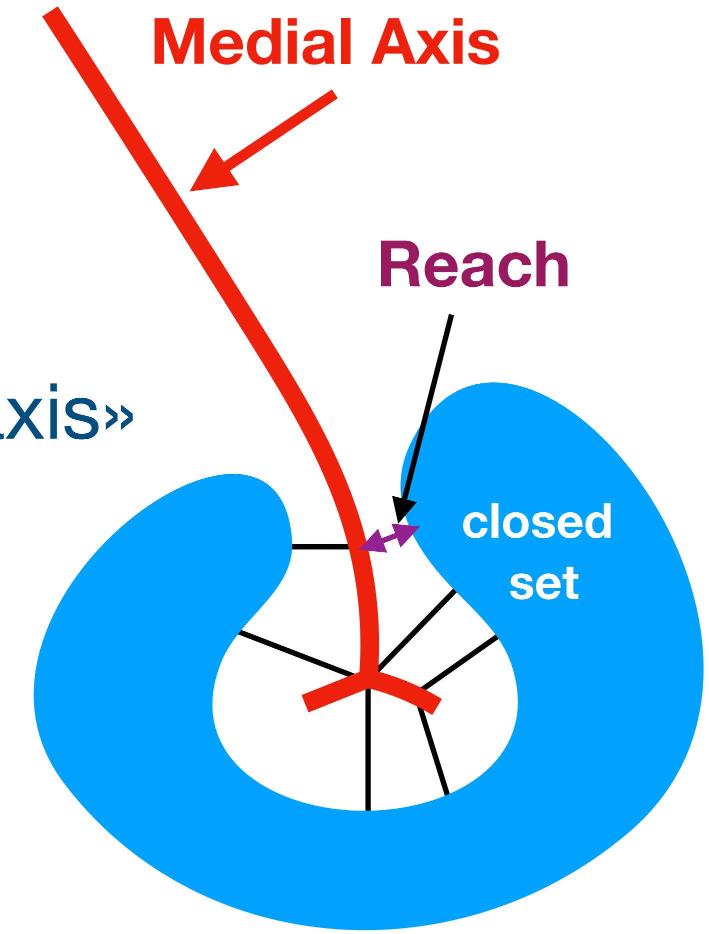
- Introduced by **Herbert Federer** (Curvature Measures 1959): class of **sets with positive reach** allow to define curvature measures beyond smooth case.



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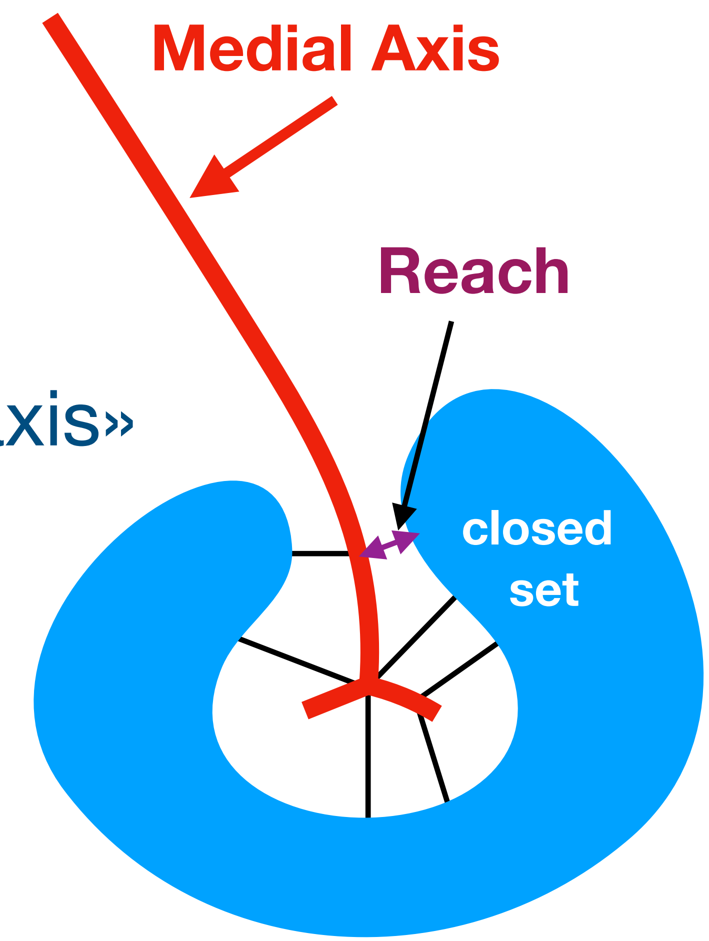

Reach of a **closed set C**

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- Introduced by **Herbert Federer** (Curvature Measures 1959): class of **sets with positive reach** allow to define curvature measures beyond smooth case.

- Used again in the context of **manifold reconstruction with topological guarantees** : Amenta et al. (lfs), Boissonnat et al., Dey et al., Niyogi et al.

Notation: offset of a set

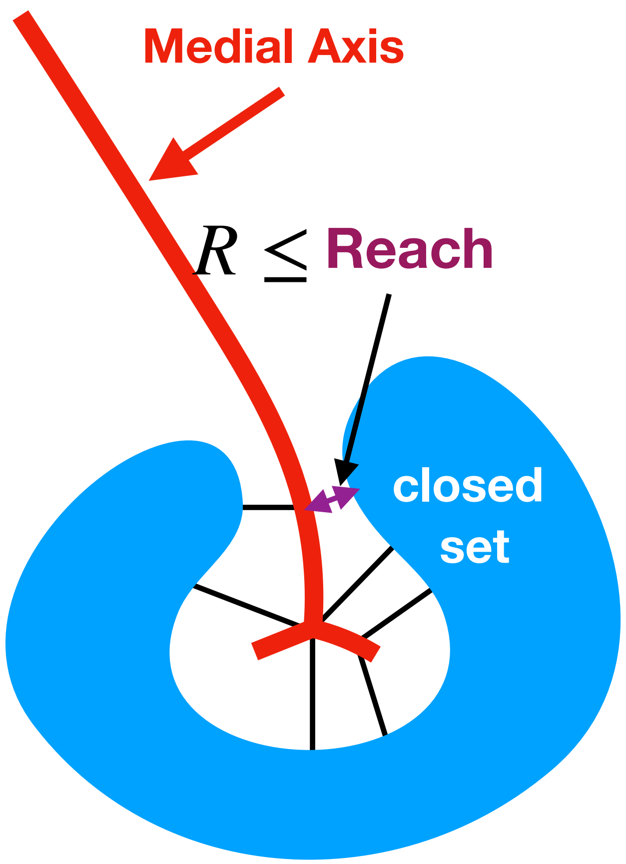
We denote by $S \oplus B(\varepsilon)$ or sometime $S^{\oplus \varepsilon}$
the Minkowski sum of S and a the ball $B(\varepsilon)$ of radius ε

In other words the ε-offset of S
In other words, S « inflated » of ε :

$$S \oplus B(\varepsilon) = S^{\oplus \varepsilon} := \bigcup_{x \in S} B(x, \varepsilon) = \left\{ y \in \mathbb{R}^d \mid d(y, S) \leq \varepsilon \right\}$$

Reconstruction Theorem for set with positive reach

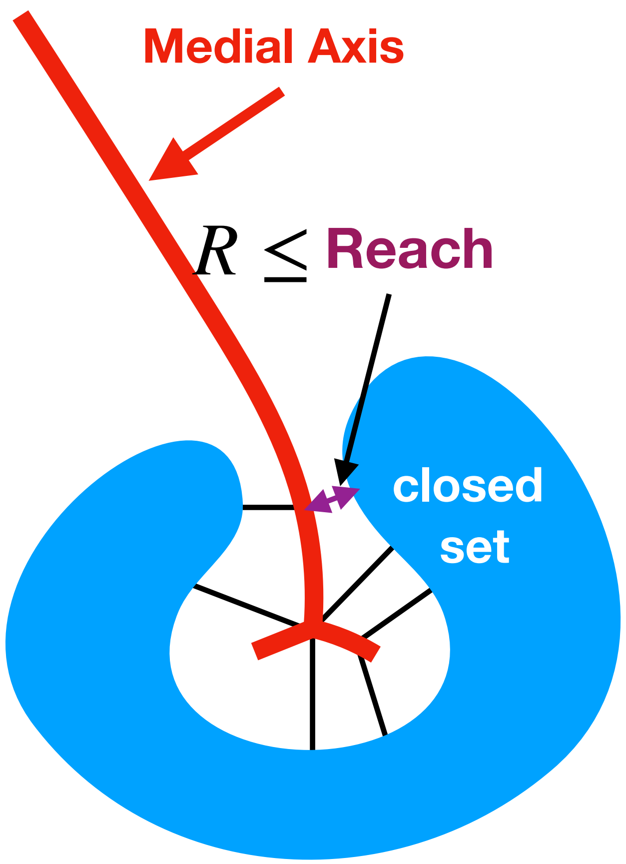
$R \leq \text{reach}(S)$

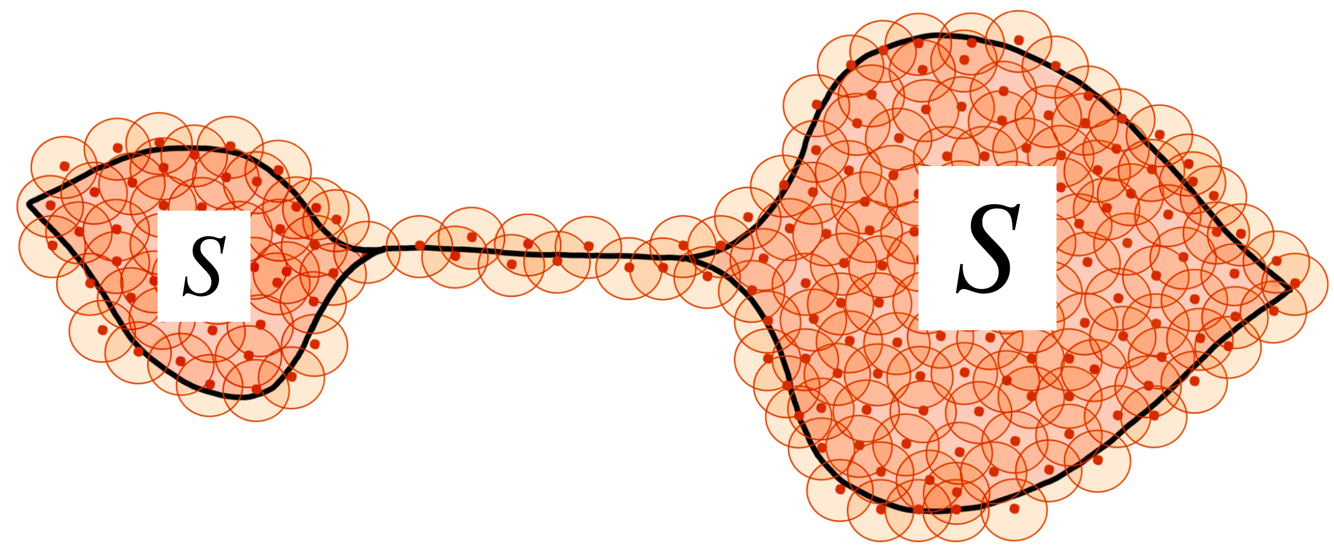


Reconstruction Theorem for set with positive reach

$R \leq \mathrm{reach}(S)$

$S \subset P \oplus B(\epsilon)$ **and** $P \subset S \oplus B(\delta)$





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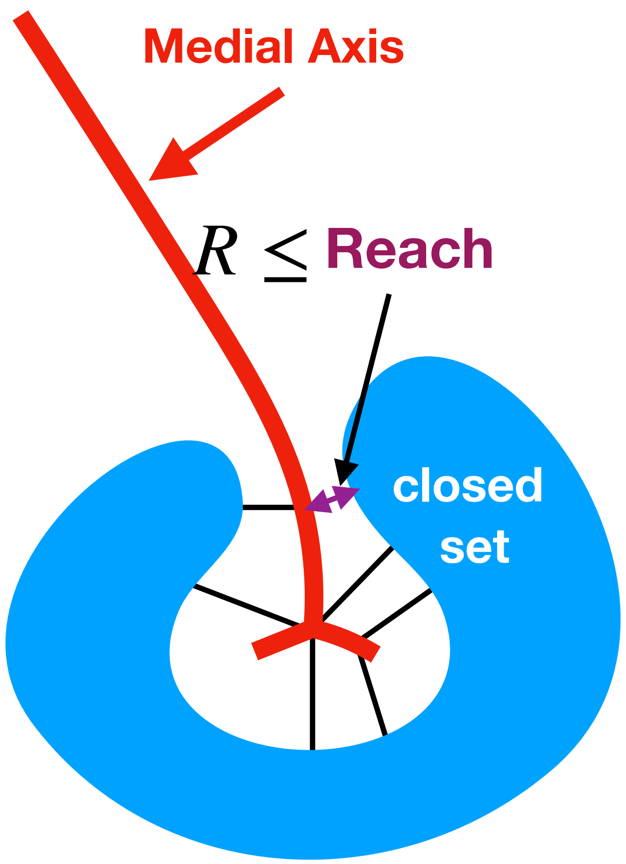
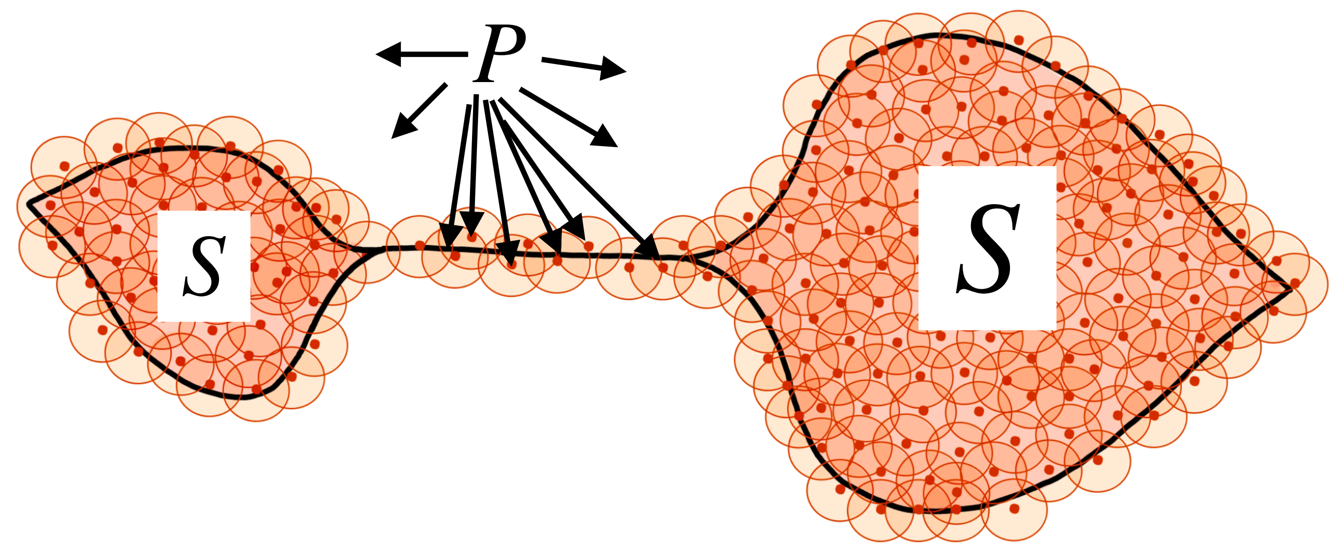




Reconstruction Theorem for set with positive reach

$R \leq \mathrm{reach}(S)$

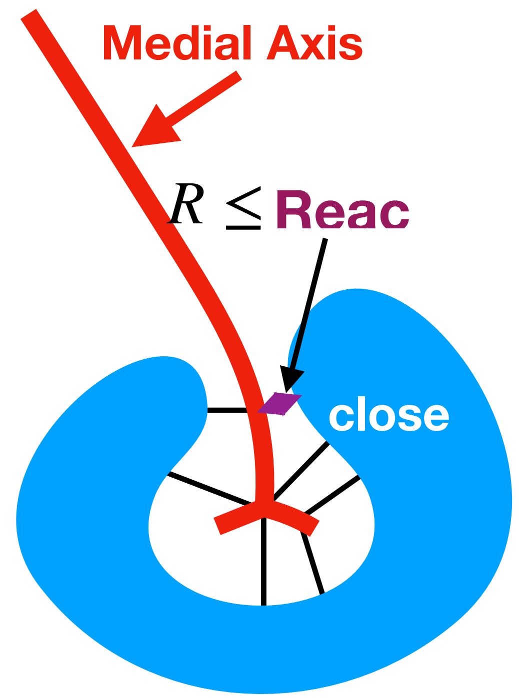
$S \subset P \oplus B(\epsilon)$ **and** $P \subset S \oplus B(\delta)$

General set of positive reach:

If ε and δ satisfy

$$\varepsilon + \sqrt{2}\,\delta \leq (\sqrt{2} - 1)R,$$

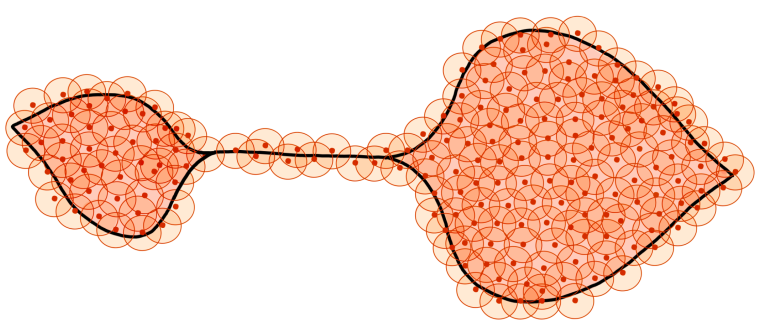
there exists a radius $r > 0$ such that the union of balls $P \oplus B(r)$ deformation-retracts onto $\mathcal{S}$ along the closest point projection. In particular, r can be chosen as $r = (R + \varepsilon)/2$



Weaker conditions for manifold of positive reach:

If ε and δ satisfy

$$(R - \delta)^2 - \varepsilon^2 \geq \left(4\sqrt{2} - 5\right) R$$



These conditions are ***tight*** *for retrieving the homology and homotopy by some offset of the sample*

Reconstruction Theorem for set with positive reach

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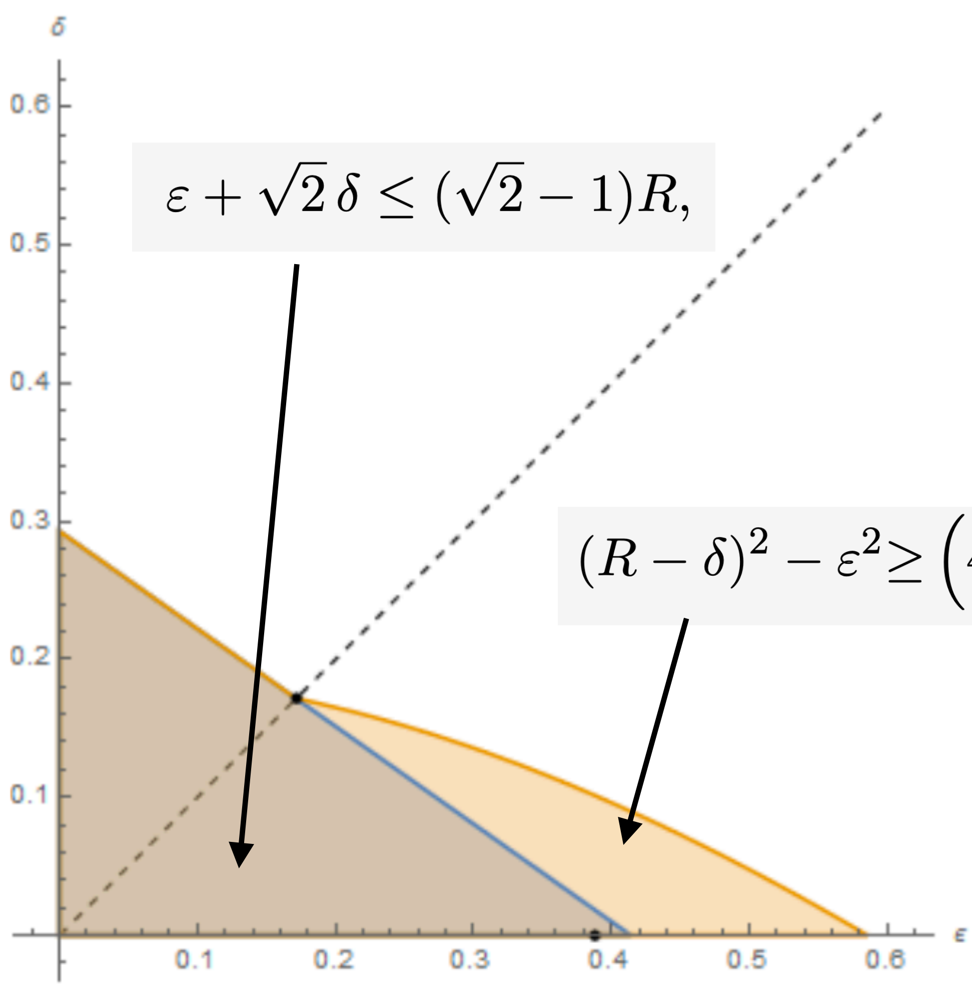
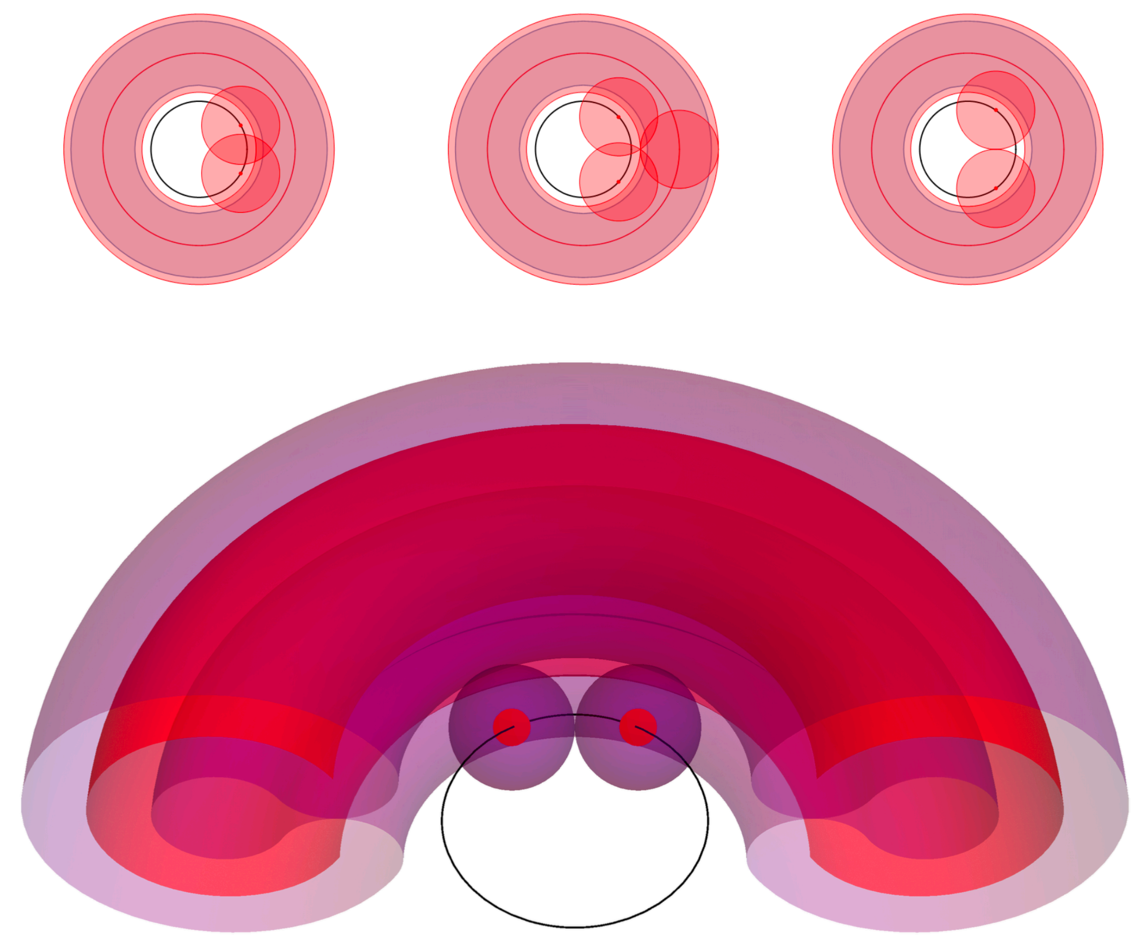
These conditions are ***tight*** *for retrieving the homology and homotopy by some offset of the sample*





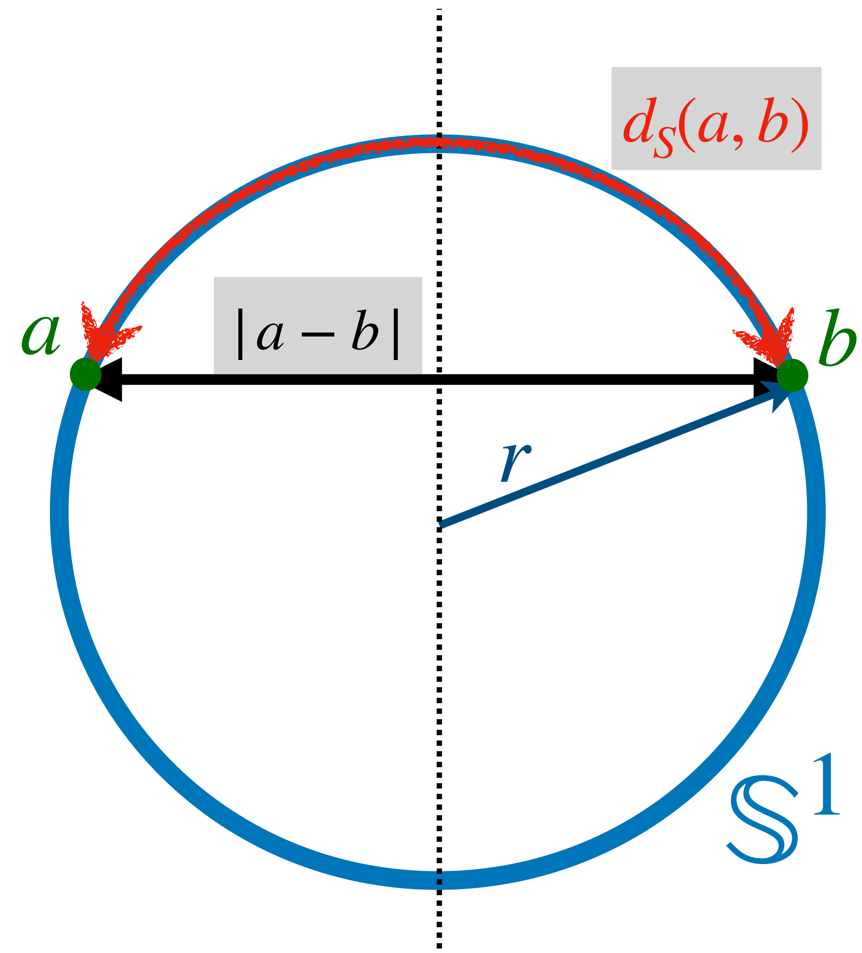
The reach can be alternatively defined by the metric distortion

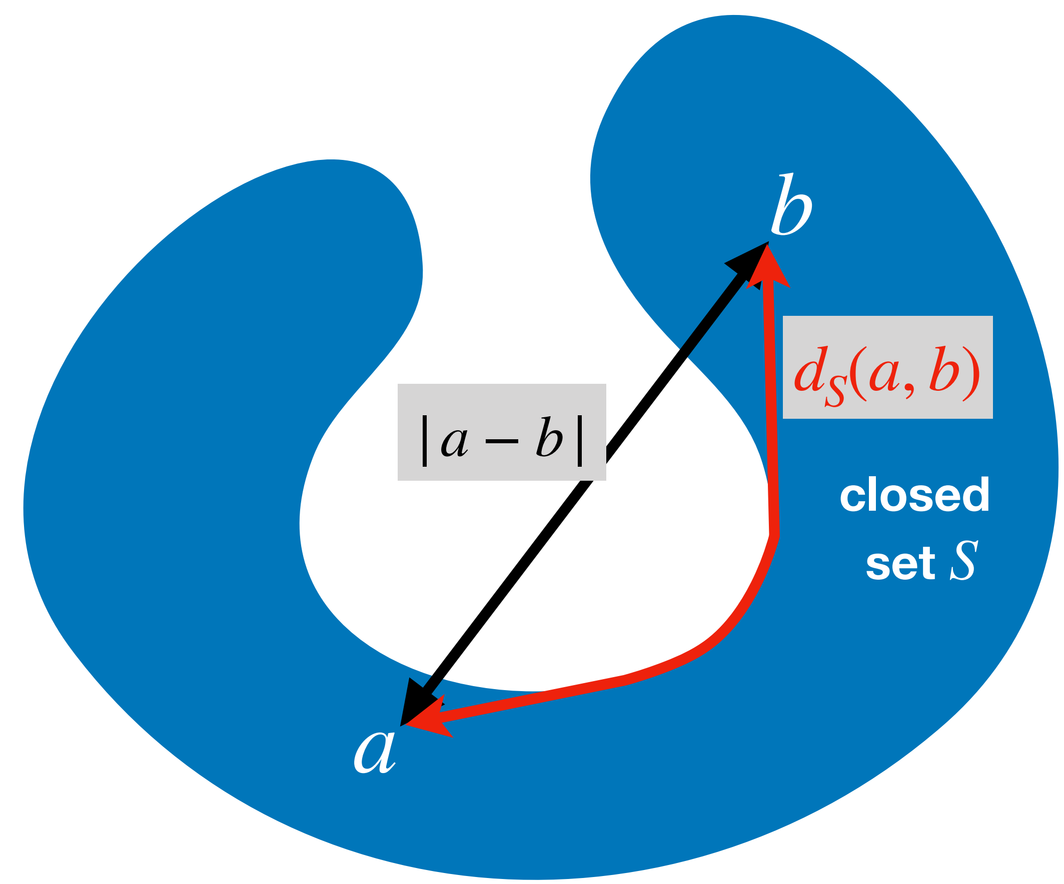
(Boissonnat, L, Wintraecken, 2017)

Theorem 1. *If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then*

$$\operatorname{rch} \mathcal{S} = \sup \left\{ r > 0,\ \forall a, b \in \mathcal{S},\ |a-b| < 2r \Rightarrow d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{|a-b|}{2r} \right\},$$

where the sup *over the empty set is* 0.





Metric distortion $\mathscr{D}_S$ as measure of regularity of a set S

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Metric distortion $\mathscr{D}_S$ as measure of regularity of a set S ?

$$t \to \mathscr{D}_S(t) = \sup_{\|a-b\| \leq t} d_S(a,b)$$

Condition above can be rewritten as:

$$\mathscr{D}_S(t) \leq 2r \arcsin \frac{t}{2r}$$

According to Gromov et Al.*:

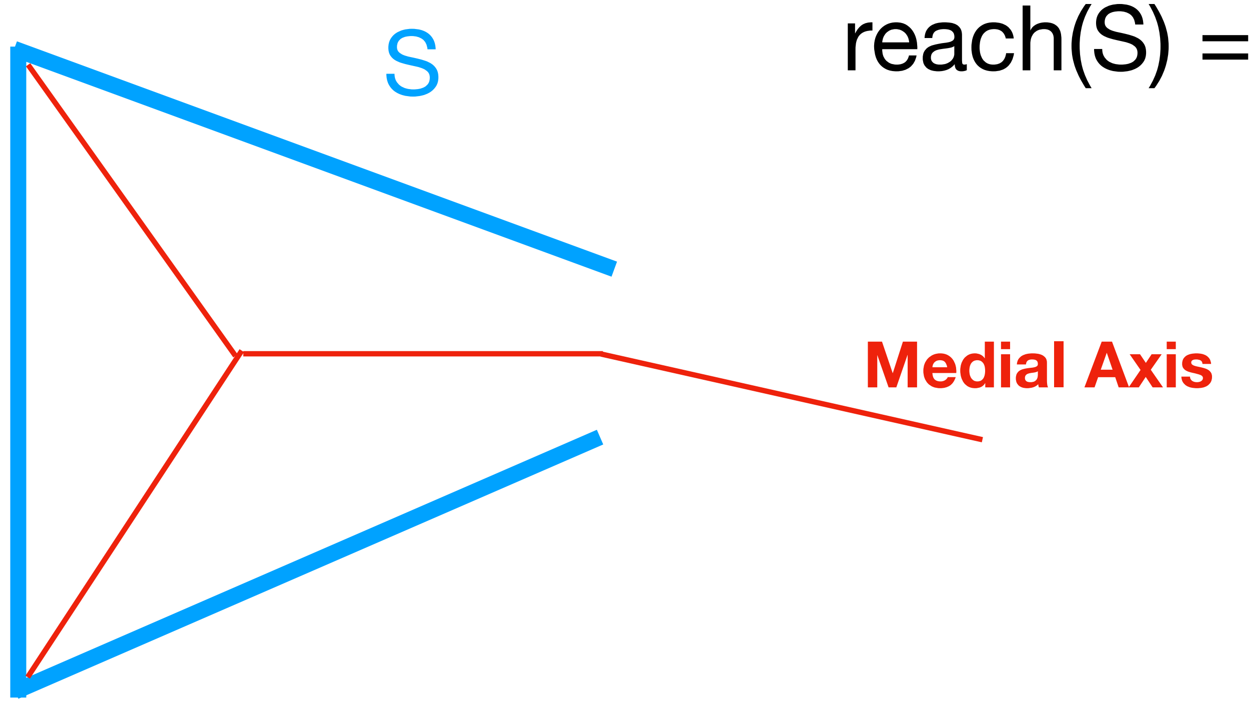
$$\mathscr{D}_S(t) \leq \frac{\pi}{2} t \Rightarrow S \text{ **is simply connected**}$$

$$\mathscr{D}_S(t) \leq \frac{2\sqrt{2}}{\pi} t \Rightarrow S \text{ **is contractible**}$$

**Metric Structures for Riemannian and Non-Riemannian Spaces, M. Gromov, M. Katz, P. Pansu, S.Semmes*

Beyond the reach

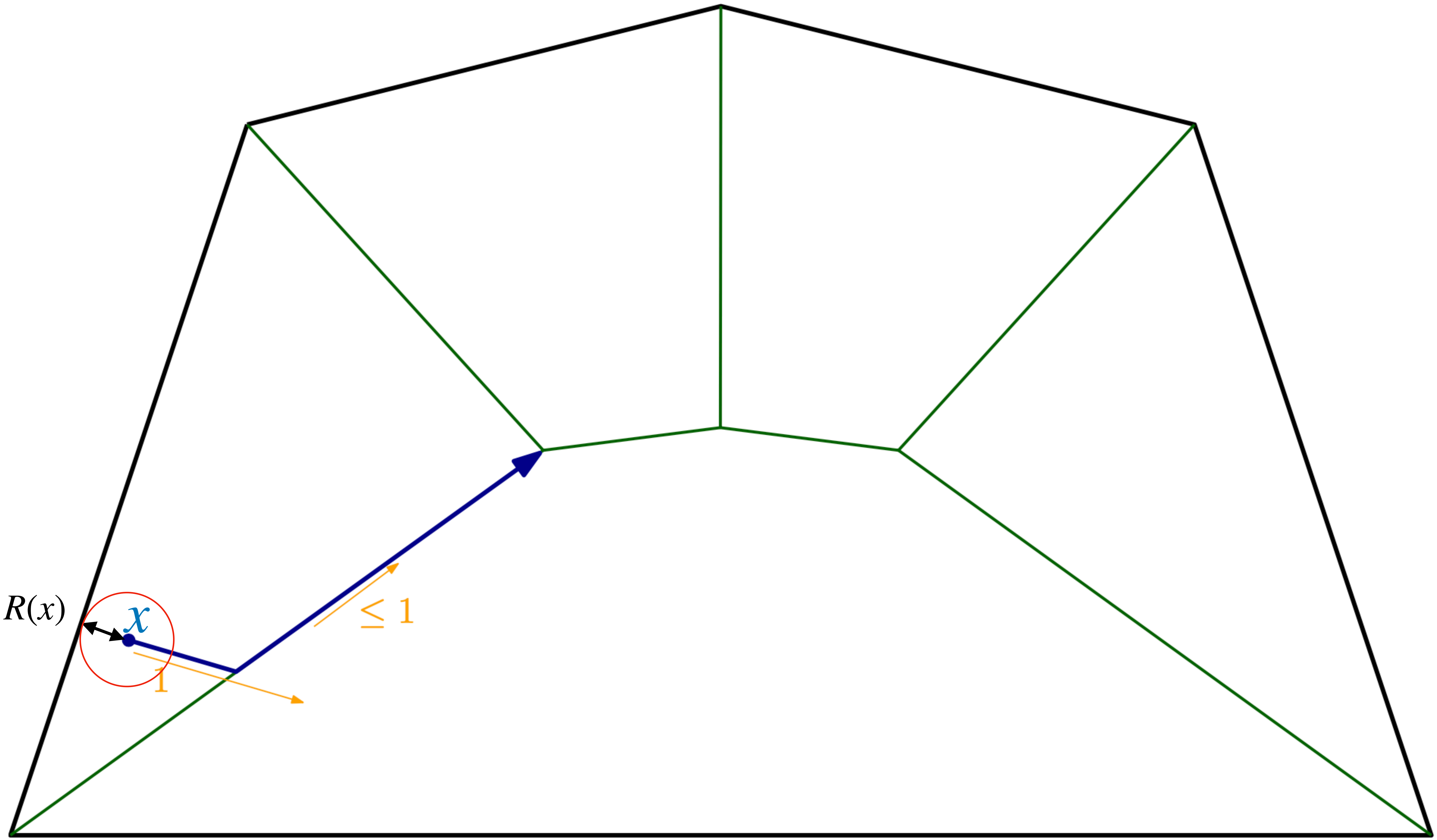
But for non smooth manifolds the reach is 0 !



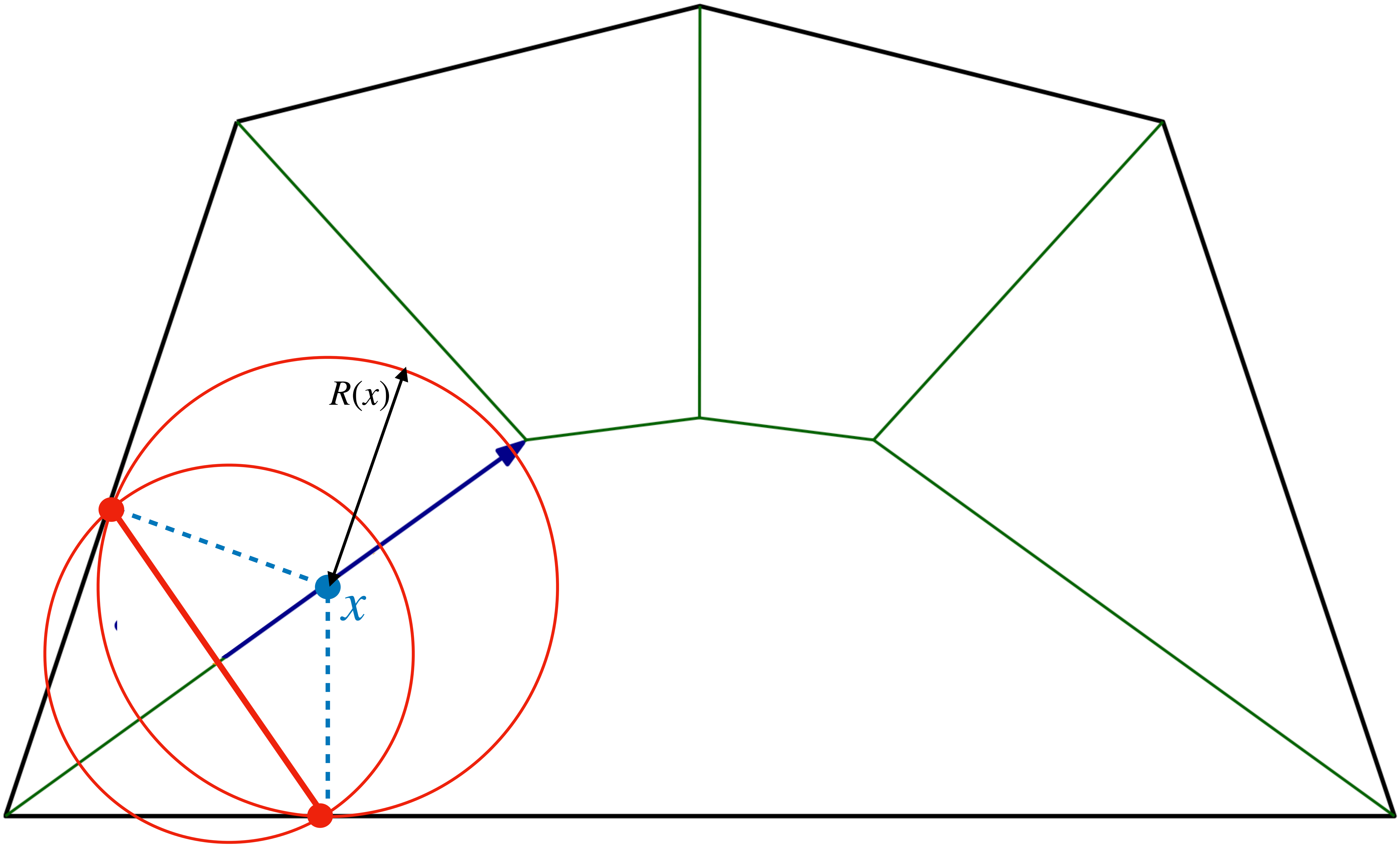
reach(S) = 0 !

Beyond the reach

Outside the medial axis, the distance fonction $x \mapsto R(x)$ is differentiable and its **gradient has unit norm**: $\|\nabla(x)\| = 1$



Beyond the reach



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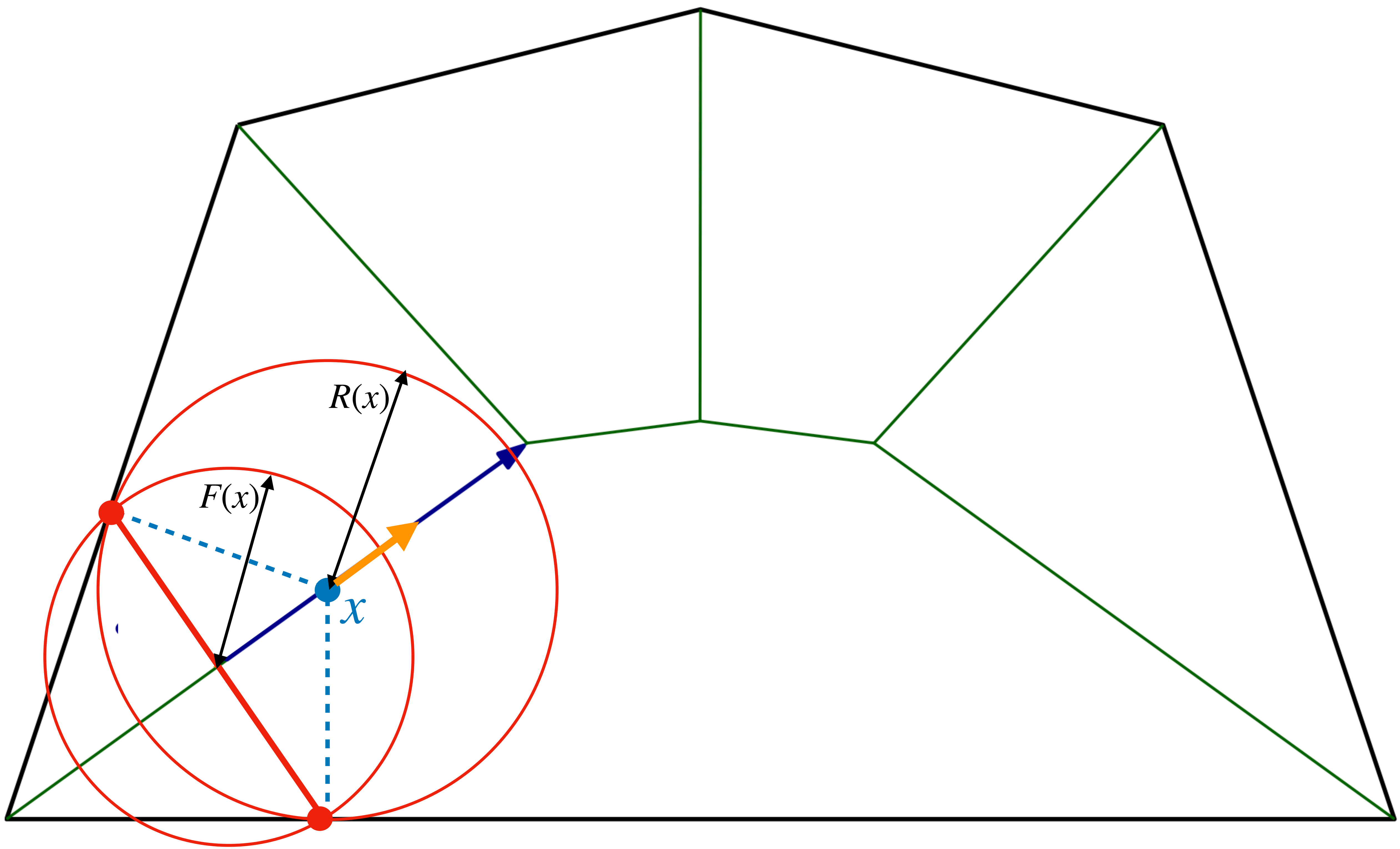

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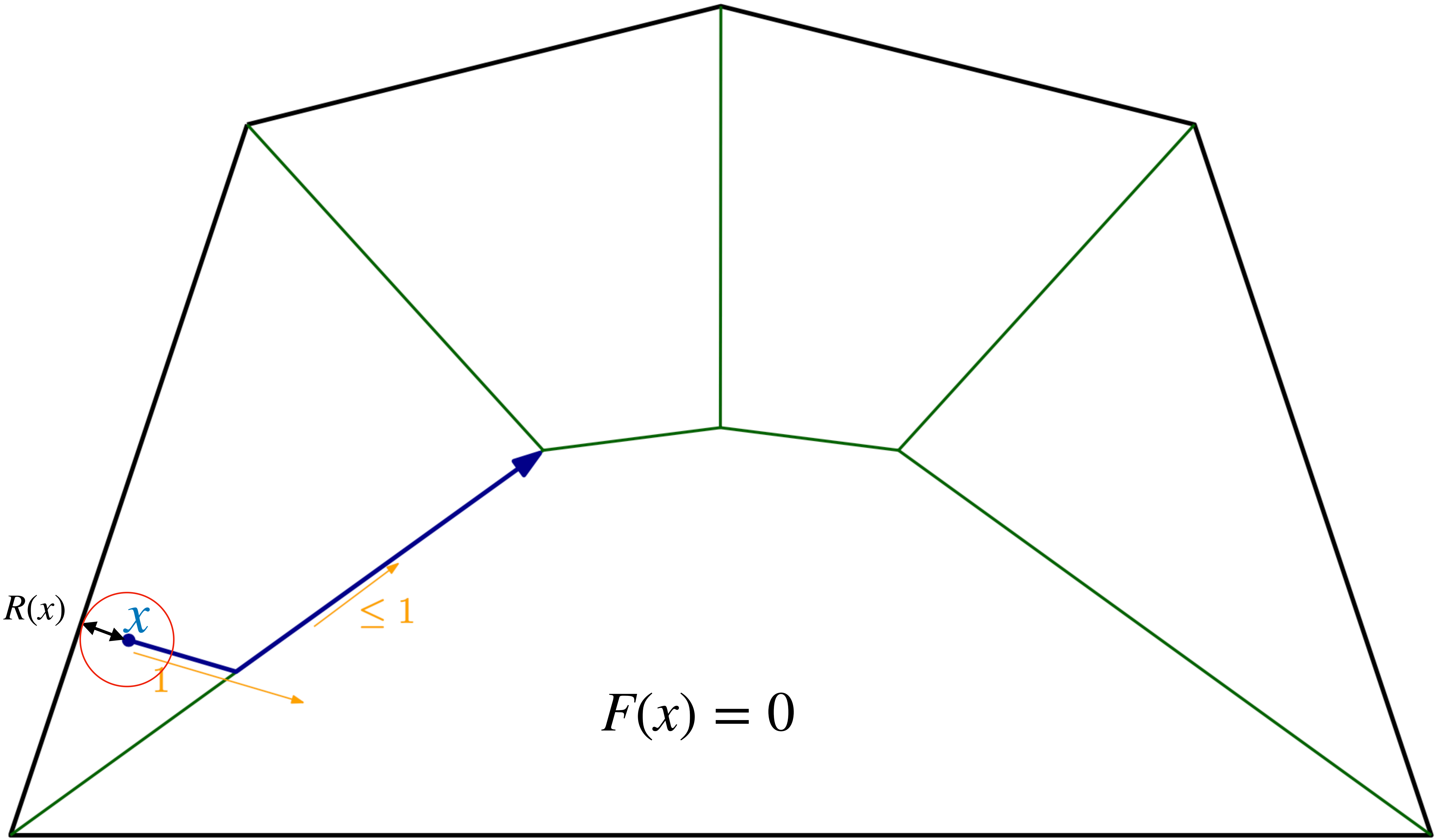

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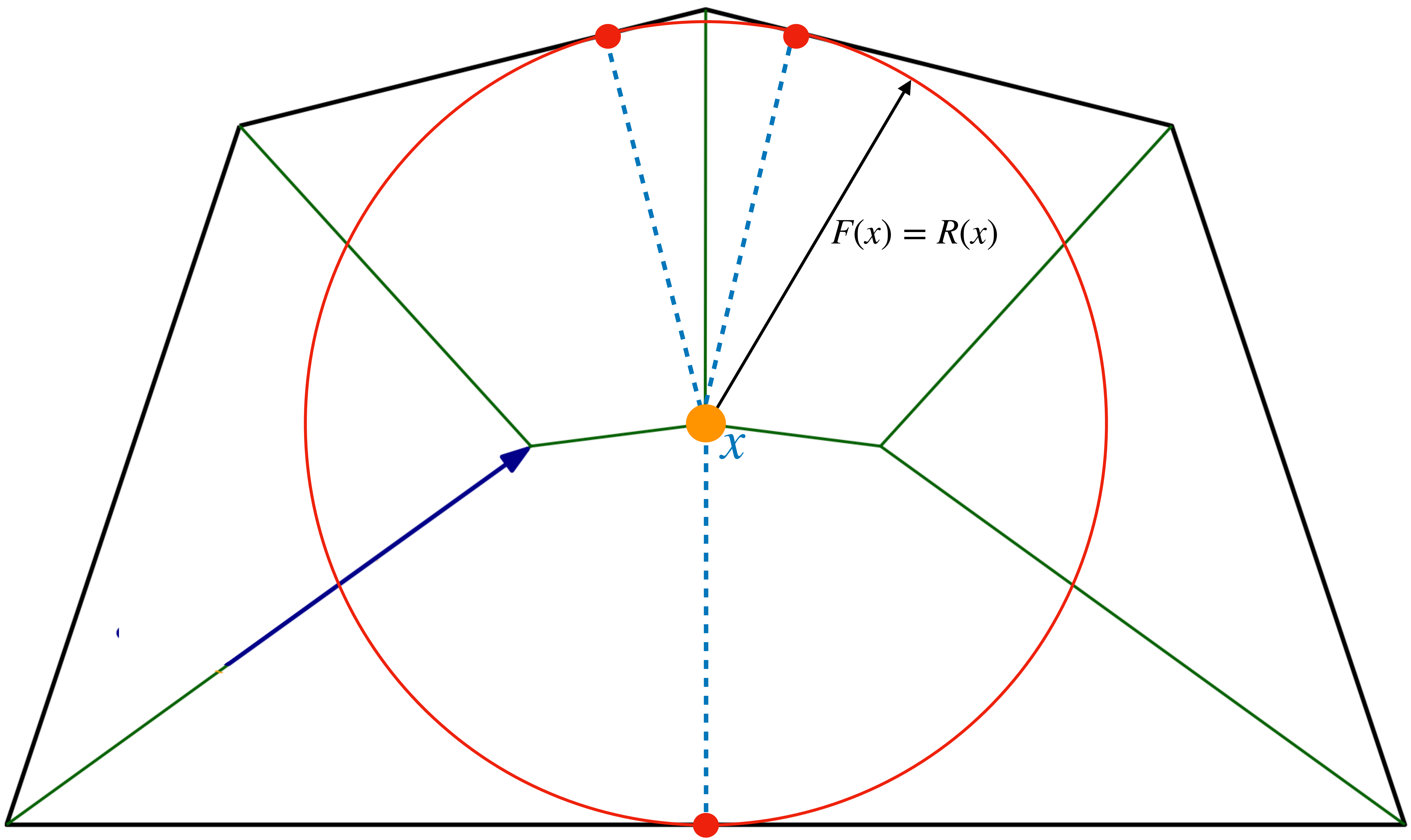


critical function

$$x \mapsto R_K(x) \underset{\text{def.}}{=} d(x, K) = \min_{y \in K} d(x, y)$$

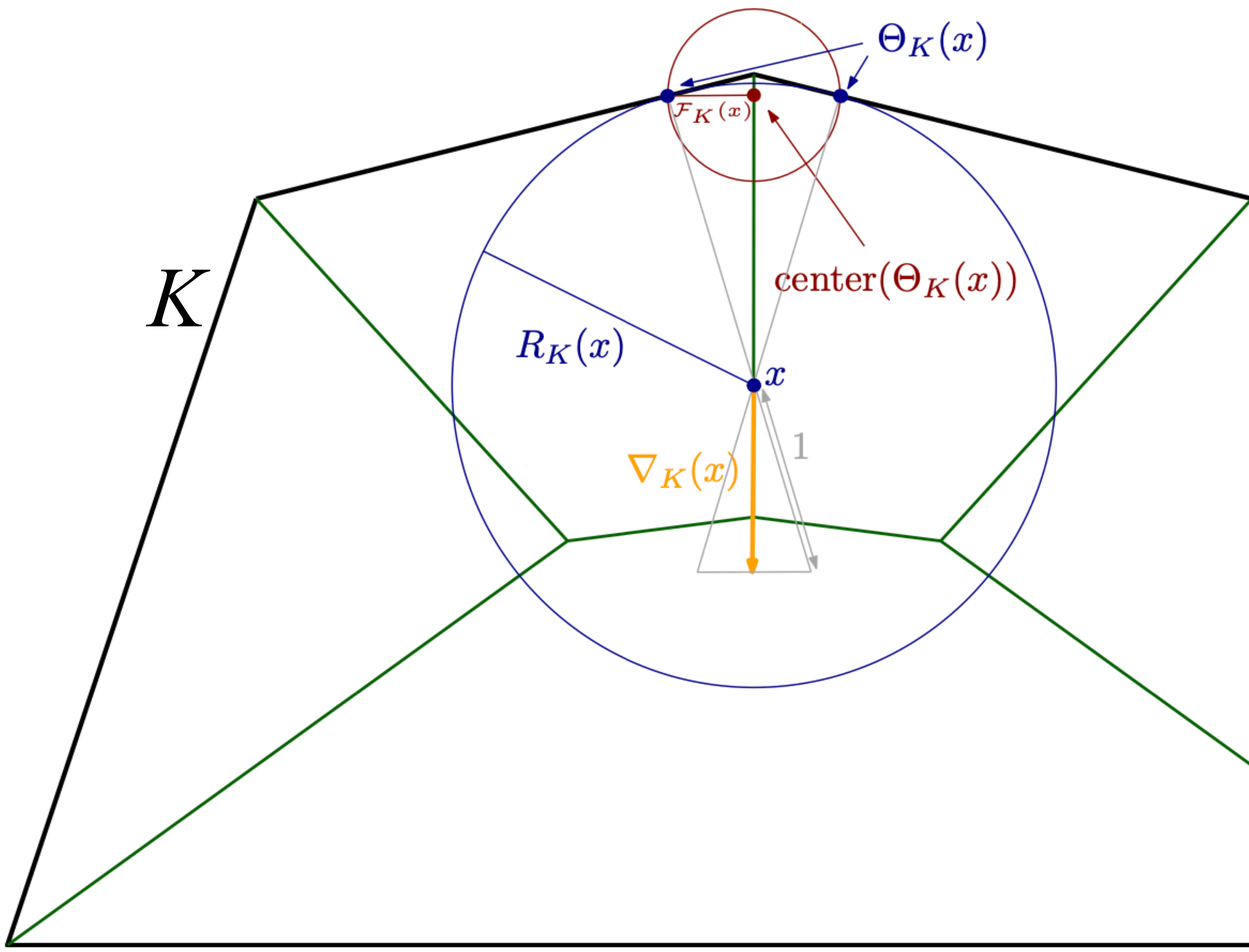
$$x \mapsto \Theta_K(x) \underset{\text{def.}}{=} \{y \in K \mid d(x, y) = R_K(x)\}\,.$$

$$\mathcal{F}_K(x) \underset{\text{def.}}{=} \text{radius}(\Theta_K(x))$$

$$\nabla_K(x) \underset{\text{def.}}{=} \frac{x - \text{center}(\Theta_K(x))}{R_K(x)}$$

$$\|\nabla_K(x)\|^2 = 1 - \left(\frac{\mathcal{F}_K(x)}{R_K(x)}\right)^2$$

$$\|\nabla(x)\| = \sqrt{1 - \frac{F(x)^2}{R(x)^2}}$$



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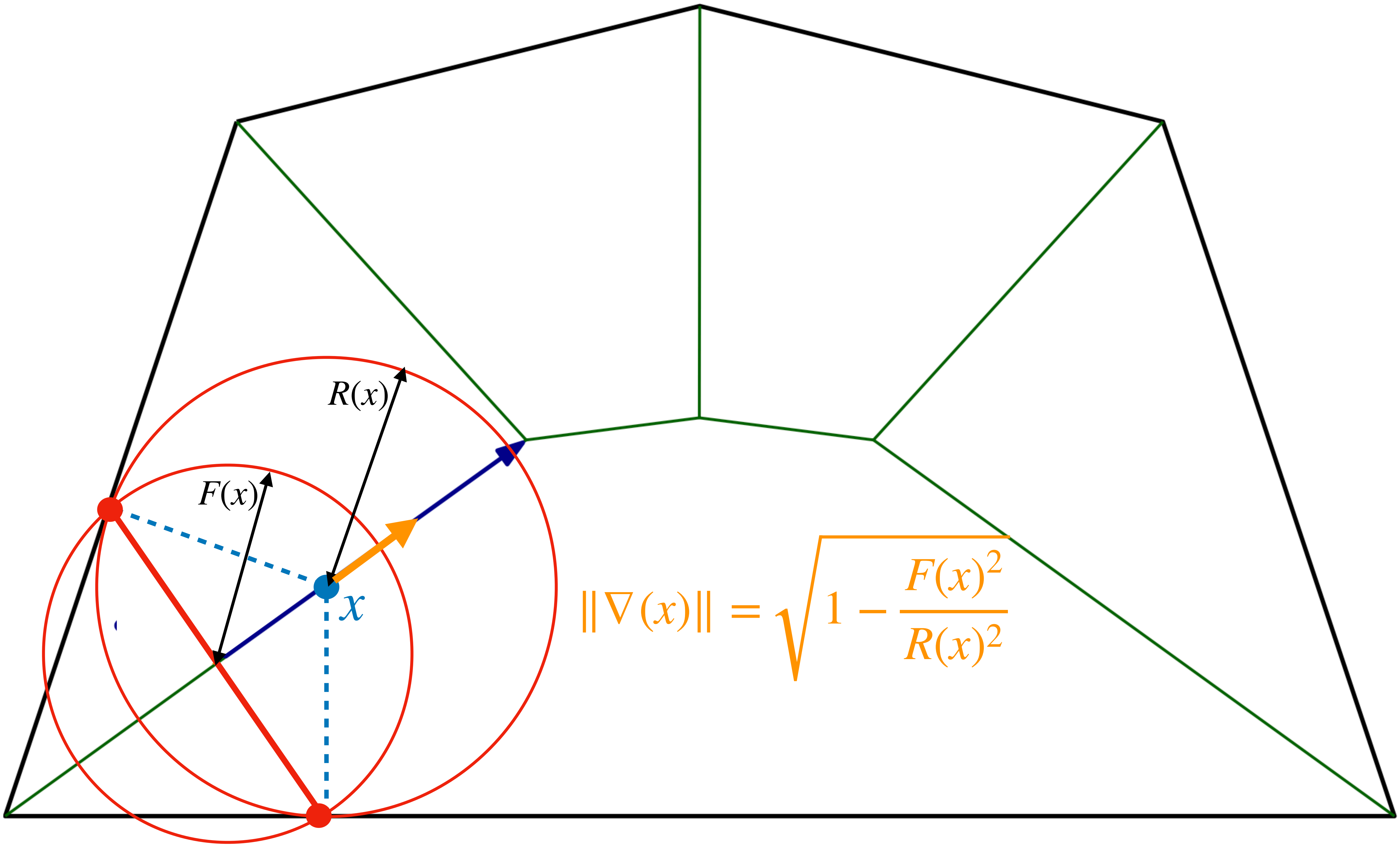

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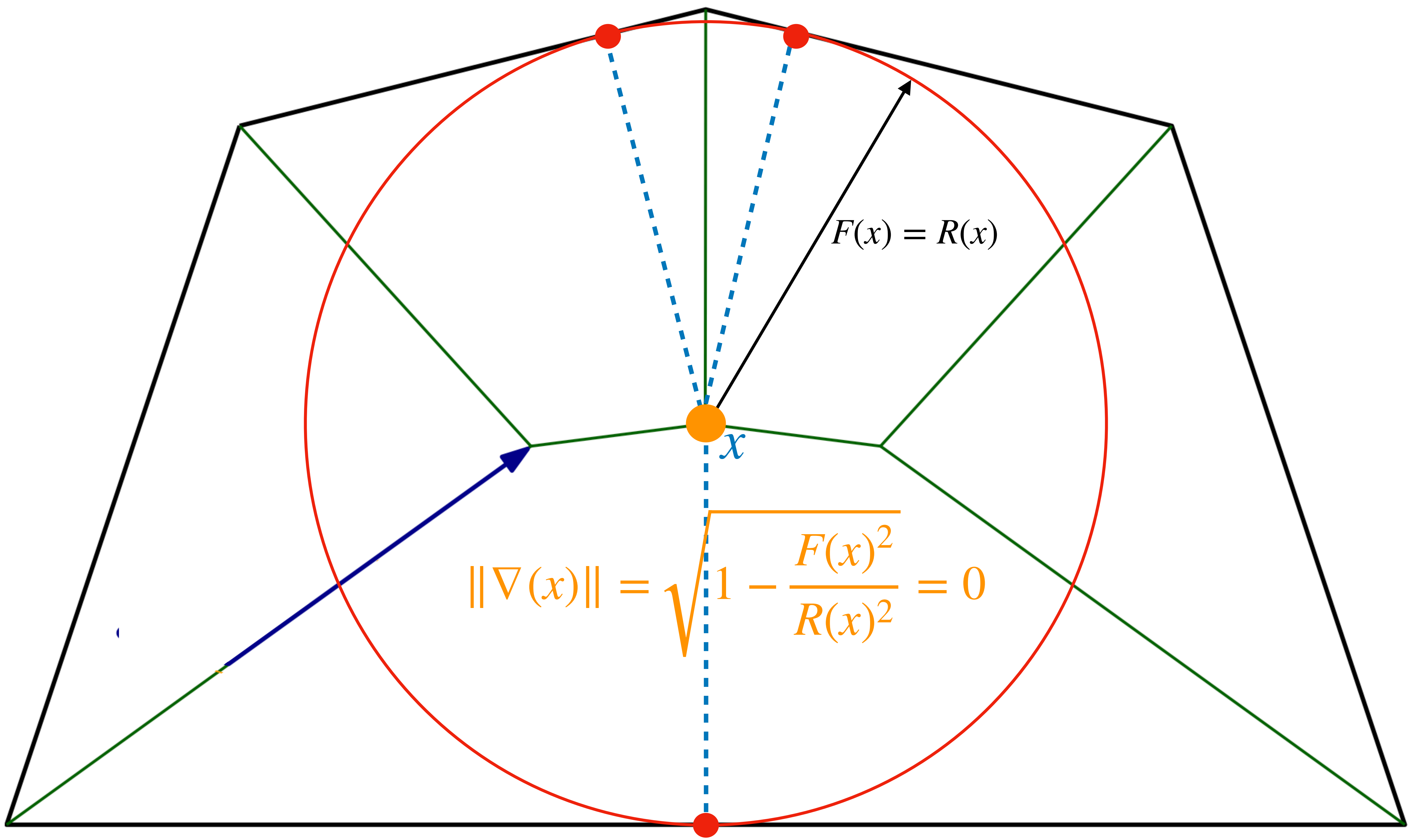

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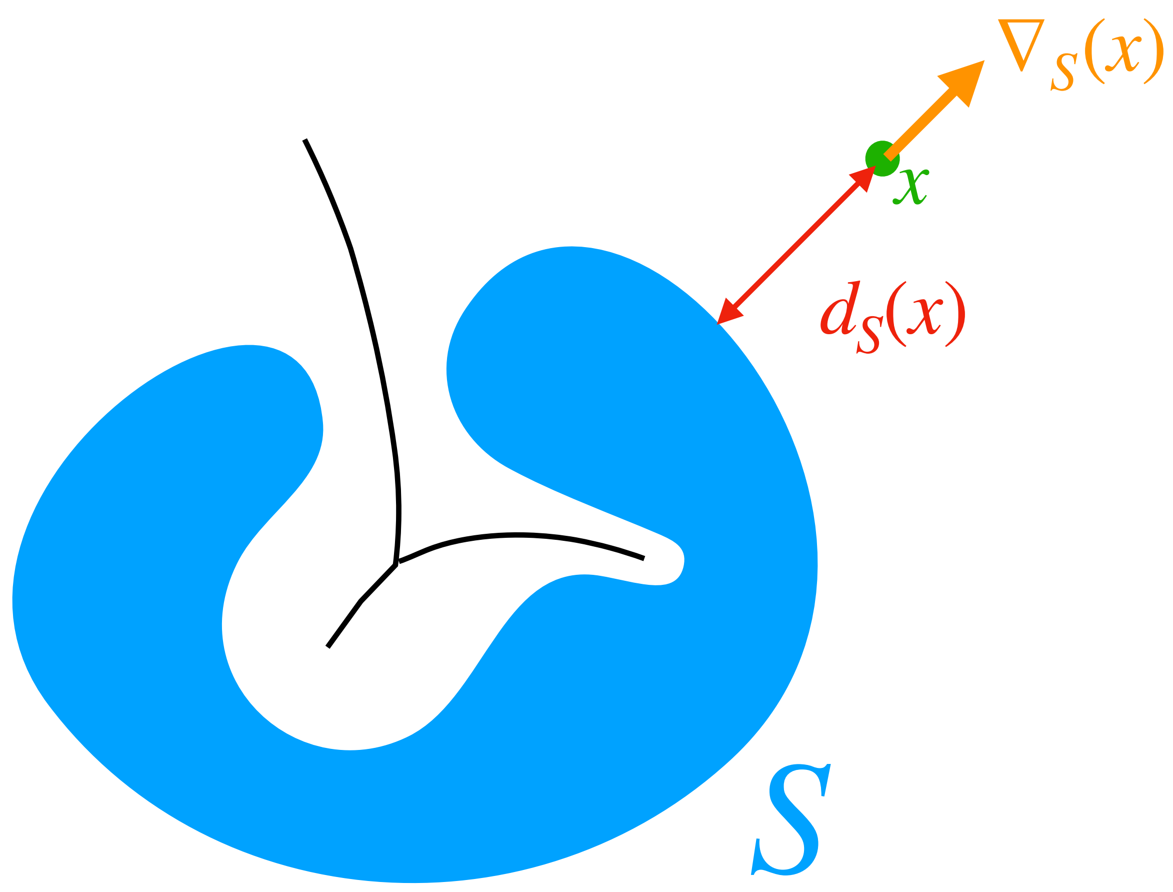


(Chazal, Cohen-Steiner, L, 2006)

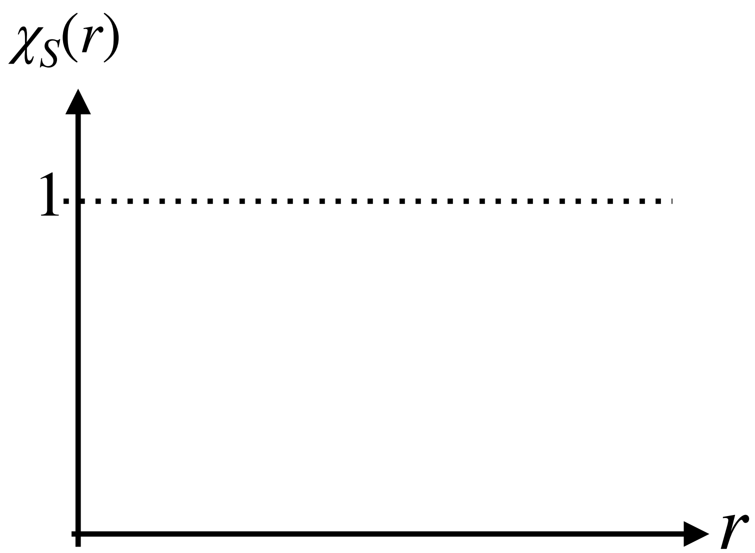
We study the properties of the **sublevel set of the distance function** d_S to the set (offsets).

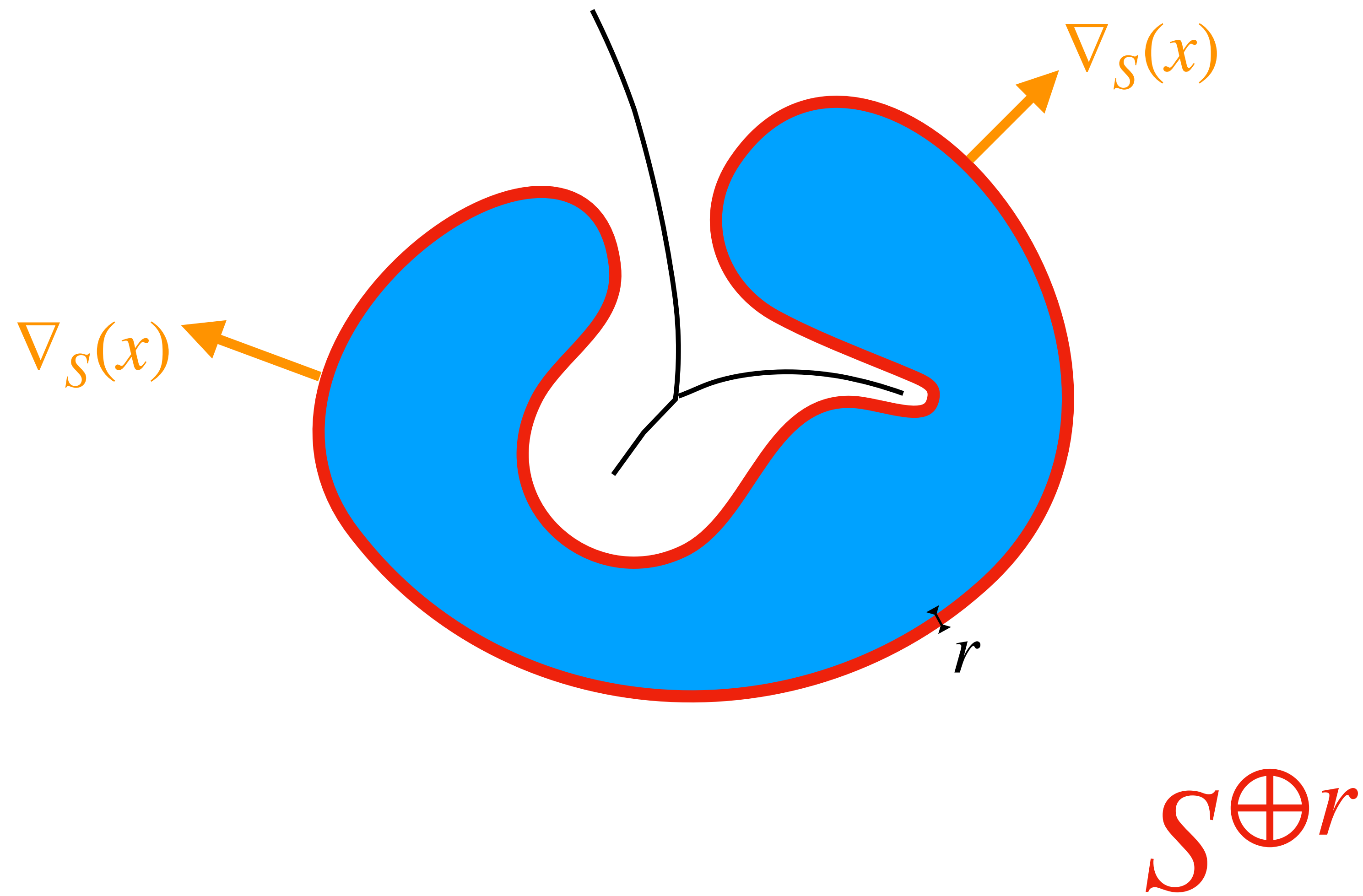
- The distance function is not smooth but admit a **generalized Gradient** ∇_S (Clarke gradient)

critical function

Value of critical function χ_S for offset r :

$$\chi_S(r) =_{def.} \inf_{d_s(x)=r} \|\nabla_S(x)\|$$





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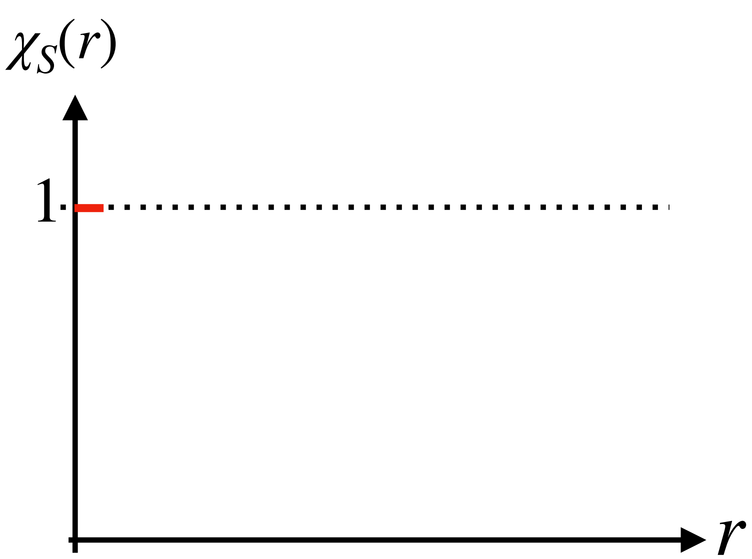

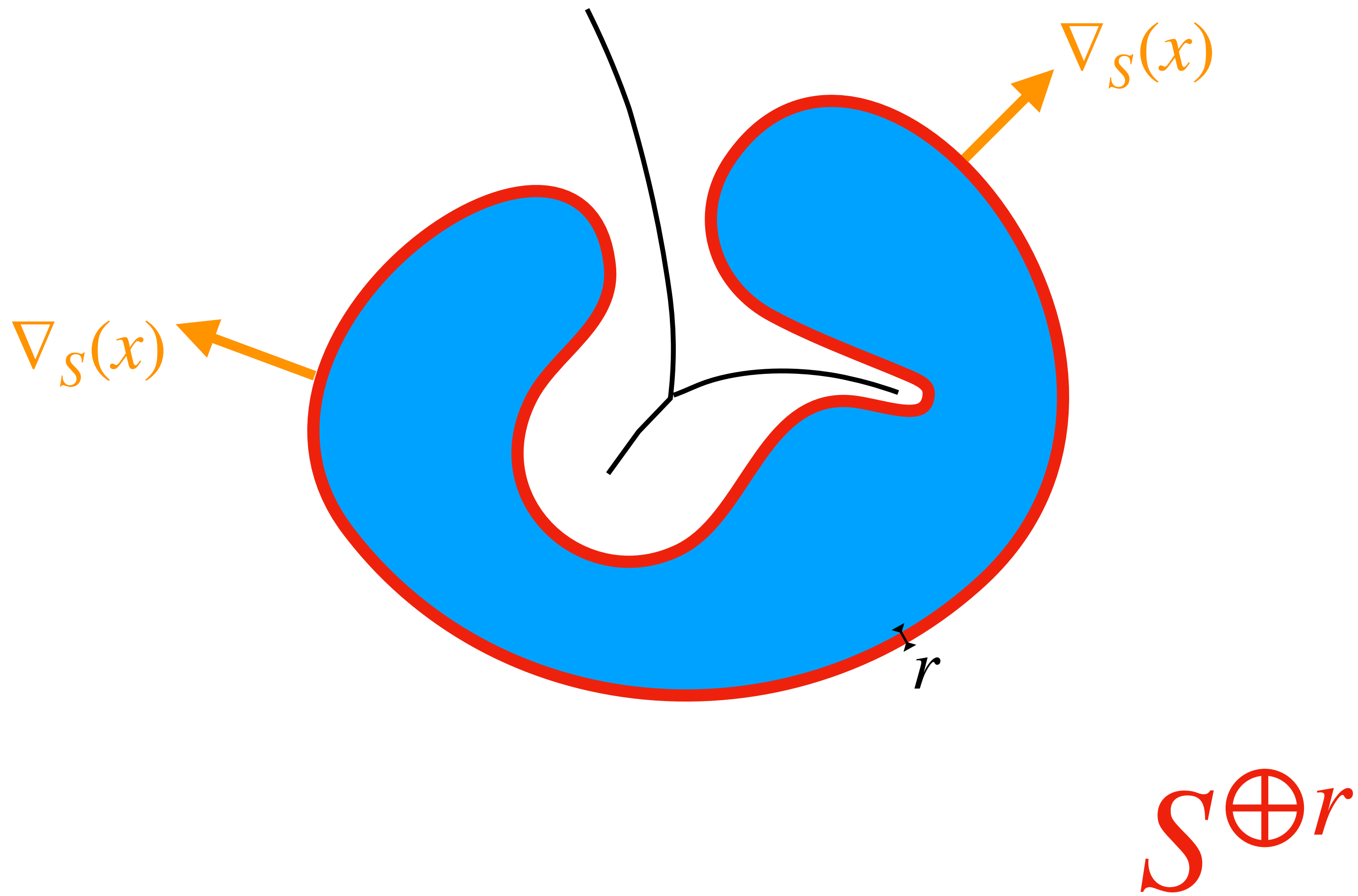

(Chazal, Cohen-Steiner, L, 2006)

We study the properties of the **sublevel set of the distance function** d_S to the set (offsets).

- The distance function is not smooth but admit a **generalized Gradient** ∇_S (Clarke gradient)

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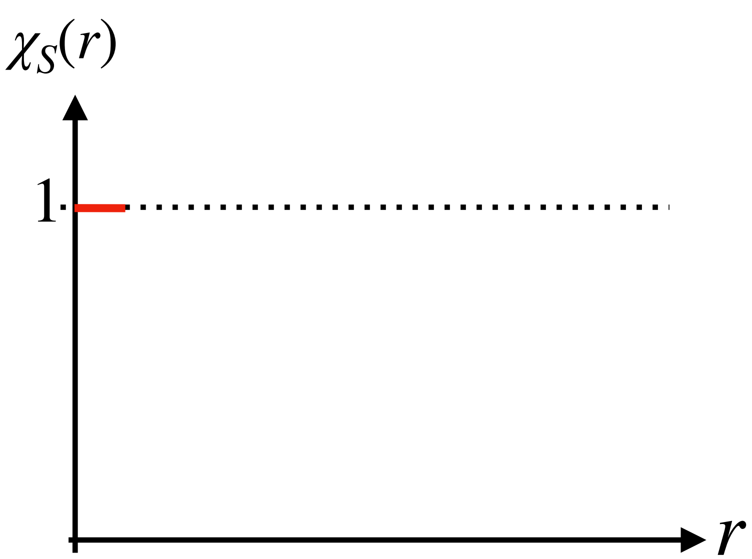

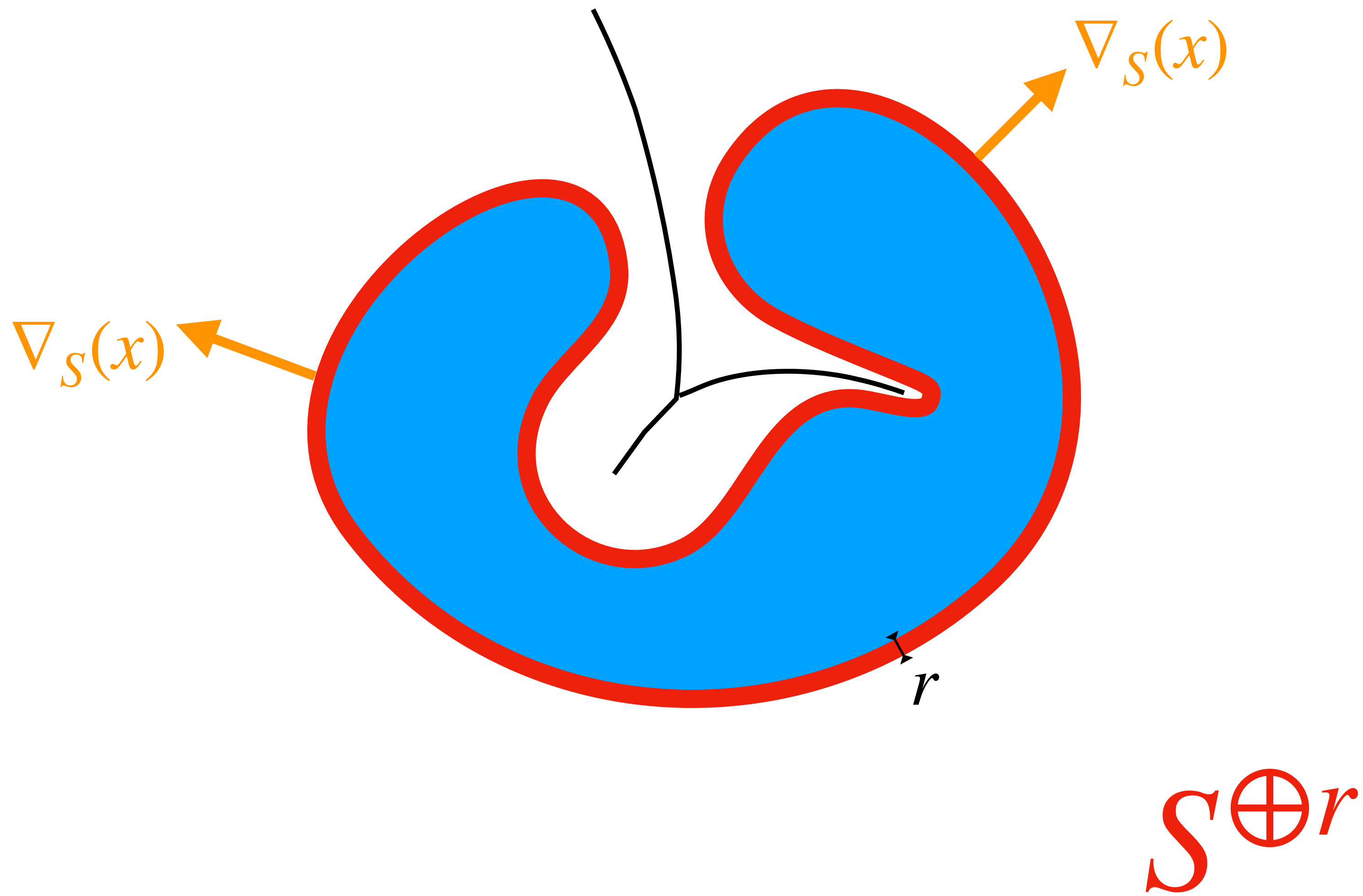

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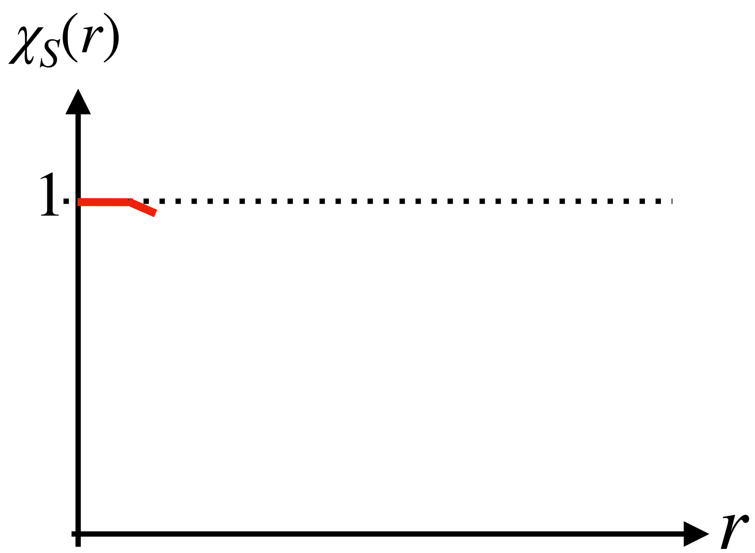

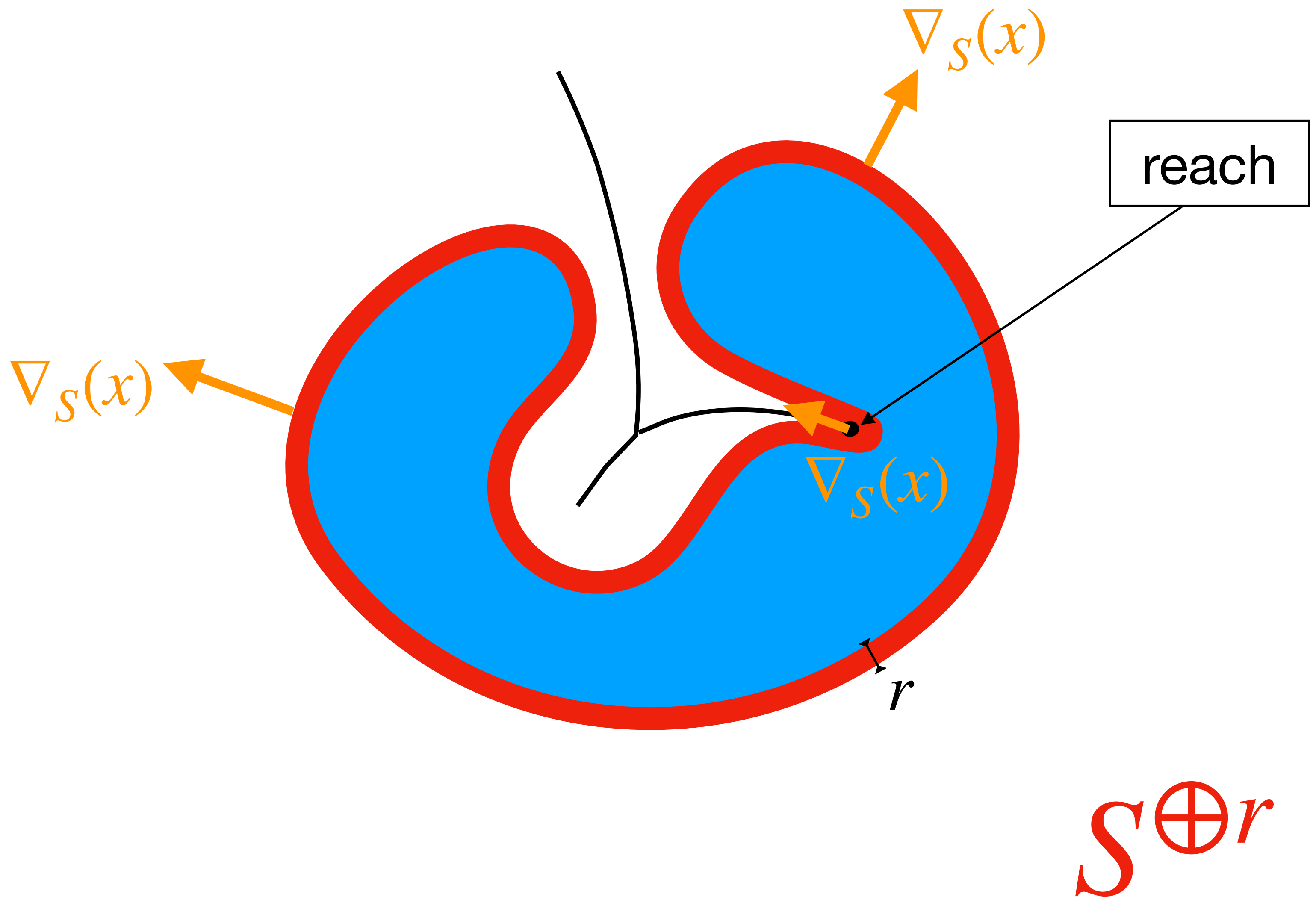

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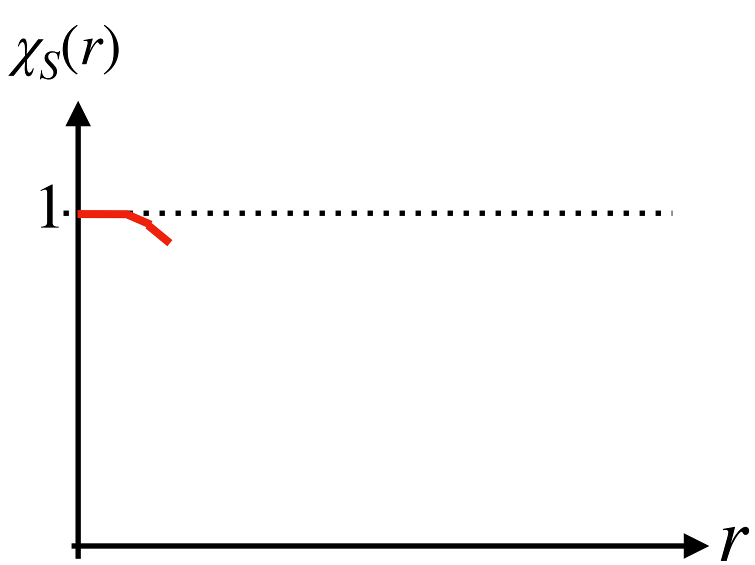

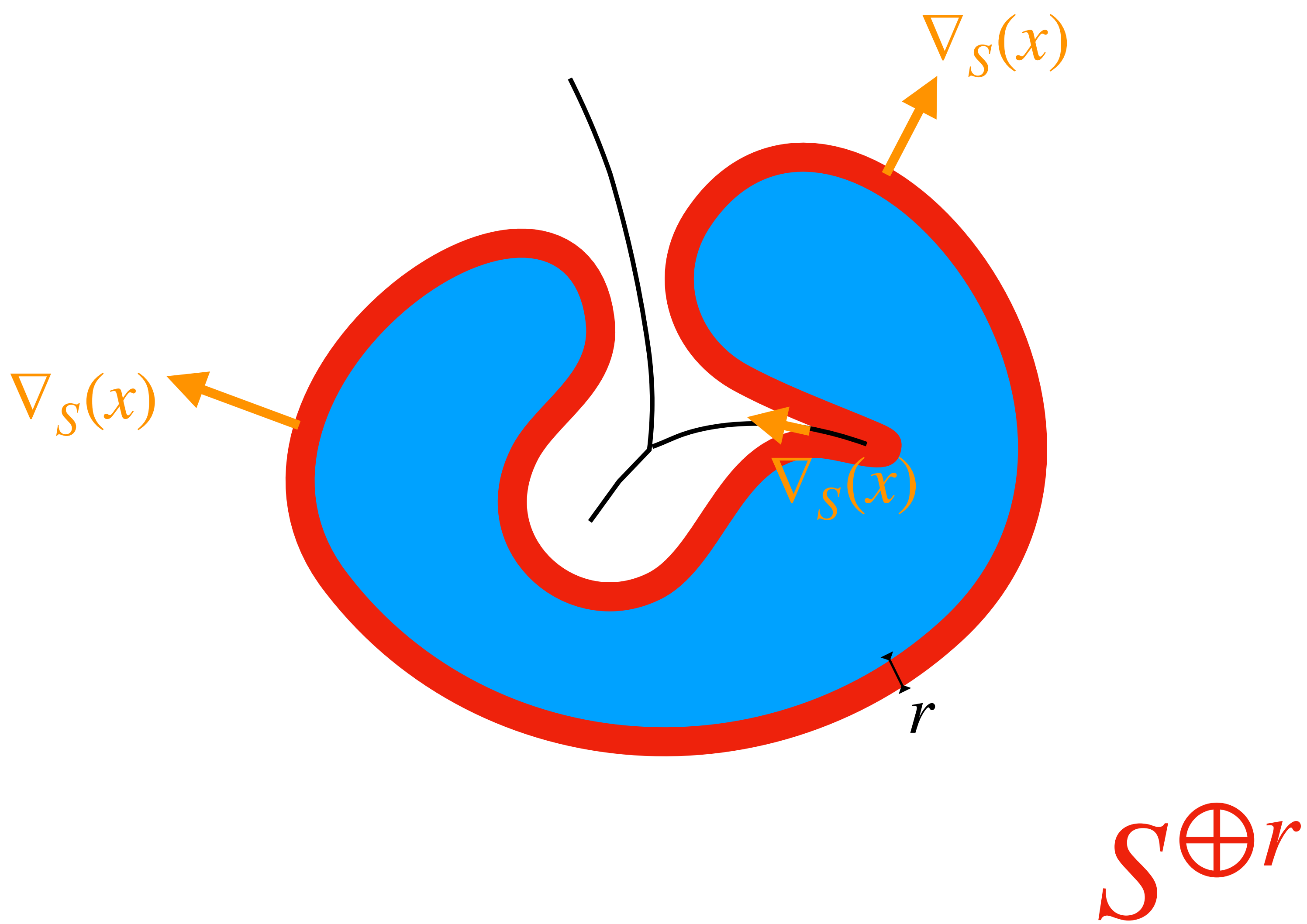

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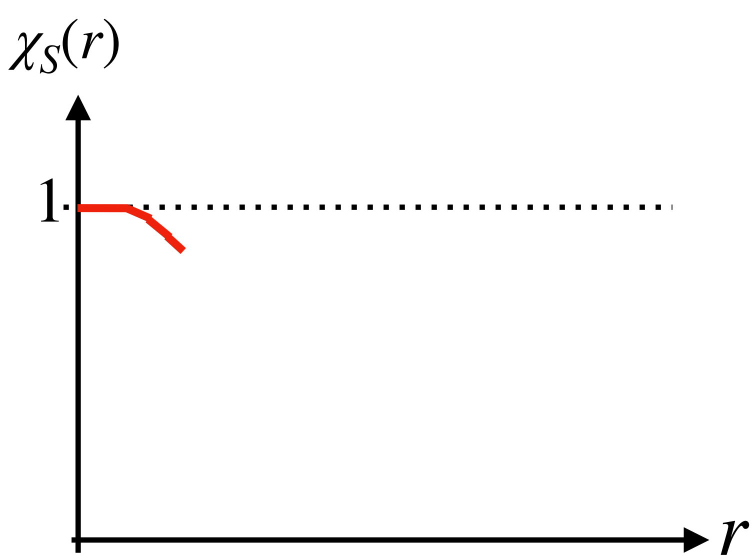

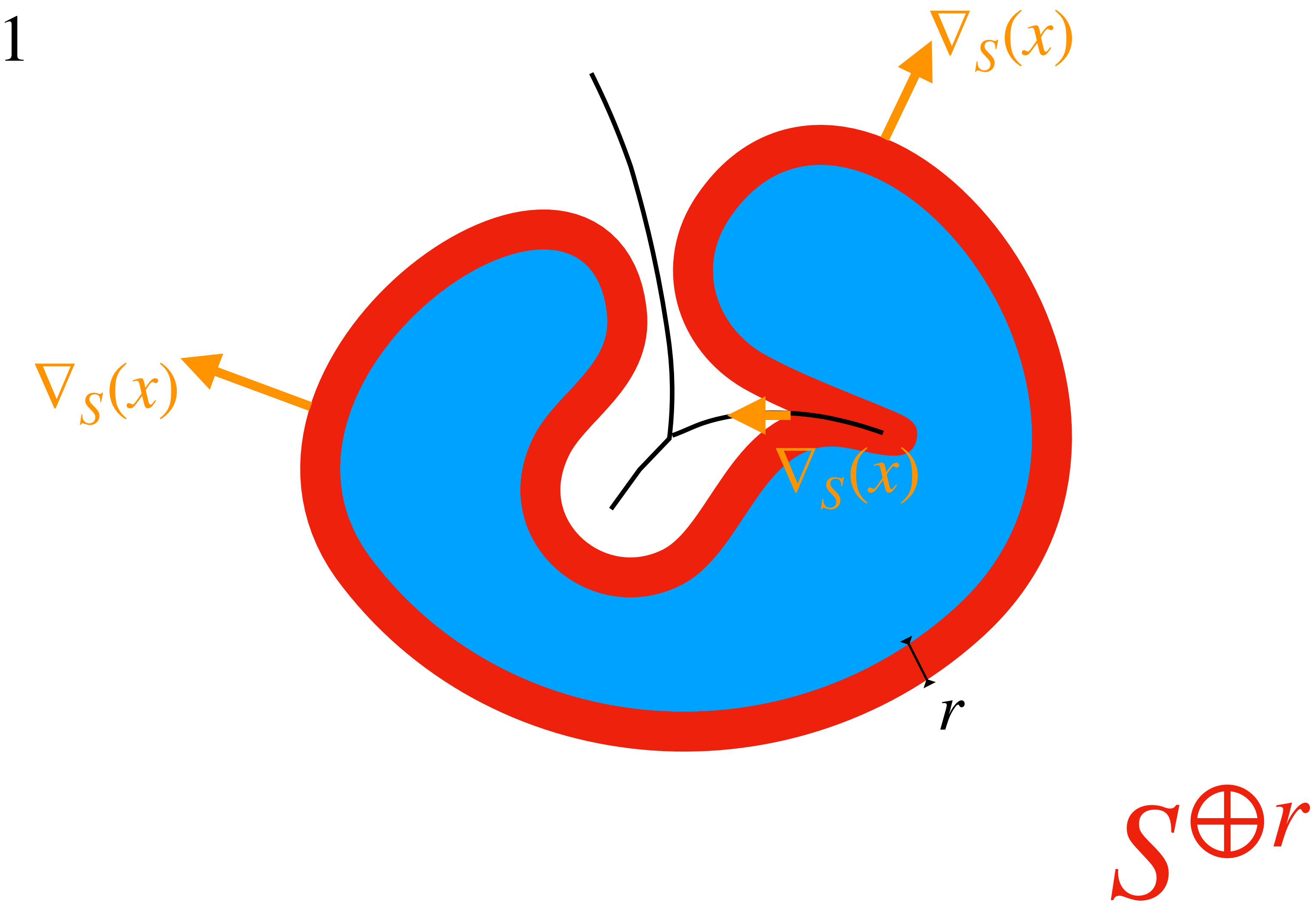

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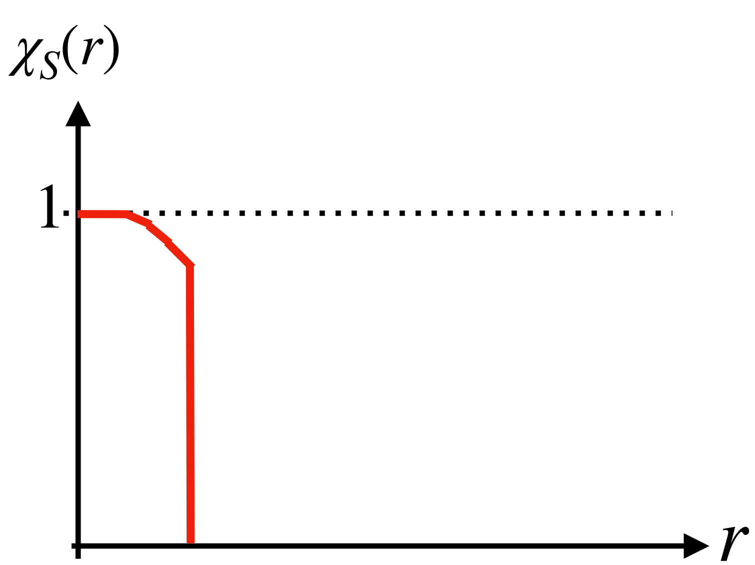

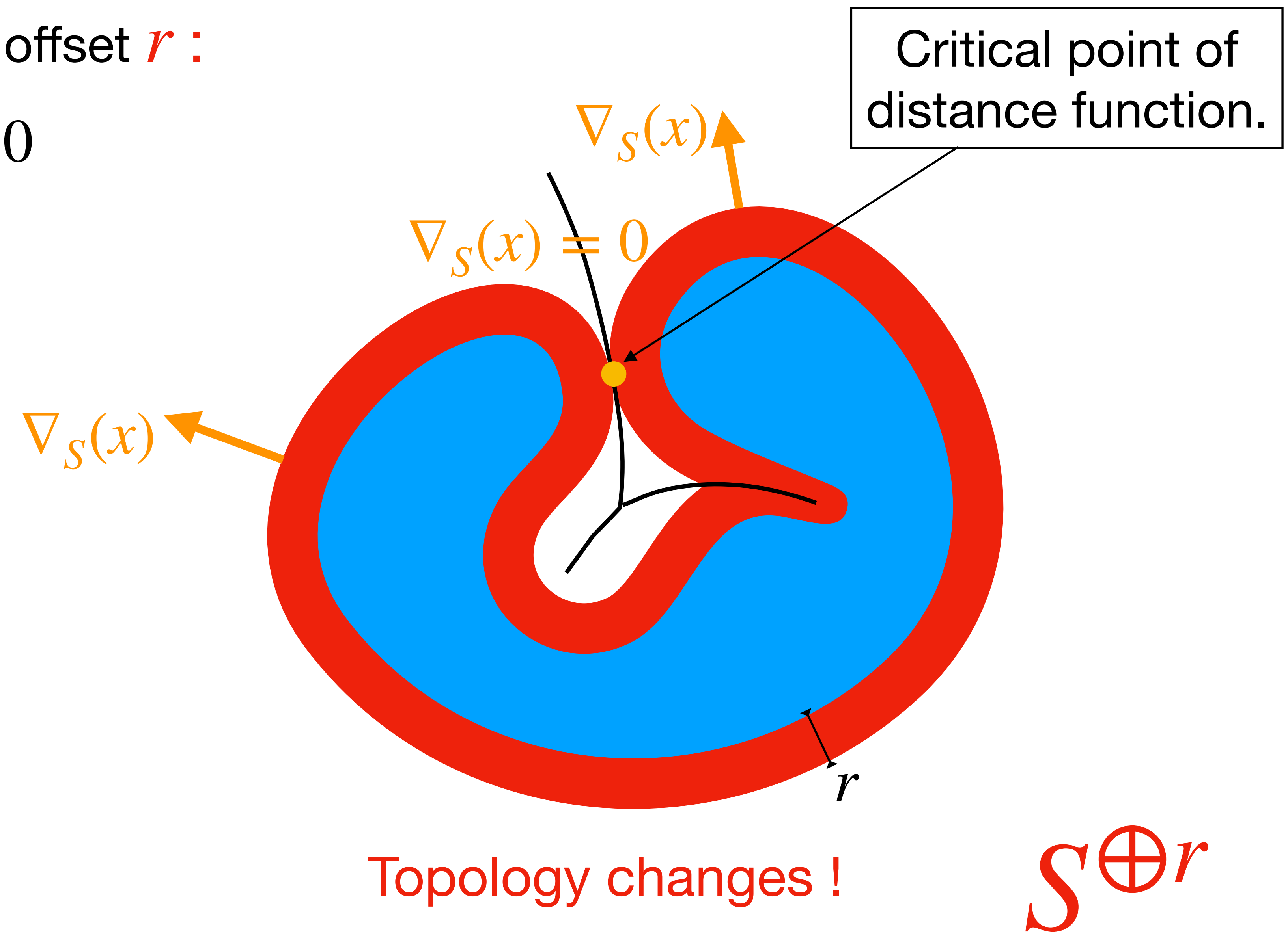

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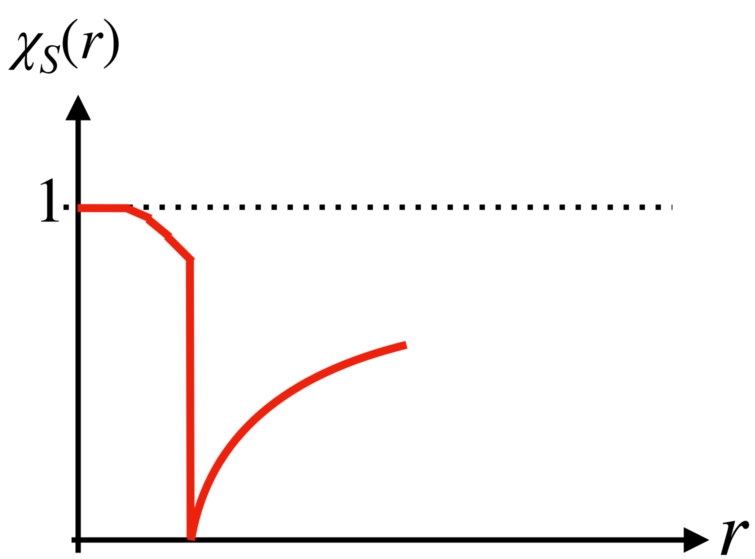

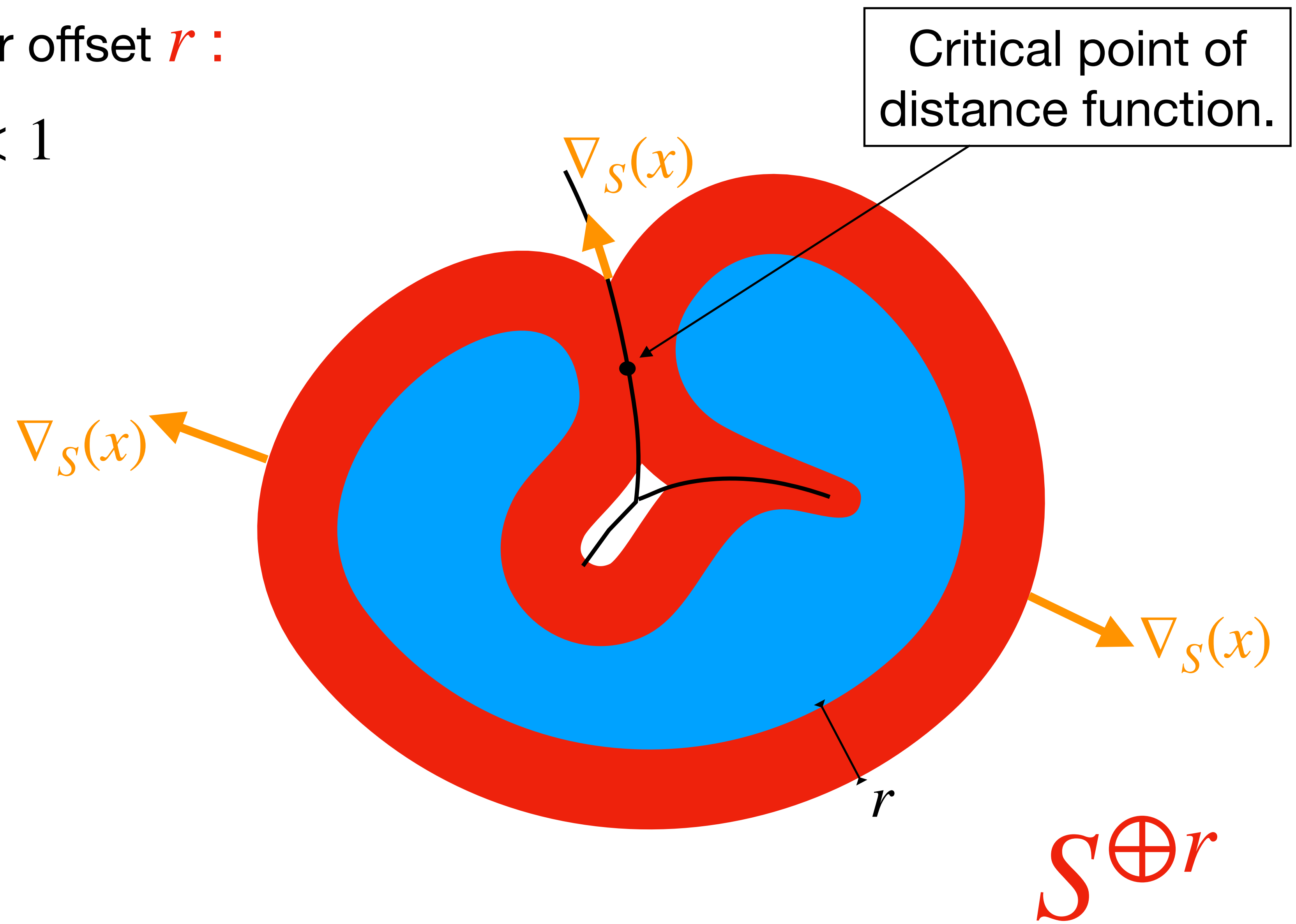

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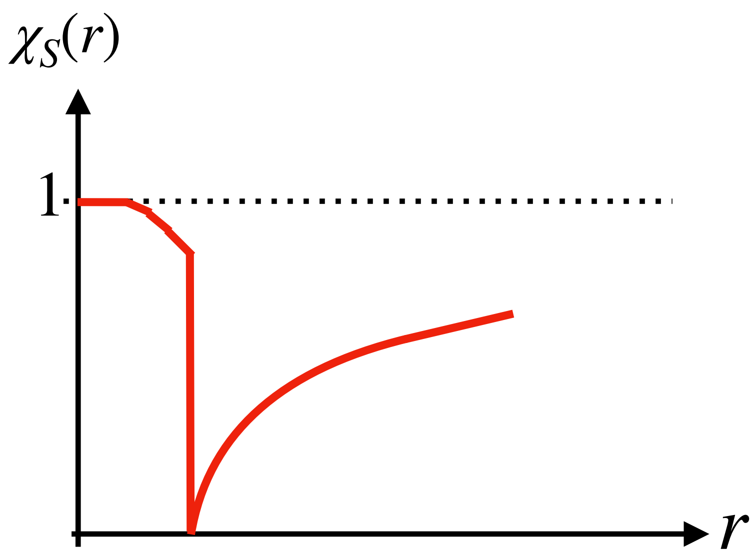

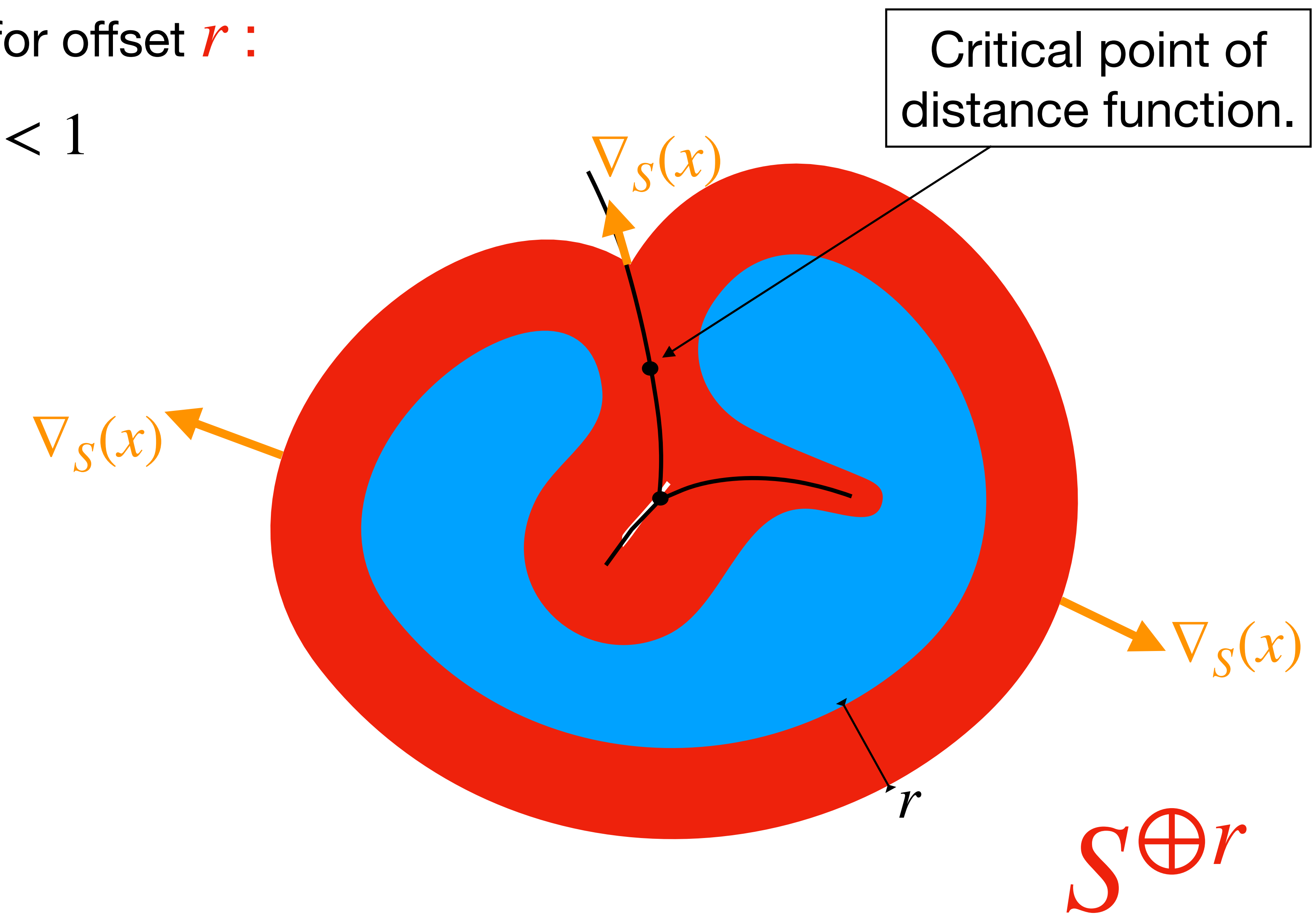

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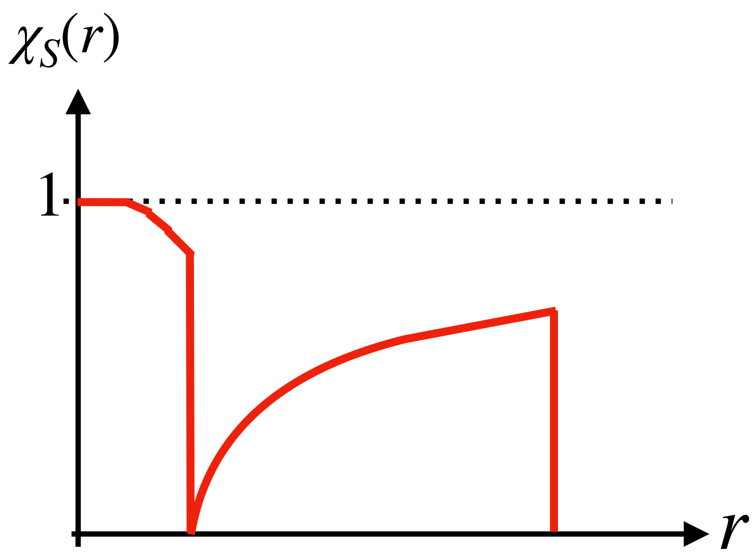

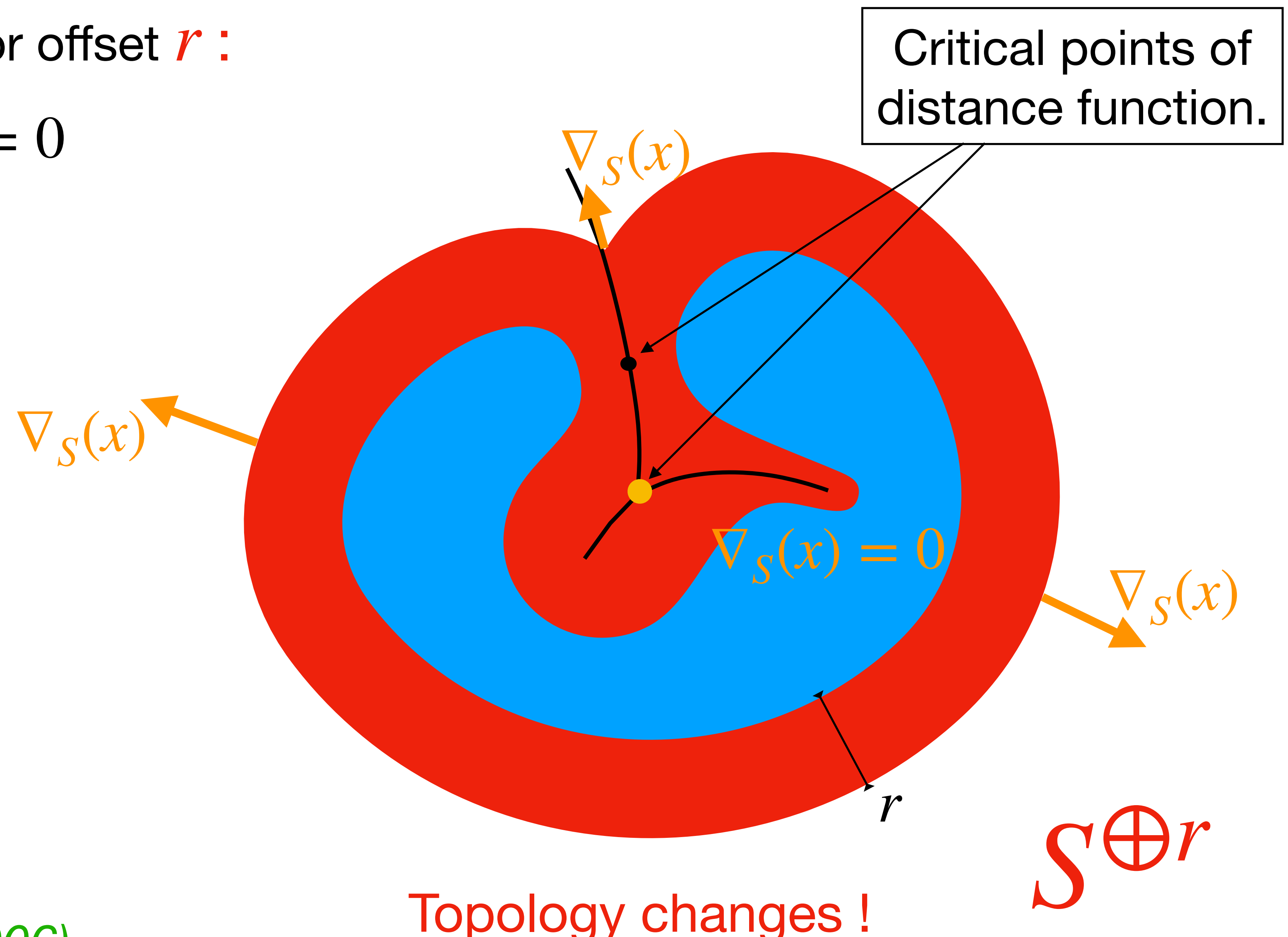

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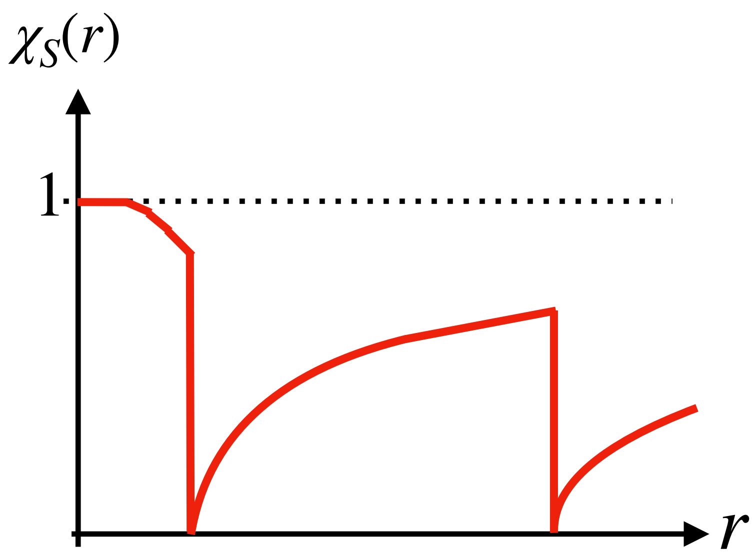

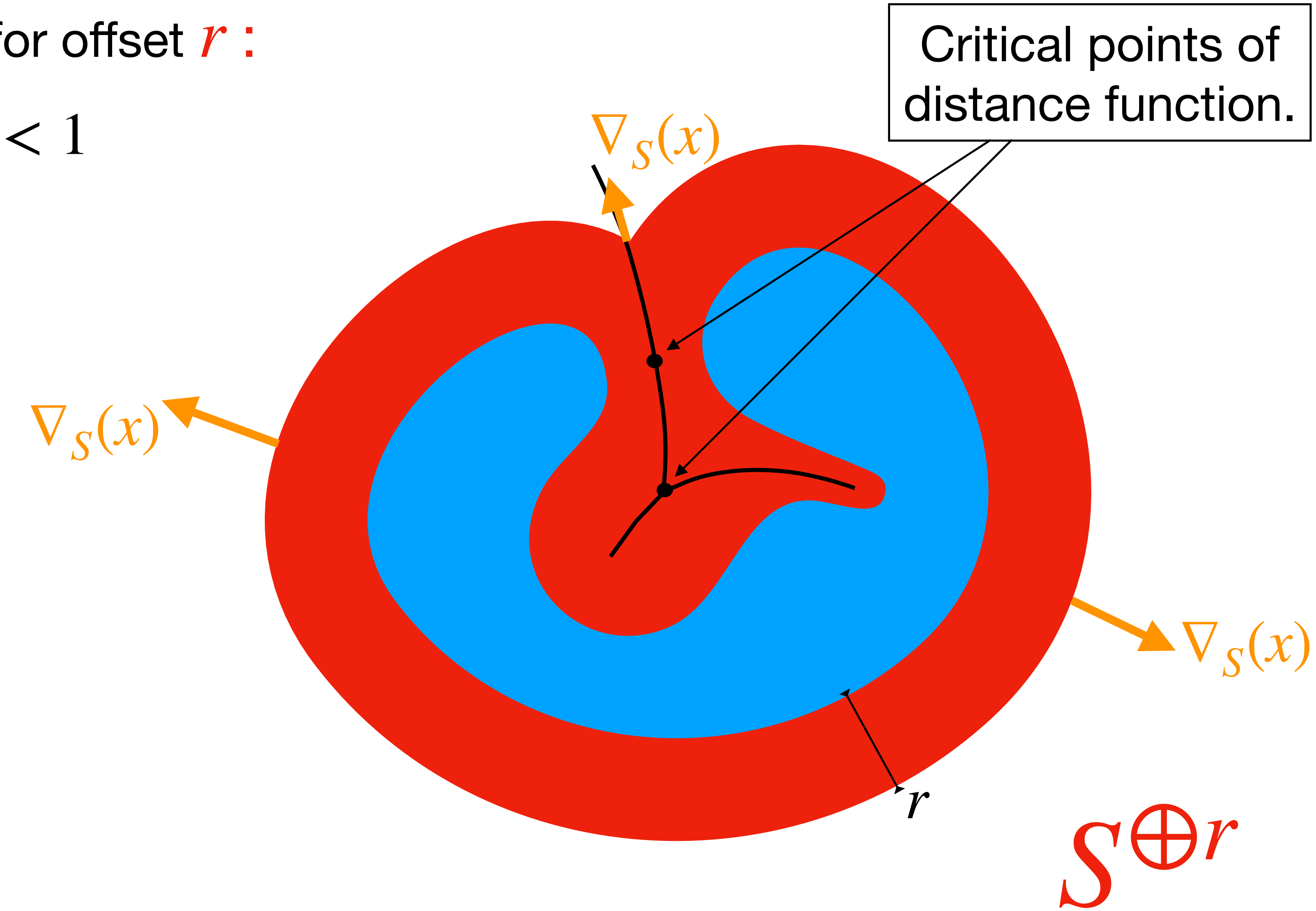

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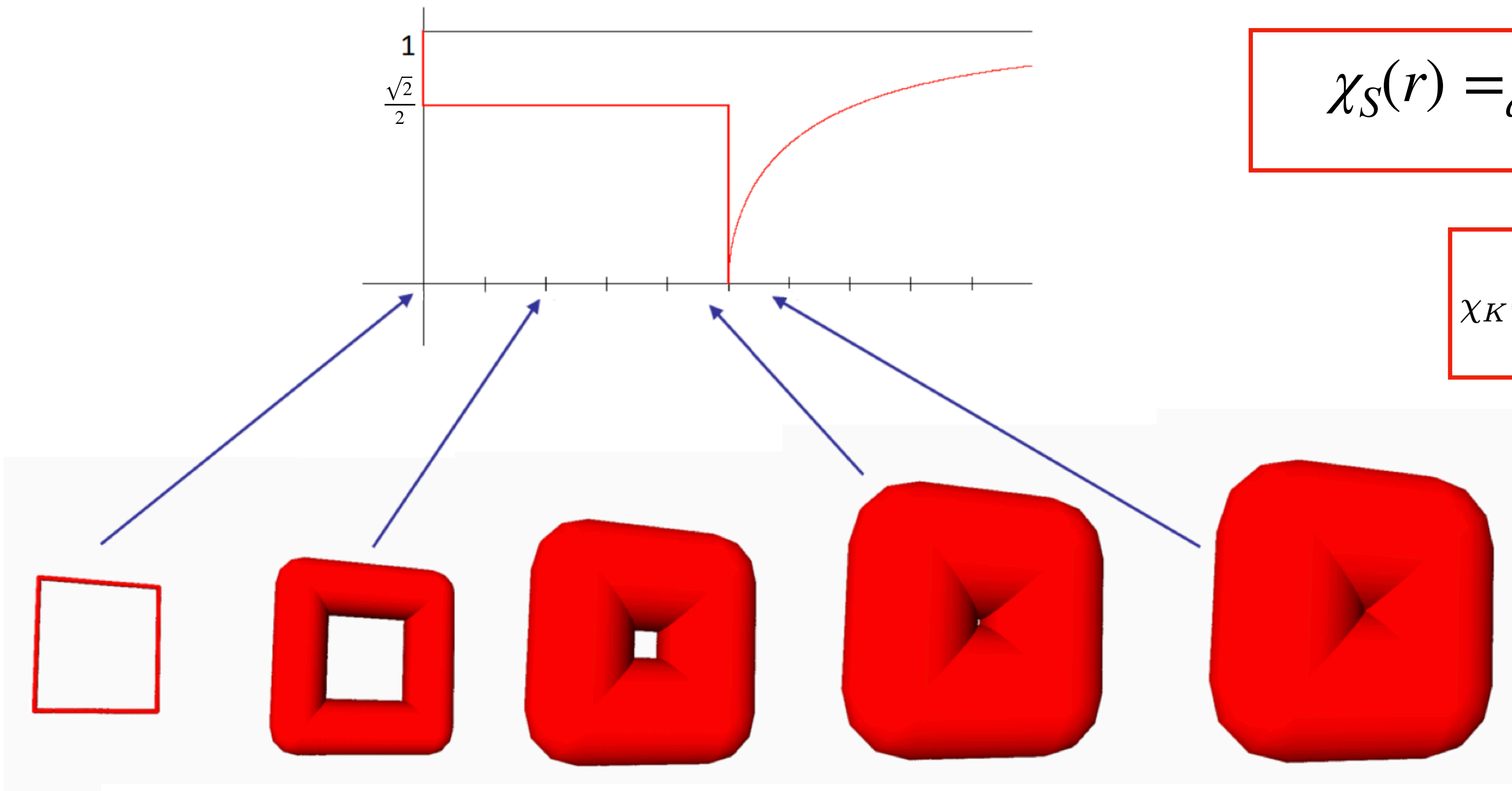

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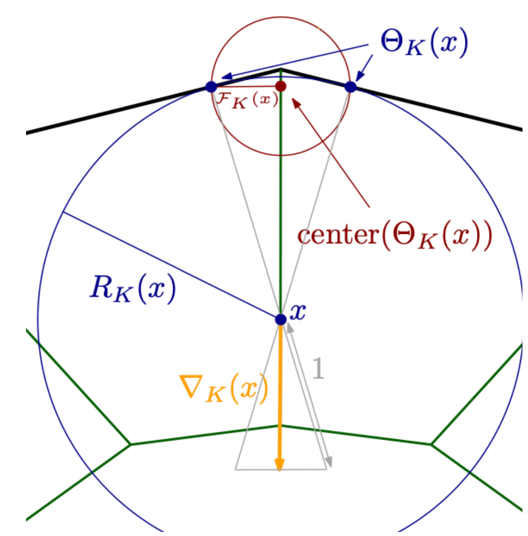

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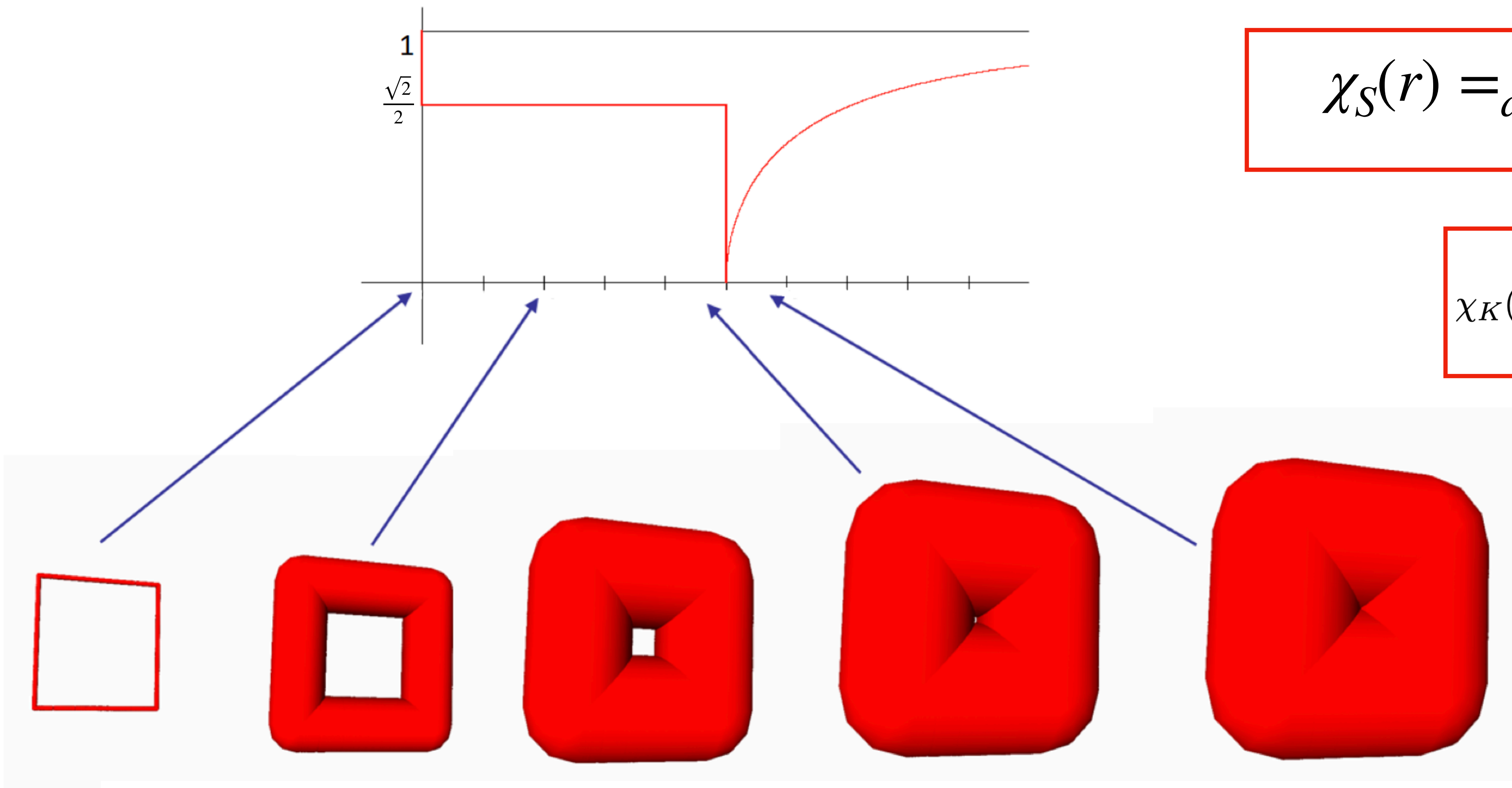

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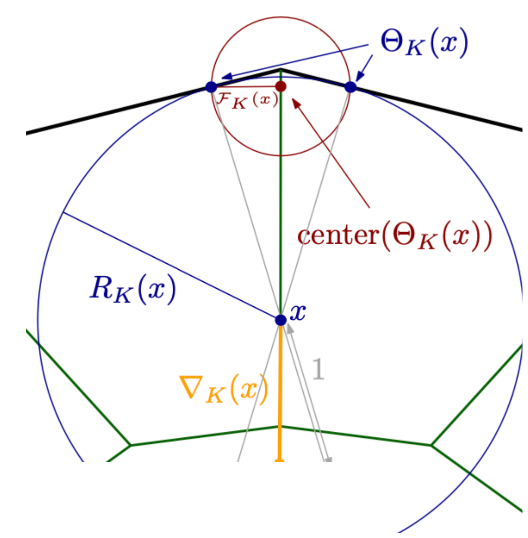

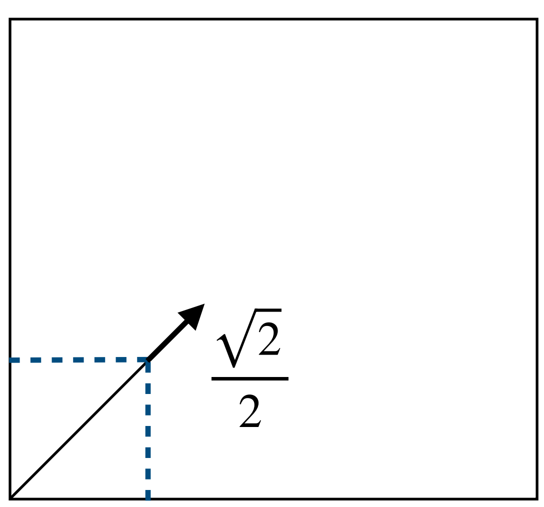

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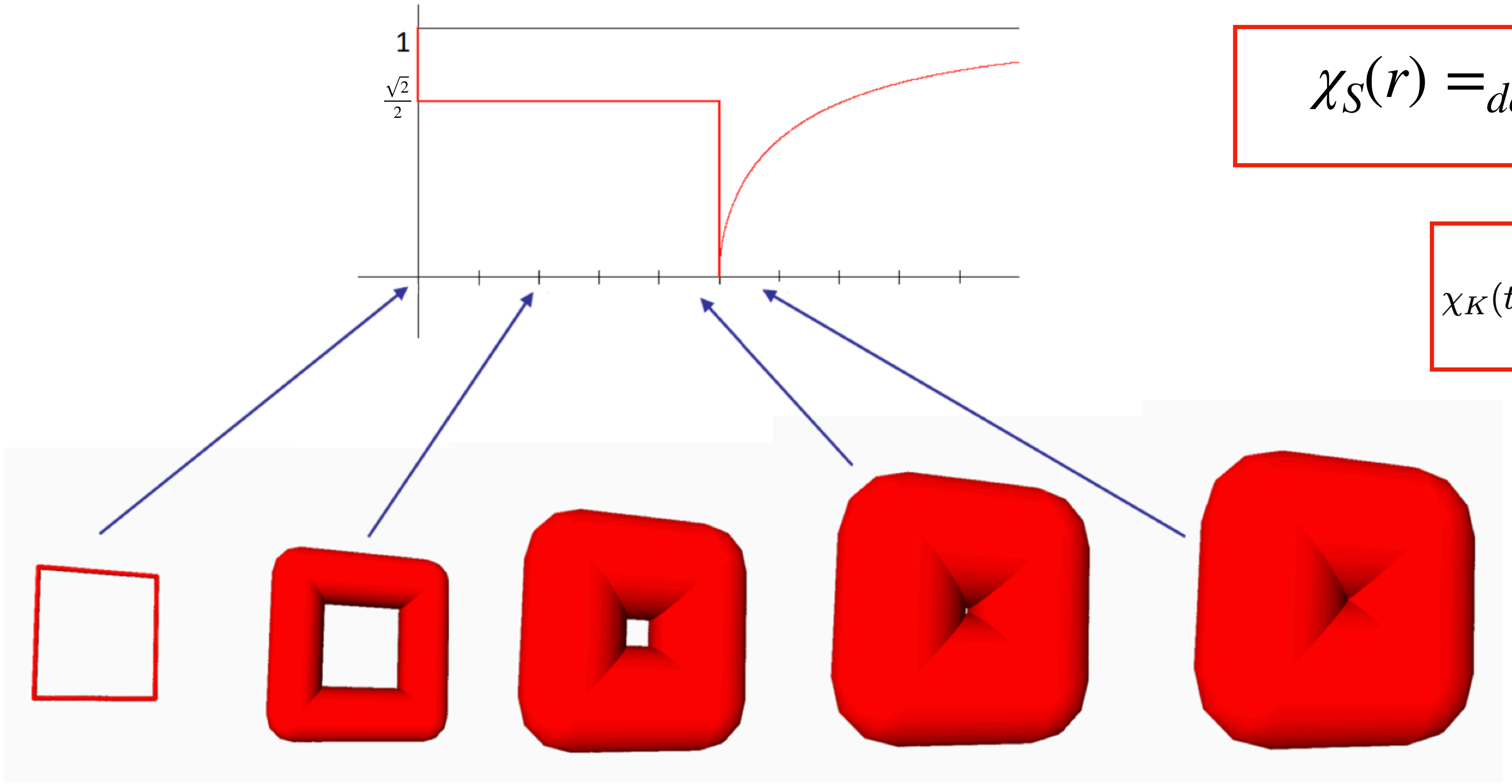

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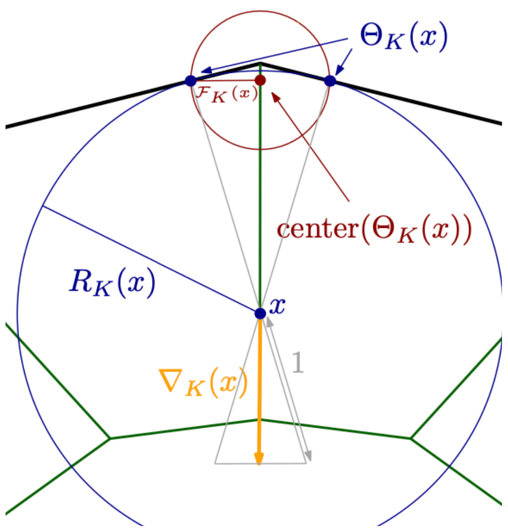
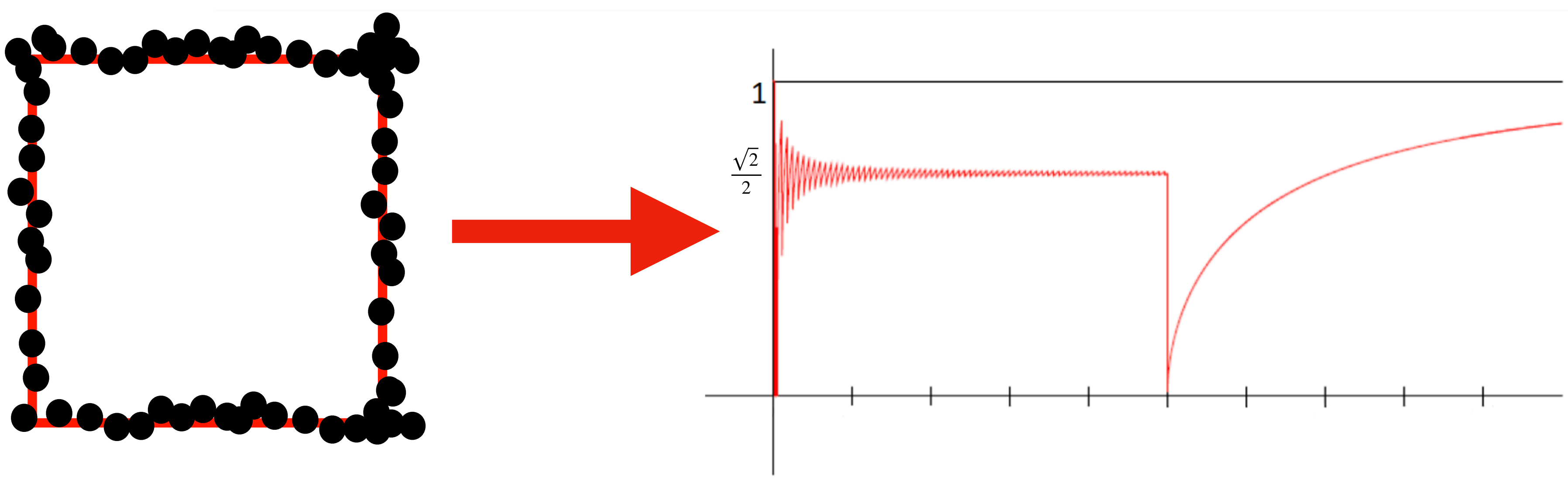




Hausdorff distance

between compact sets

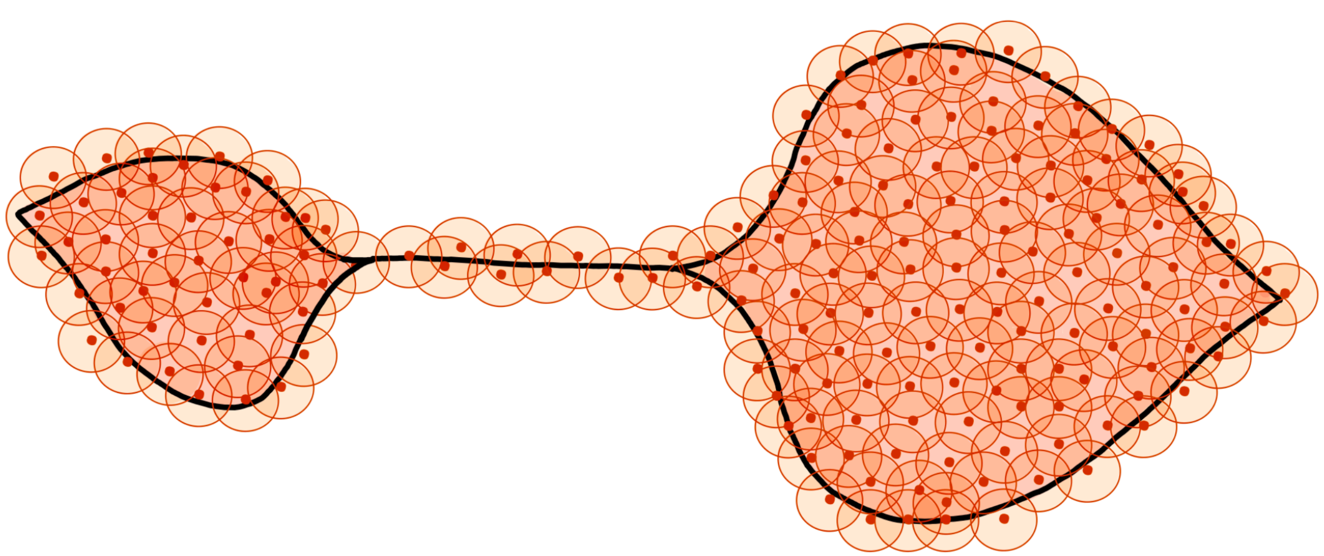
$$d_H(X, Y) := \max \left(\sup_{x \in x} d(x, Y),\ \sup_{y \in Y} d(y, X) \right)$$

Where: $d(x, Y) := \sup_{y \in Y} d(x, y)$

Equivalently:

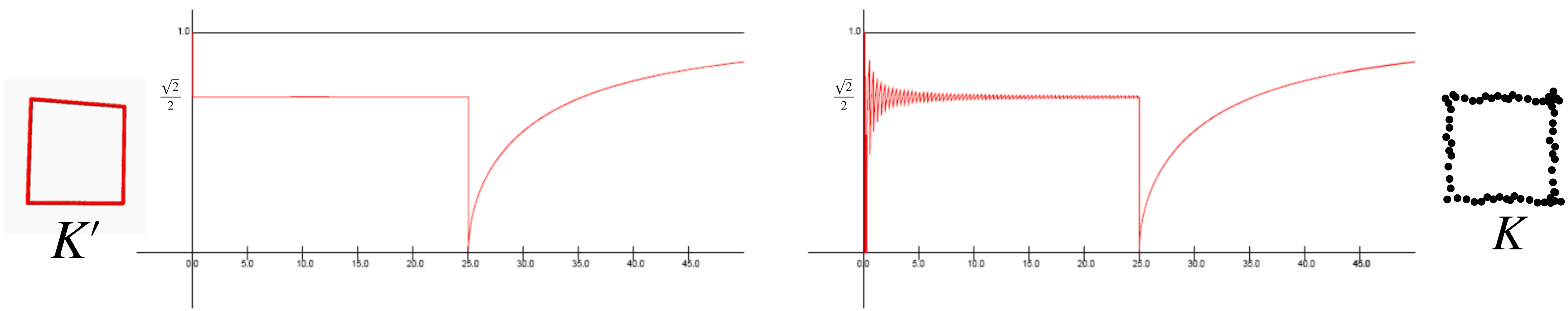
$$d_H(X, Y) = \sup \left\{ \rho \geq 0 \mid X \subset Y^{\oplus \rho} \ \ \textbf{and} \ \ Y \subset X^{\oplus \rho} \right\}$$

$$= \| d(\,.\,, X) - d(\,.\,, Y) \|_\infty = \sup_{z \in \mathbb{R}^d} | d(z, X) - d(z, Y) |$$



critical function

Stability of the critical function



Theorem:[critical function stability theorem CCSL'06] Let K and K' be two compact subsets of $\mathbb{R}^d$ s. t. $d_H(K, K') \leq \varepsilon$. For all $r \geq 0$, we have:

$$\inf\{\chi_{K'}(u) \mid u \in I(r, \varepsilon)\} - 2\sqrt{\frac{\varepsilon}{r}} \leq \chi_K(r)$$

where $I(r, \varepsilon) = [r - \varepsilon, r + 2\chi_K(r)\sqrt{\varepsilon r} + 3\varepsilon]$

critical function

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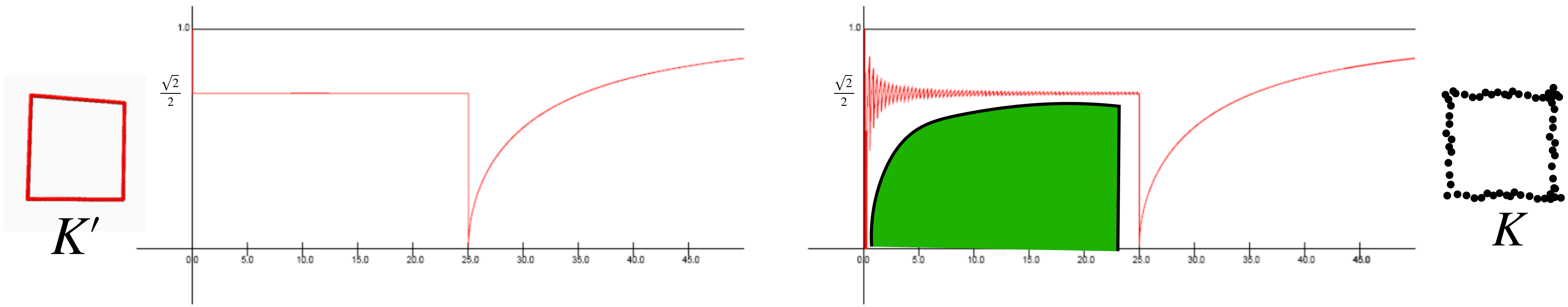

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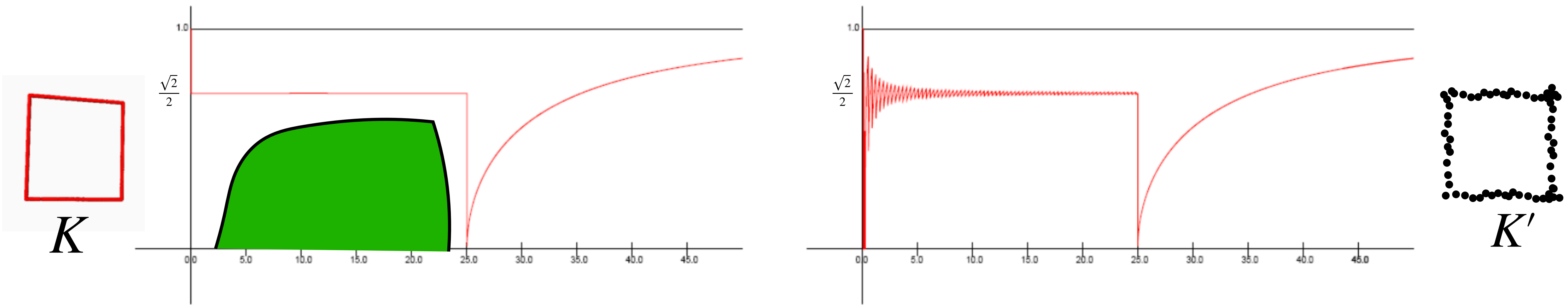

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Homotopy reconstruction of non smooth sets

$$d_H(K', K) < \varepsilon \Rightarrow K^{\oplus\alpha} \subset K'^{\oplus\alpha+\varepsilon} \subset K^{\oplus\alpha+2\varepsilon}$$

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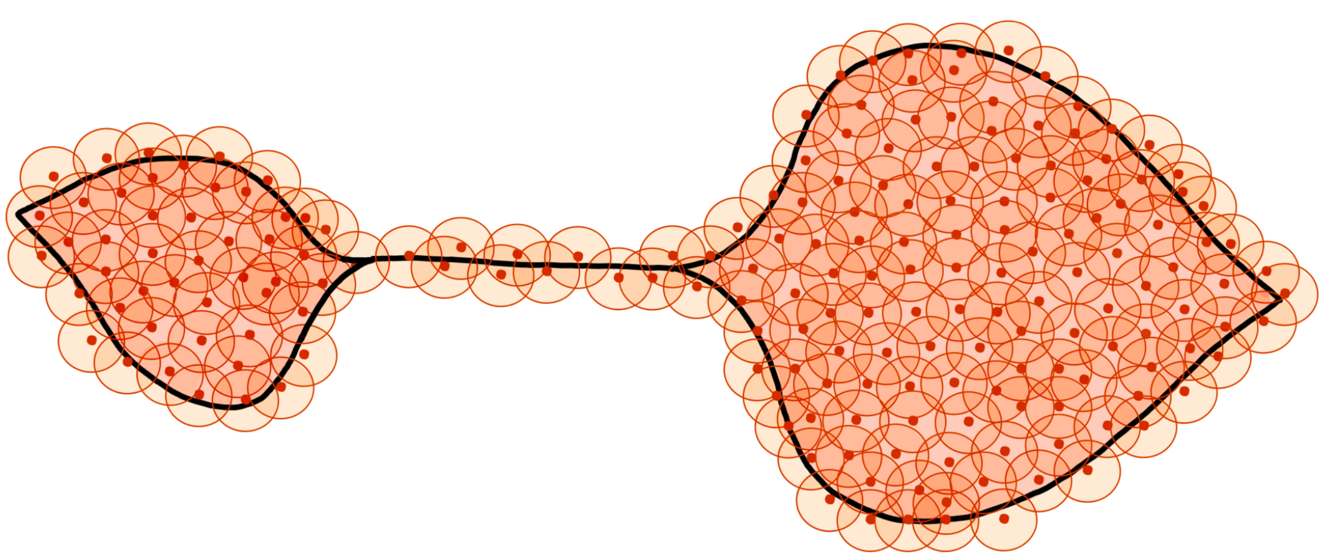

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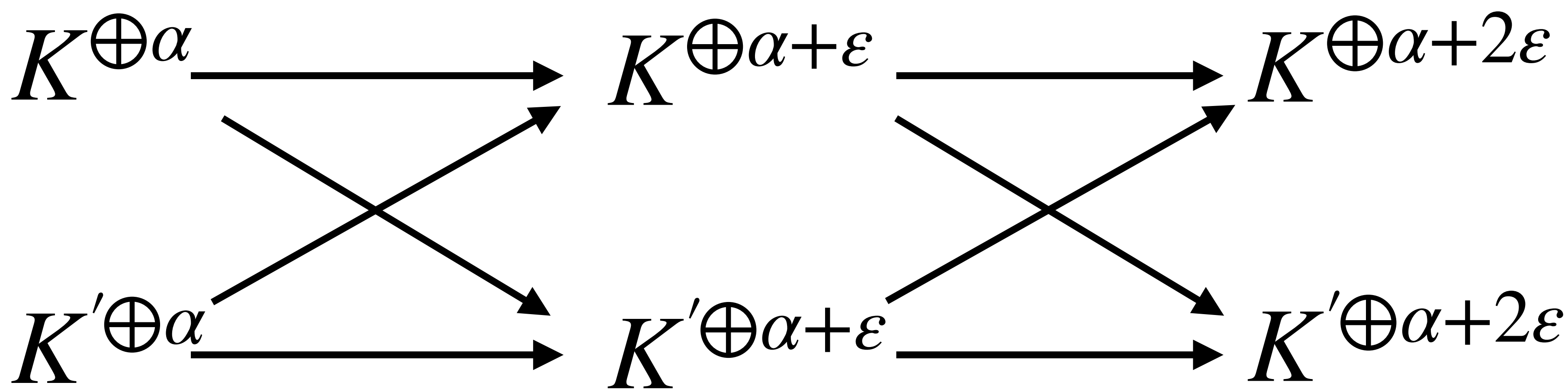

- Inclusion commutes

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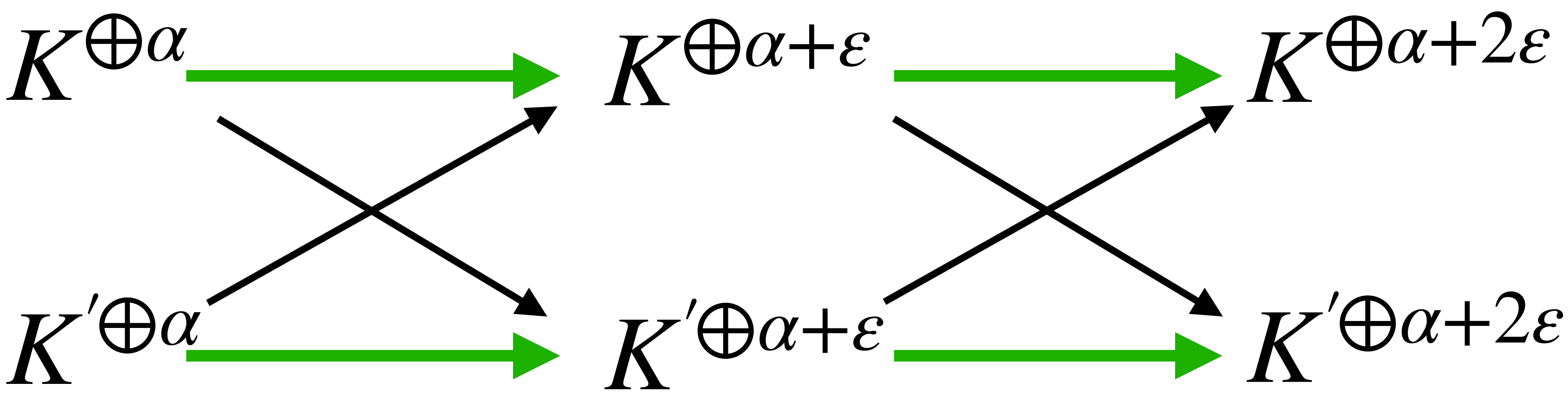
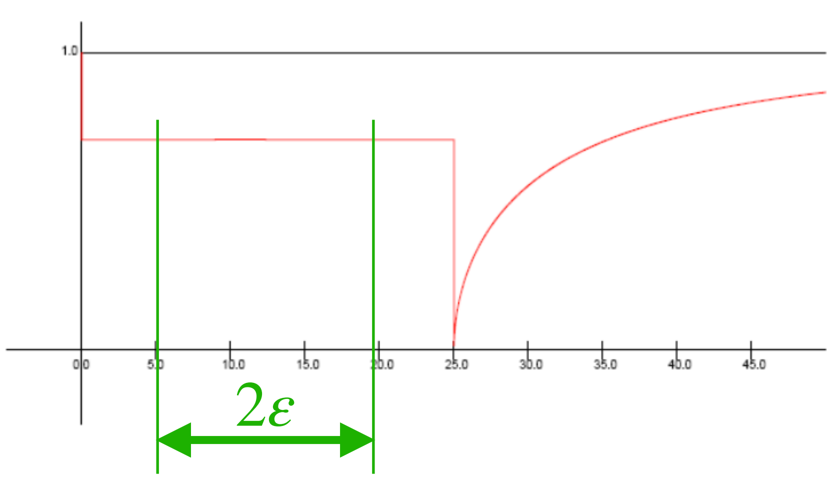
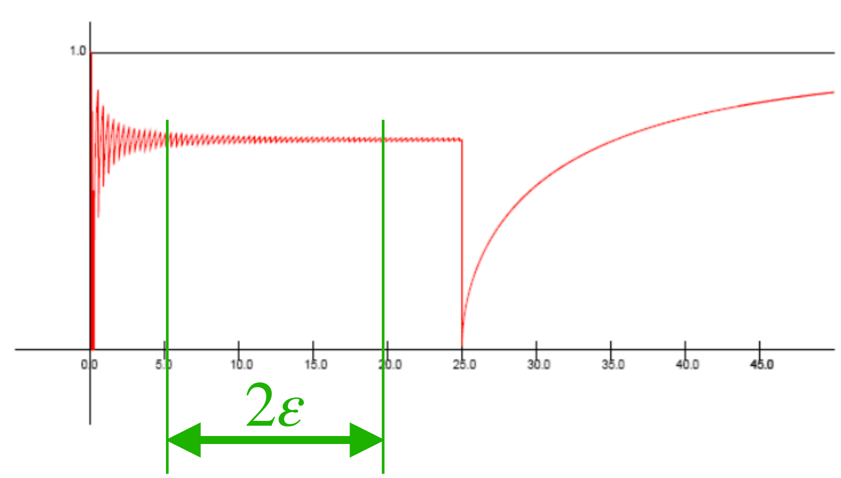


- Inclusion commutes
- Horizontal inclusions are homotopy equivalences





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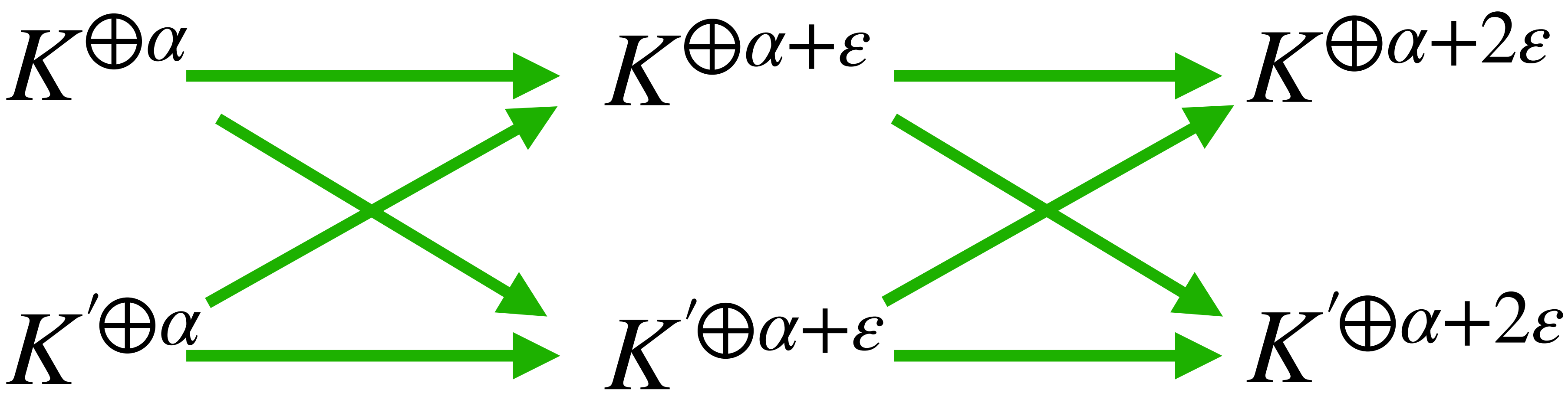
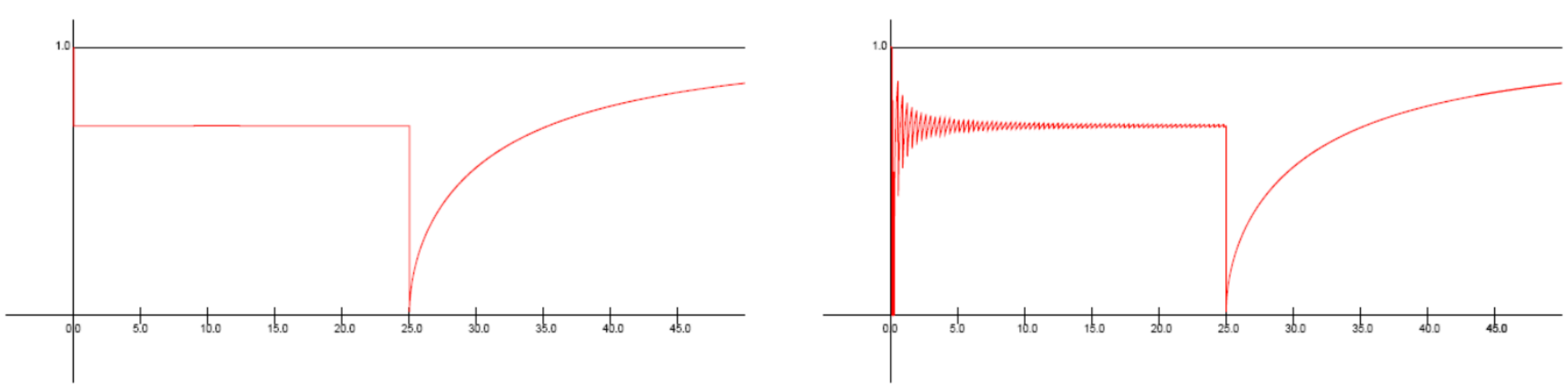


- => all inclusions are homotopy equivalences



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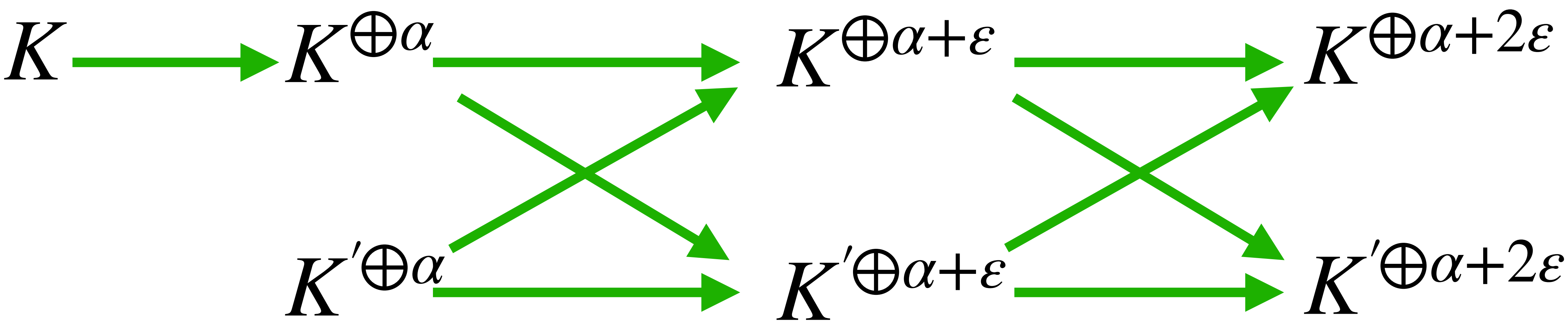

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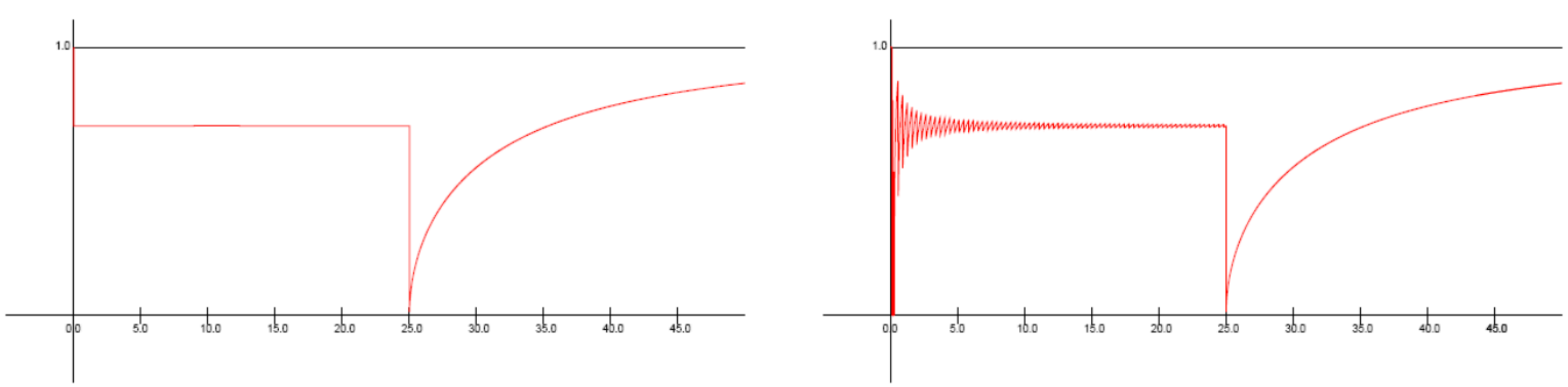
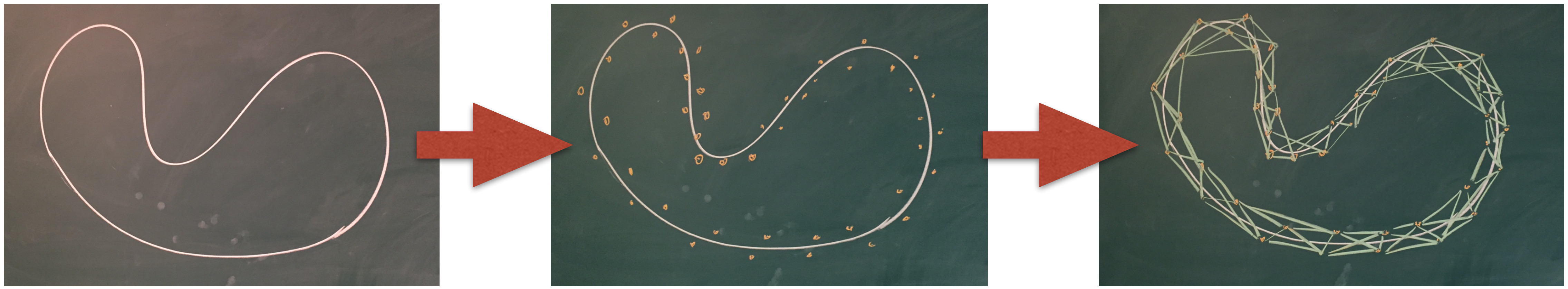


When a simplicial complex over a point sample recovers the homotopy type

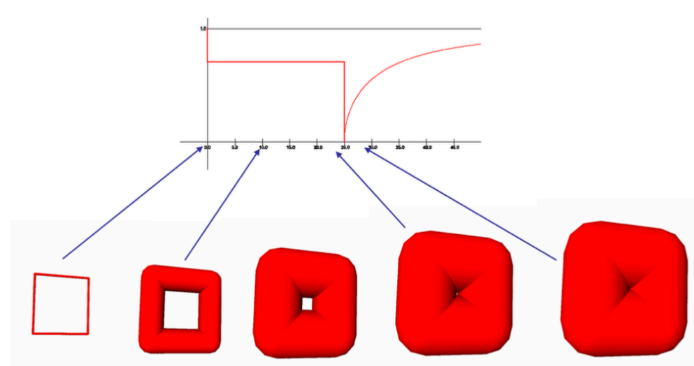


By **quantifying the stability of the critical function** with respect to the change in **Hausdorff distance** we get:

F. Chazal, D. Cohen-Steiner, and A. Lieutier. A sampling theory for compact sets in Euclidean space. *Discete Comput. Geom.*, 41:461–479, 2009.

Cech complex, non-smooth

The critical function of a compact set



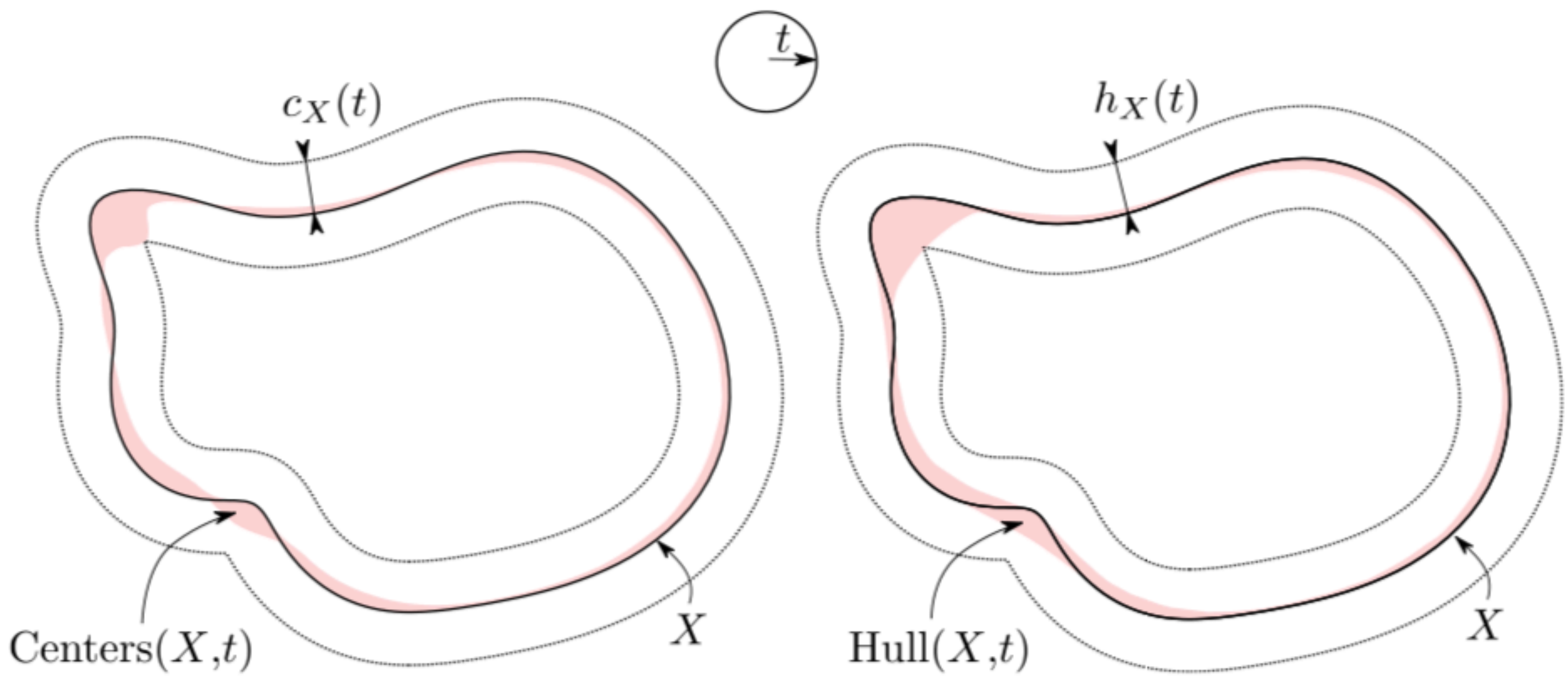
Definition: The critical function $\chi_K : (0, +\infty) \to \mathbb{R}_+$ of a compact set K is the function defined by

$$\chi_K(r) = \inf_{x \in d_K^{-1}(r)} \|\nabla_K(x)\|$$

Beyond the reach

Convexity defect approach

(Attali, L, Salinas, 2011)



$$\mathrm{Centers}(X,t) = \bigcup_{\substack{\emptyset \neq \sigma \subset X \\ \mathrm{Rad}(\sigma) \leq t}} \{\mathrm{Center}(\sigma)\}.$$

$$\mathrm{Hull}(X,t) = \bigcup_{\substack{\emptyset \neq \sigma \subset X \\ \mathrm{Rad}(\sigma) \leq t}} \mathrm{Hull}(\sigma).$$

$$c_X(t) = d_H(Centers(X,t), X)$$

$$h_X(t) = d_H(Hull(X,t), X)$$

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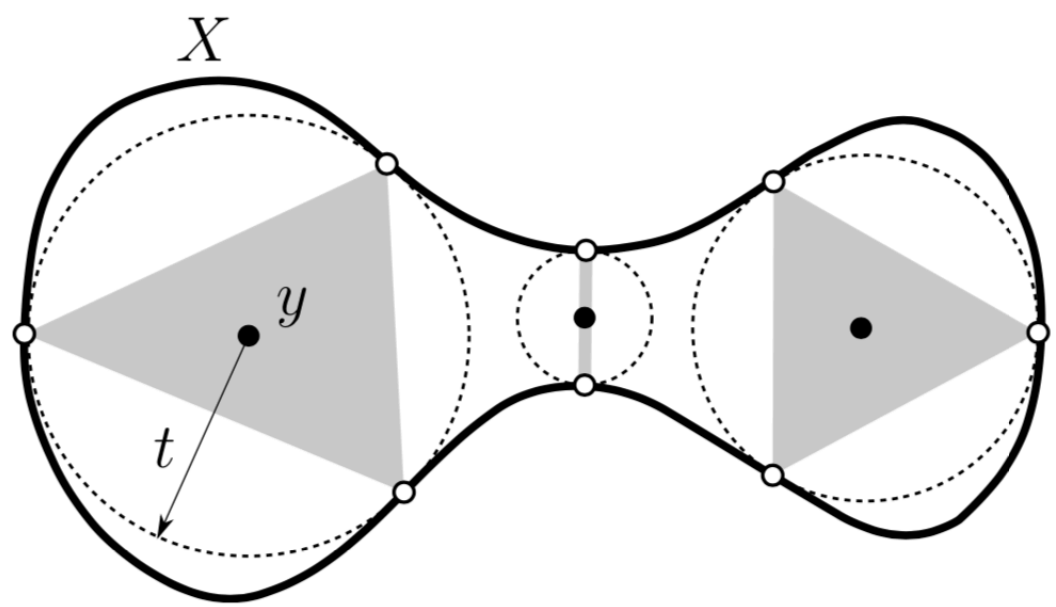
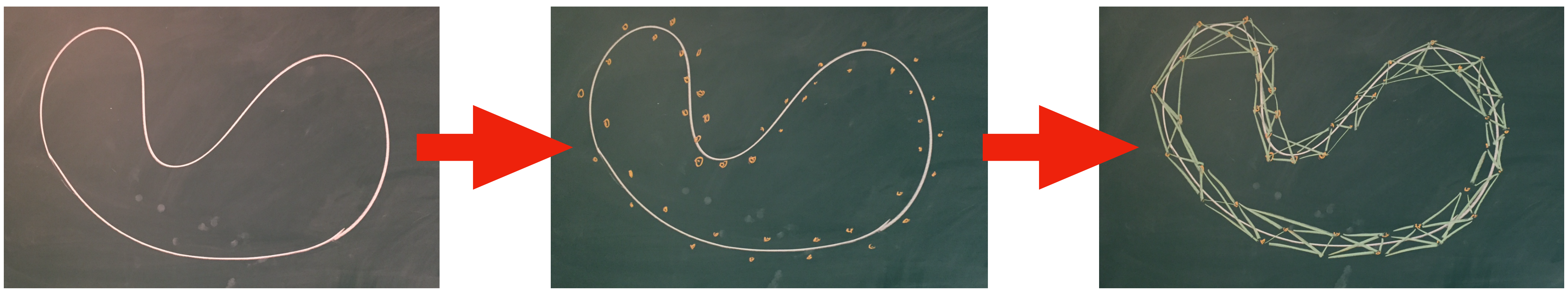


$$c_X(t) = d_H(Centers(X,t), X)$$

$$h_X(t) = d_H(Hull(X,t), X)$$

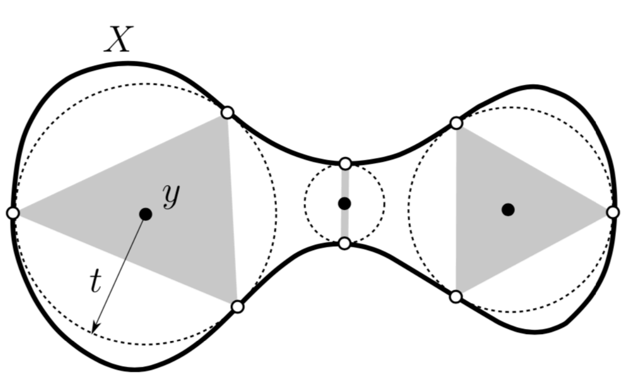
Lemma 2. *For any compact set $X \subset \mathbb{R}^n$ and any real number $t > 0$, the following three conditions are equivalent: (1) t is a critical value of $d(\cdot, X)$; (2) $c_X(t) = t$; (3) $h_X(t) = t$.*

When a simplicial complex over a point sample recovers the homotopy type



By **quantifying the stability of the convexity defect** with respect to the change in **Hausdorff distance** we get:

D. Attali, A. Lieutier, and D. Salinas. Vietorisrips complexes also provide topologically correct reconstructions of sampled shapes. *Comput. Geom.*, 46:448–465, 2013.



For **Rips** Complex: a kind of **geometric** and **effective** (i.e. quantified) version of a result by **J. Latschev** (2001)

When a simplicial complex over a point sample recovers the homotopy type

P. Niyogi, S. Smale, and S. Weinberger. Finding the homology of submanifolds with high confidence from random samples. *Discrete Comput. Geom.*, 39:419–441, 2008.

Dominique Attali, Hana Dal Poz Kouřimská, Christopher Fillmore, Ishika Ghosh, André Lieutier, Elizabeth Stephenson, and Mathijs Wintraecken. Optimal homotopy reconstruction results\a la niyogi, smale, and weinberger. *arXiv preprint arXiv:2206.10485*, 2022. **(optimal)**

reach

Critical function & μ-reach

F. Chazal, D. Cohen-Steiner, and A. Lieutier. A sampling theory for compact sets in Euclidean space. *Discete Comput. Geom.*, 41:461–479, 2009.

D. Attali, A. Lieutier, and D. Salinas. Vietorisrips complexes also provide topologically correct reconstructions of sampled shapes. *Comput. Geom.*, 46:448–465, 2013.

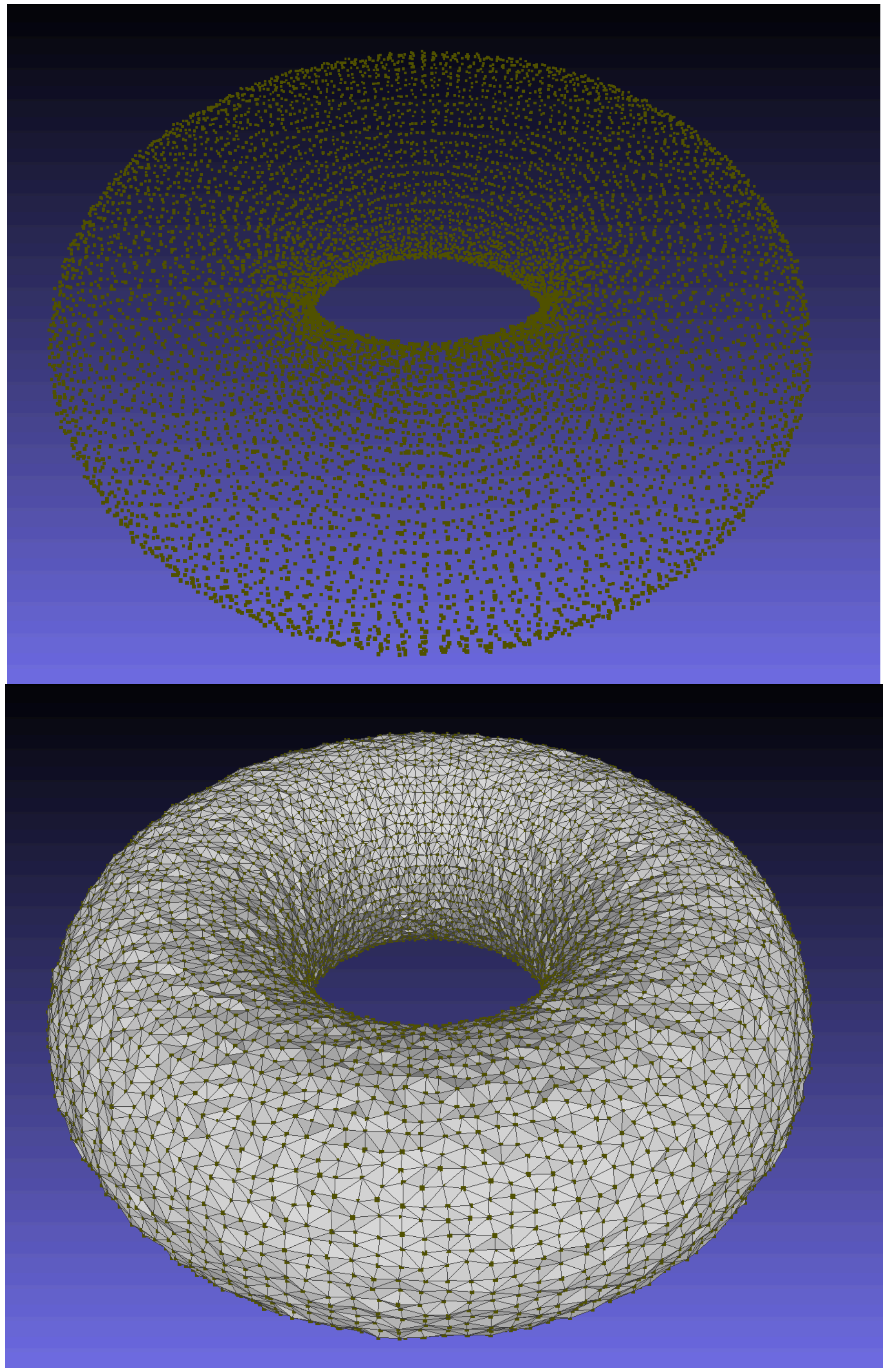
(best known constant for Vietoris-Rips complexes)

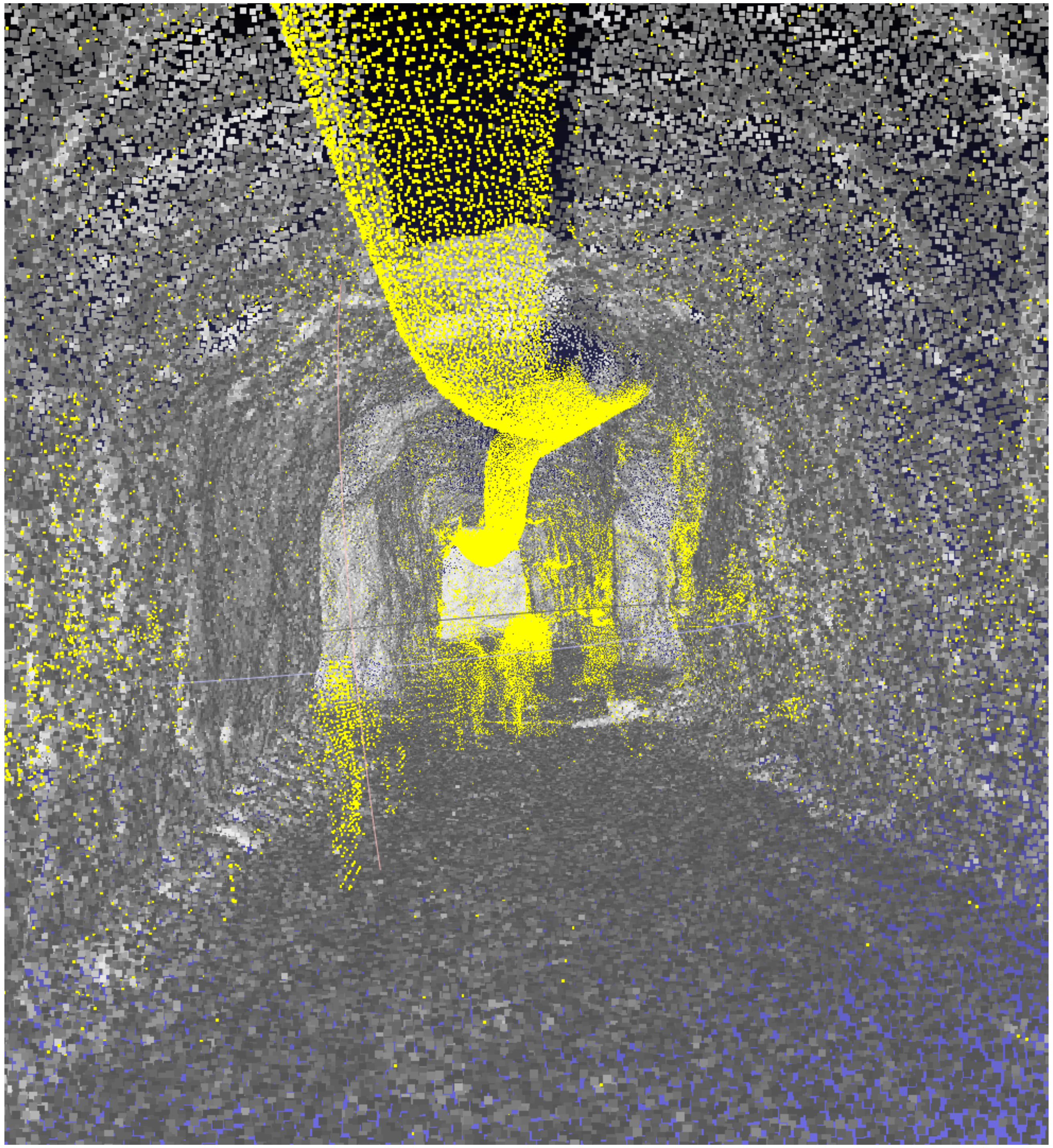
Convexity defects

Jisu Kim, Jaehyeok Shin, Frédéric Chazal, Alessandro Rinaldo, and Larry Wasserman. Homotopy Reconstruction via the Cech Complex and the Vietoris-Rips Complex. *(SoCG 2020)*.

(best known constant for Cech complexes)

Part 2:
Triangulation by minimal chains



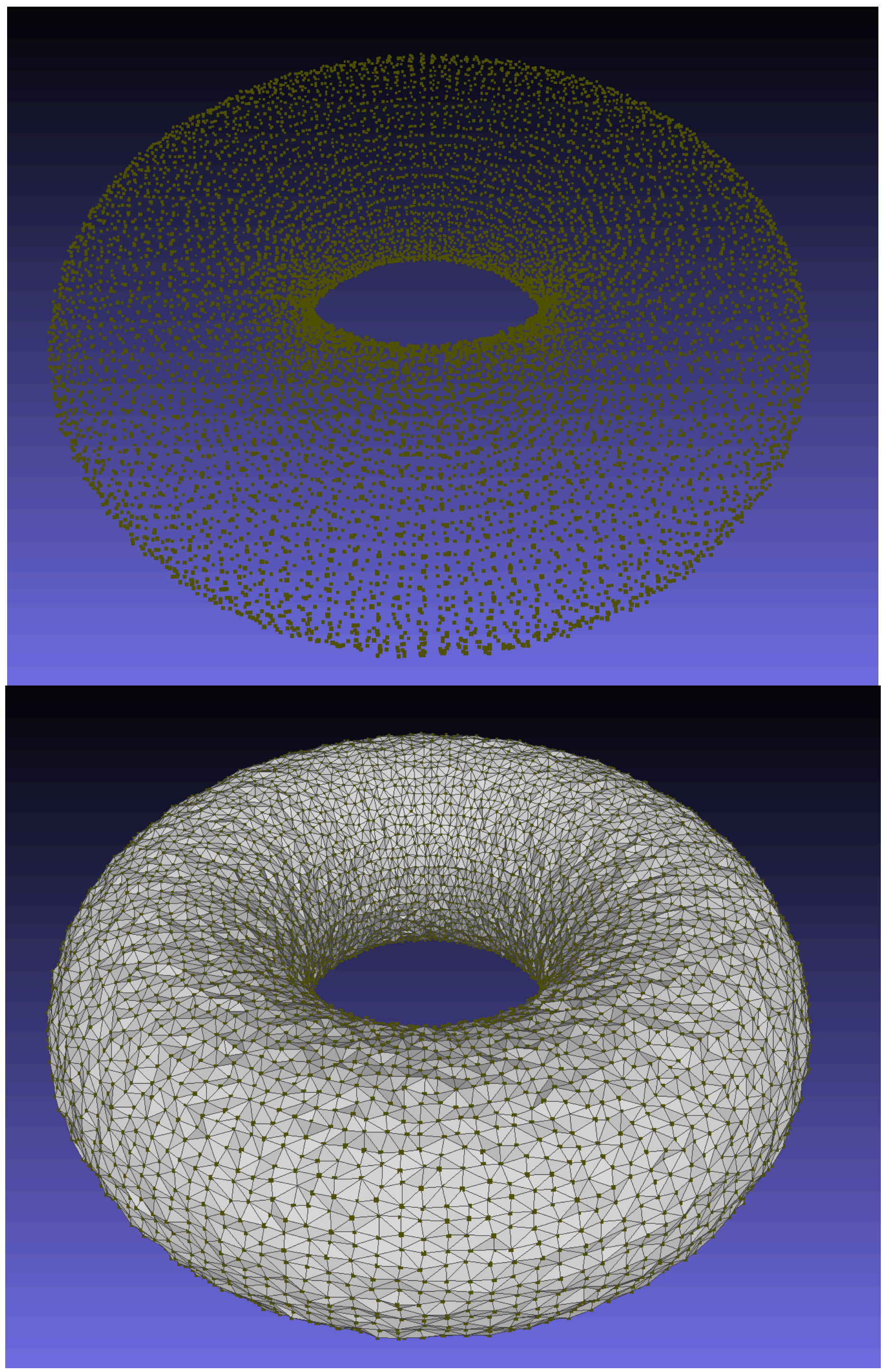


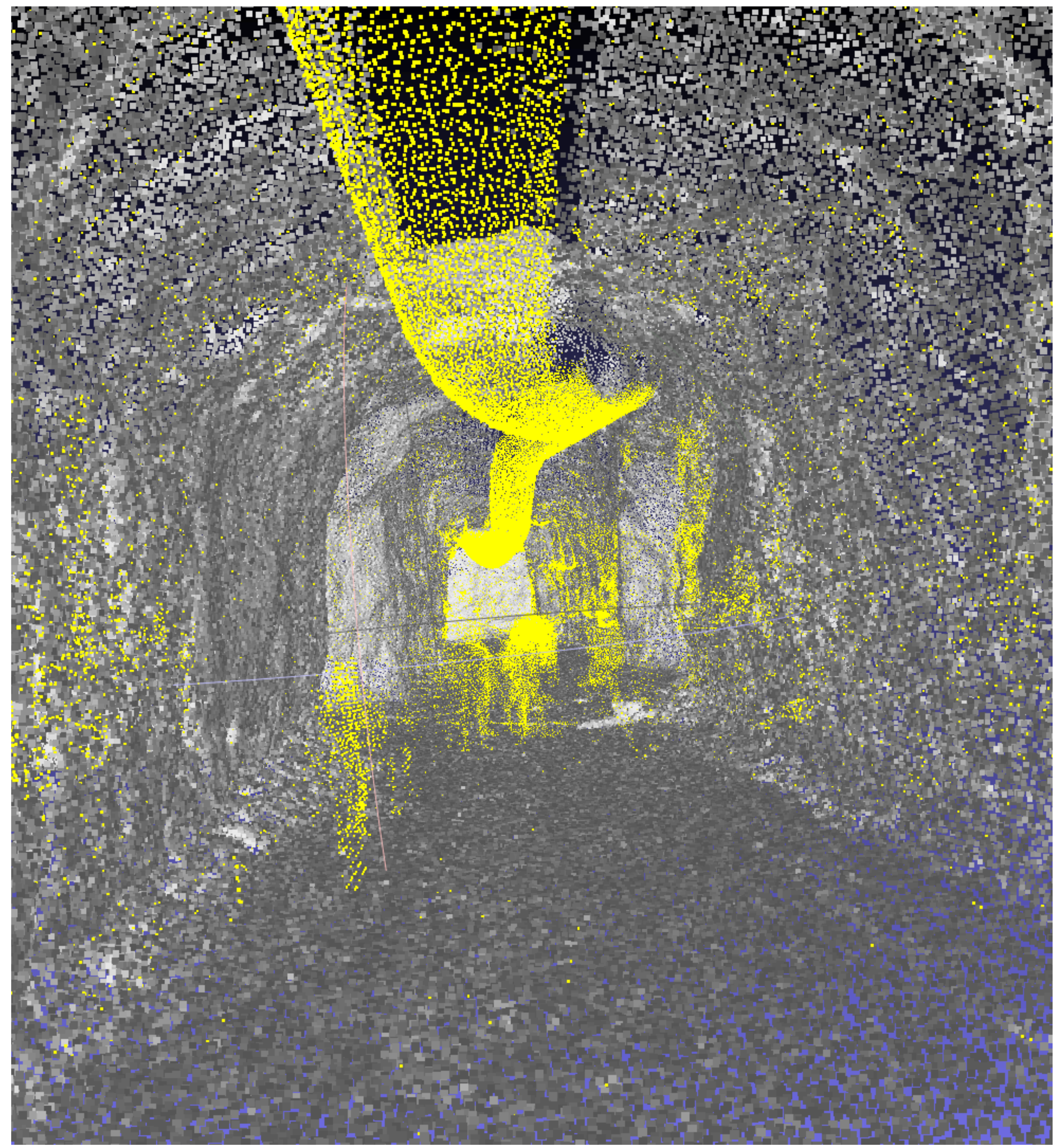
Part 2:
Triangulation by minimal chains

Part 1 has focused on the computation of a simplicial complex which reproduce the **homotopy type**.

In **part 2** we consider the computation of **homeomorphic** simplicial complexes, in other words **Triangulations**

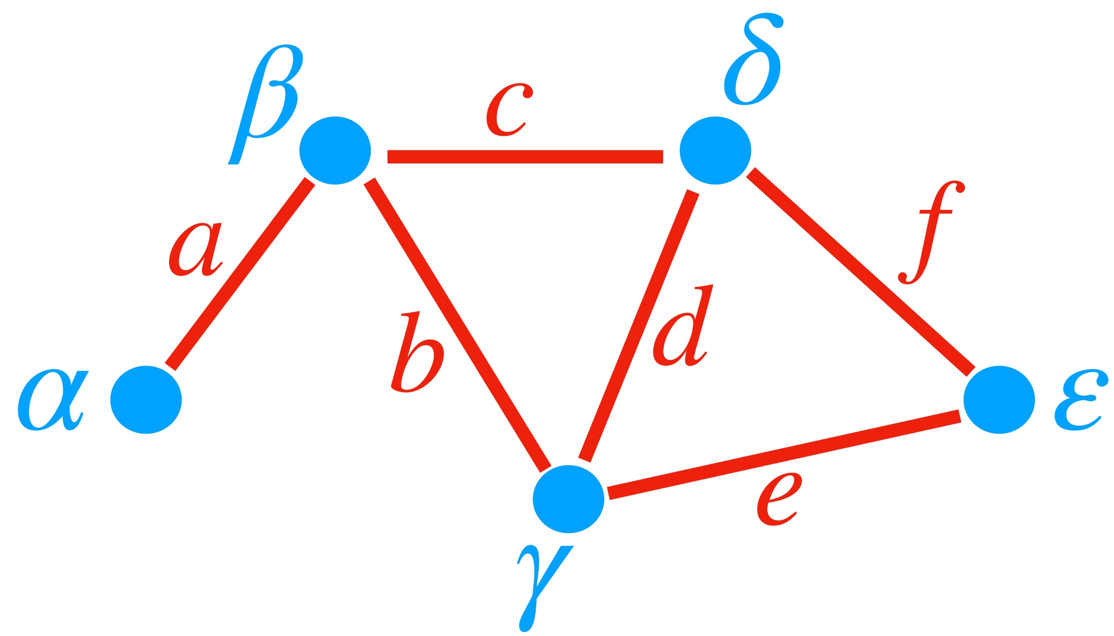
Part 2:
Triangulation by minimal chains





Minimal homology notions:

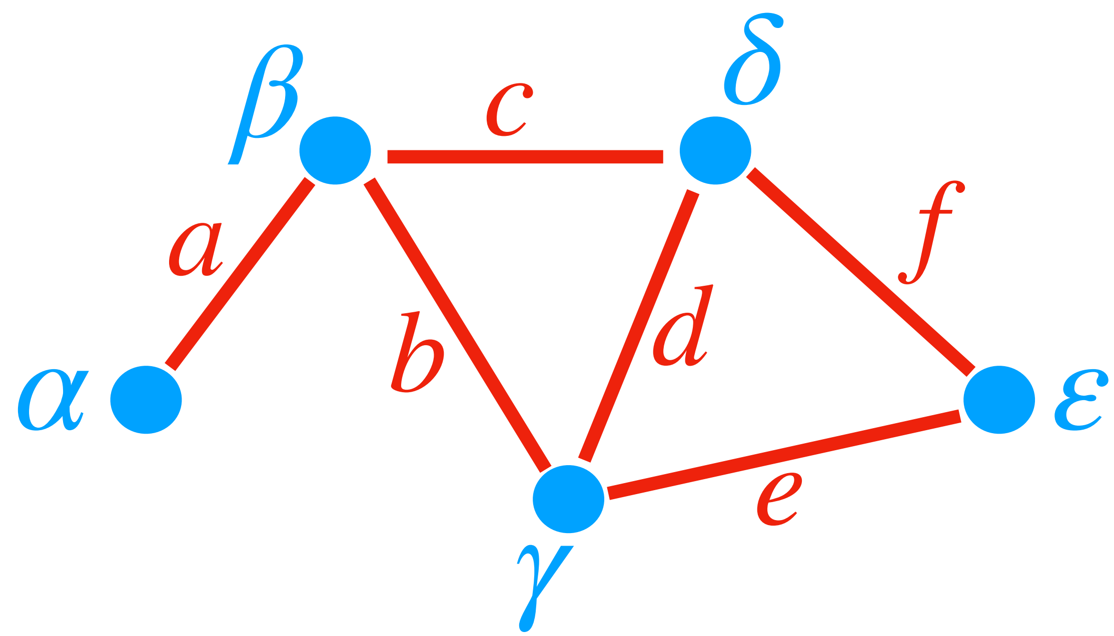
Algebraic formulation of topological properties



Is there a path between α and ε ?

Minimal homology notions:

Algebraic formulation of topological properties

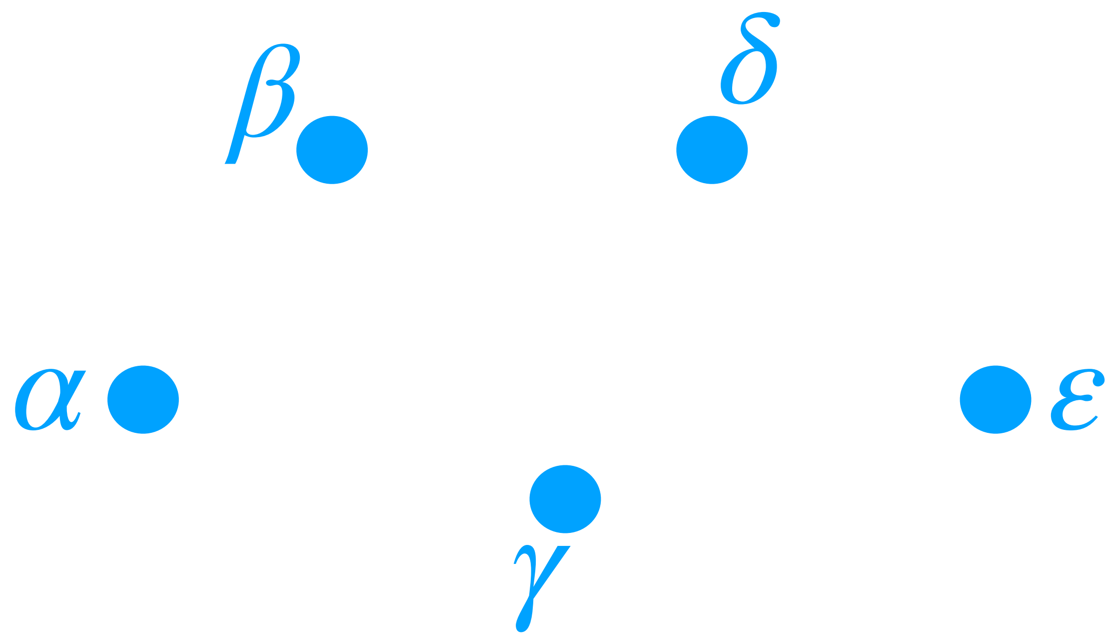


Is there a path between α and ε ?

A (linear) **algebra** formulation of this question ?

Minimal homology notions:

Algebraic formulation of topological properties

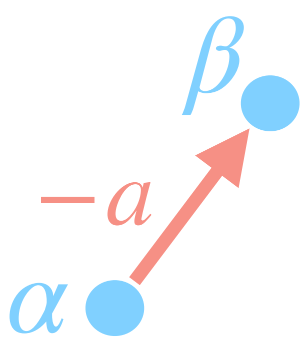


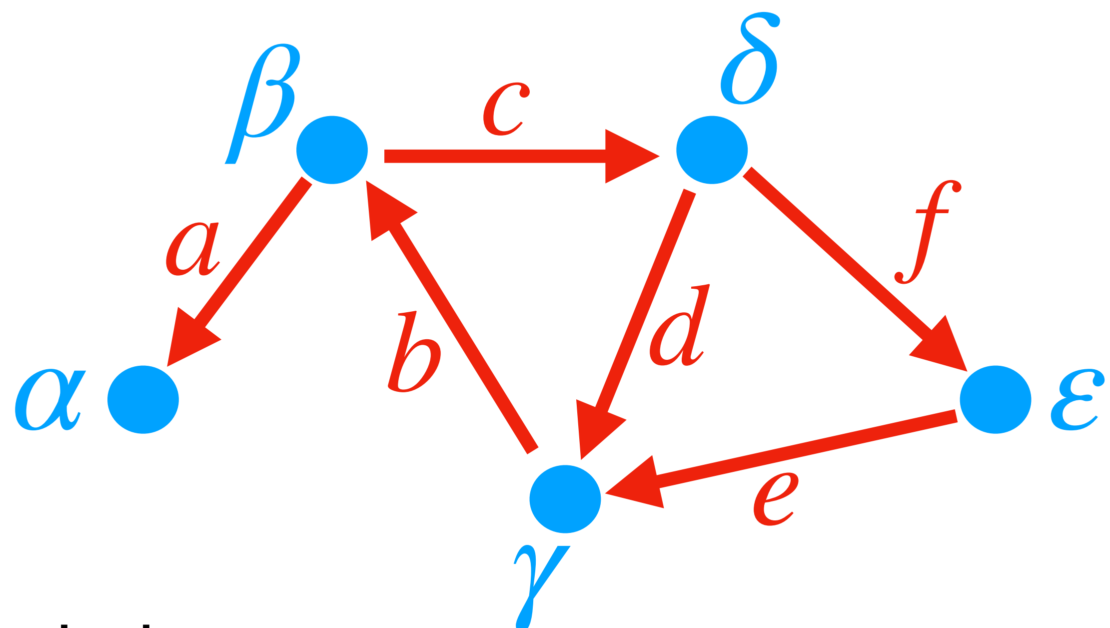
Vector space of 0-chains:

$$C_0 = \left\{ Y_\alpha \alpha + Y_\beta \beta + Y_\gamma \gamma + Y_\delta \delta + Y_\varepsilon \varepsilon \mid Y \in \mathbb{R}^5 \right\}$$

Minimal homology notions:

Algebraic formulation of topological properties





Vector space of 0-chains:

$$C_0 = \left\{ Y_\alpha \alpha + Y_\beta \beta + Y_\gamma \gamma + Y_\delta \delta + Y_\varepsilon \varepsilon \mid Y \in \mathbb{R}^5 \right\}$$

(basis = 0-simplices)

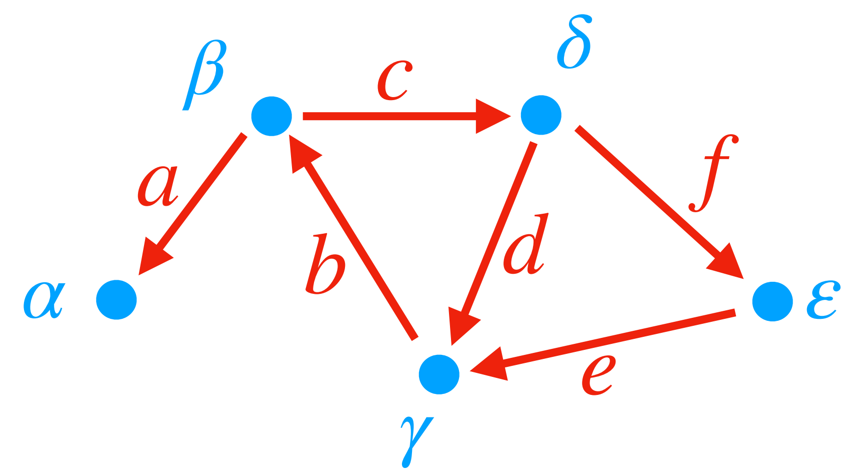
Vector space of 1-chains:

$$C_1 = \left\{ X_a a + X_b b + X_c c + X_d d + X_e e + X_f f \mid X \in \mathbb{R}^6 \right\}$$

(basis = ``oriented” 1-simplices)

Minimal homology notions:

Algebraic formulation of topological properties



$$C_0 = \left\{ Y_\alpha \alpha + Y_\beta \beta + Y_\gamma \gamma + Y_\delta \delta + Y_\varepsilon \varepsilon \mid Y \in \mathbb{R}^5 \right\}$$

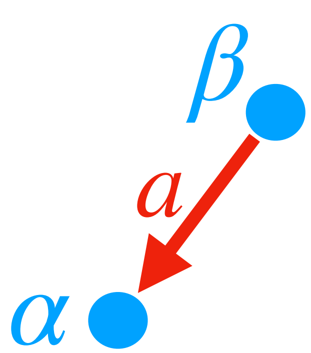
(basis = 0-simplices)

$$C_1 = \left\{ X_a a + X_b b + X_c c + X_d d + X_e e + X_f f \mid X \in \mathbb{R}^6 \right\}$$

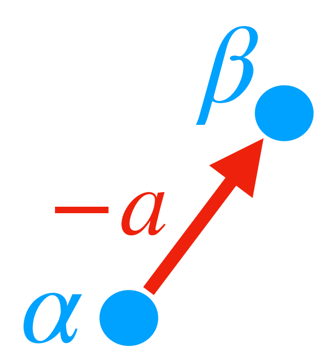
(basis = ``oriented” 1-simplices)

Boundary linear operator:

$$\boxed{\partial : C_1 \rightarrow C_0}$$



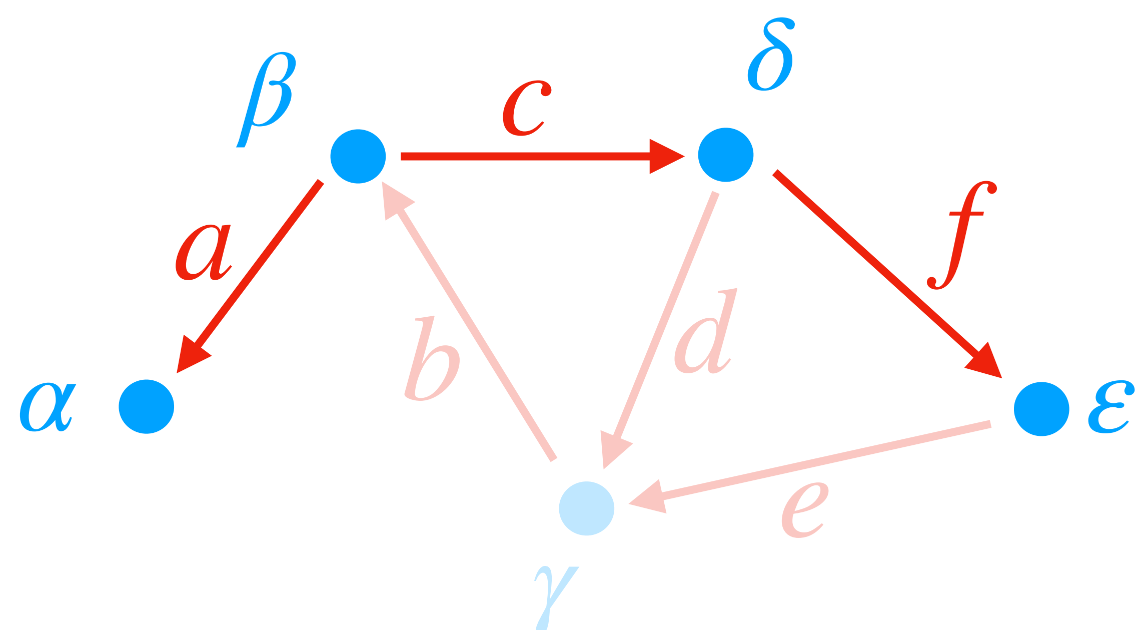
$$\partial a = \alpha - \beta$$



$$\partial(-a) = -\partial a = \beta - \alpha$$

Minimal homology notions:
Algebraic formulation of topological properties

$$\partial : C_1 \to C_0$$



$$\partial(-a + c + f) = (\beta - \alpha) + (\delta - \beta) + (\varepsilon - \delta)$$
$$= \varepsilon - \alpha$$

Is there a path between α and ε ?

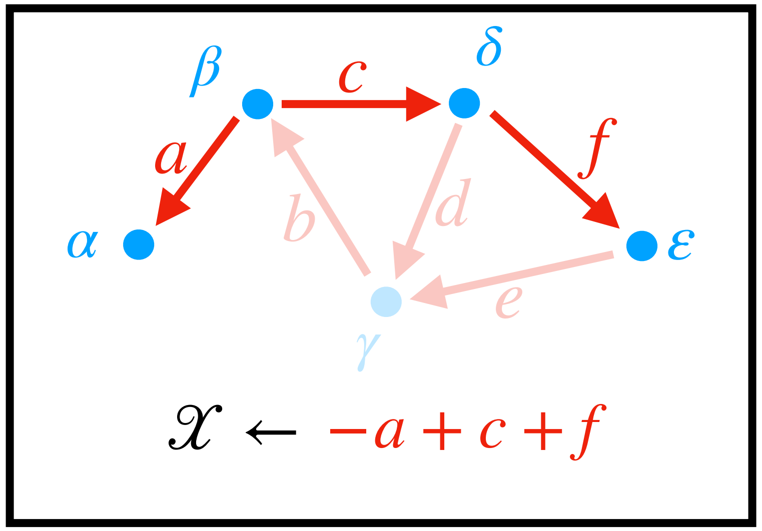
Yes: $-a + c + f$

Minimal homology notions:

Algebraic formulation of topological properties

There a path between α and ε

$$\iff \exists \mathscr{X} \in C_1 \mid \partial \mathscr{X} = \varepsilon - \alpha$$



Minimal homology notions:
Algebraic formulation of topological properties

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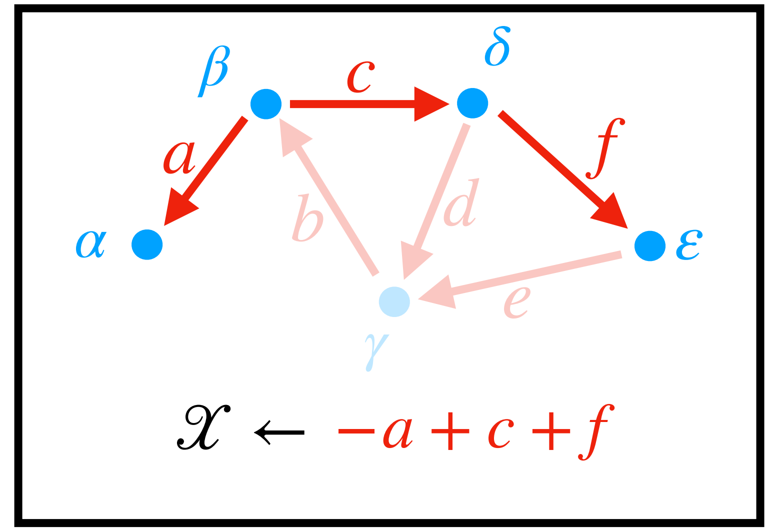
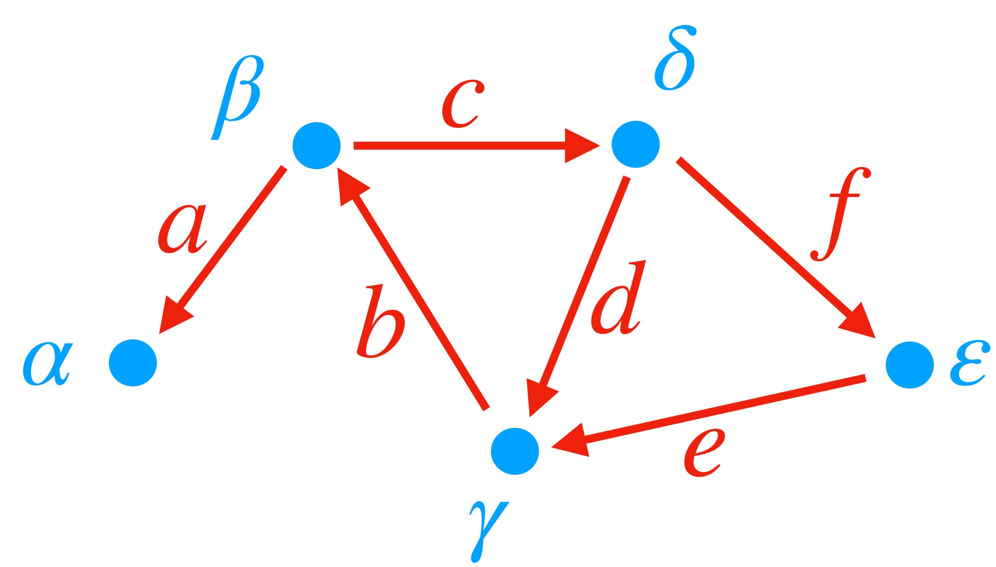


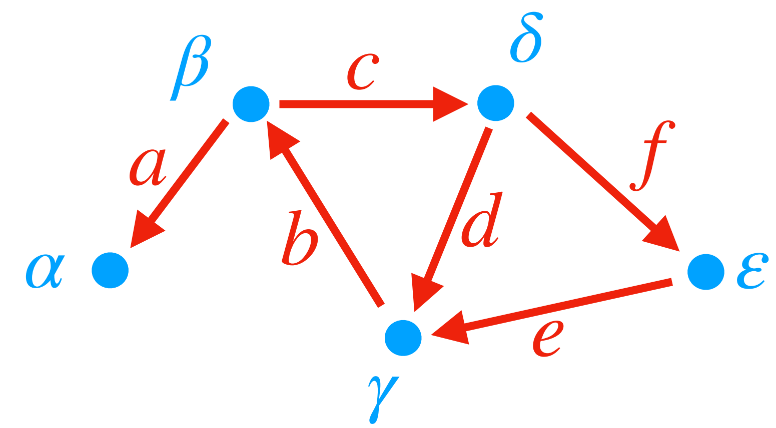
$\partial =$

	a	b	c	d	e	f
α	1	0	0	0	0	0
β	-1	1	-1	0	0	0
γ	0	-1	0	1	0	0
δ	0	0	1	-1	-1	-1
ε	0	0	0	0	1	1



Minimal homology notions:

Algebraic formulation of topological properties



There a path between α and ε

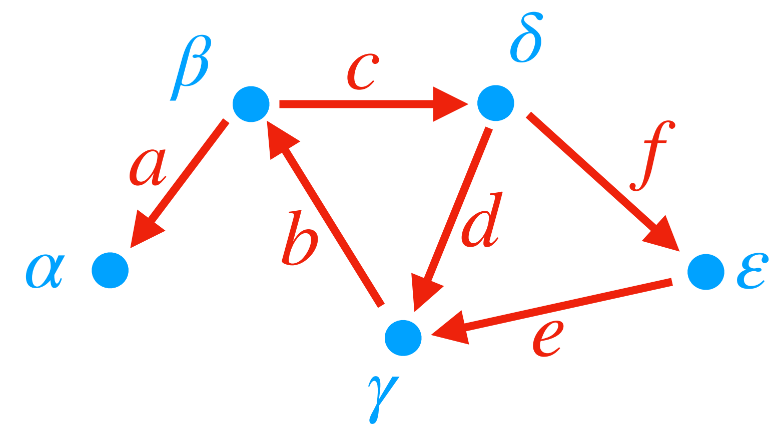
$$\iff \exists \mathscr{X} \in C_1 \mid \partial \mathscr{X} = \varepsilon - \alpha$$

$$\iff \exists \mathscr{X} \in C_1 \mid \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_a \\ X_b \\ X_c \\ X_d \\ X_e \\ X_f \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Minimal homology notions:

Algebraic formulation of topological properties

There a path between α and ε



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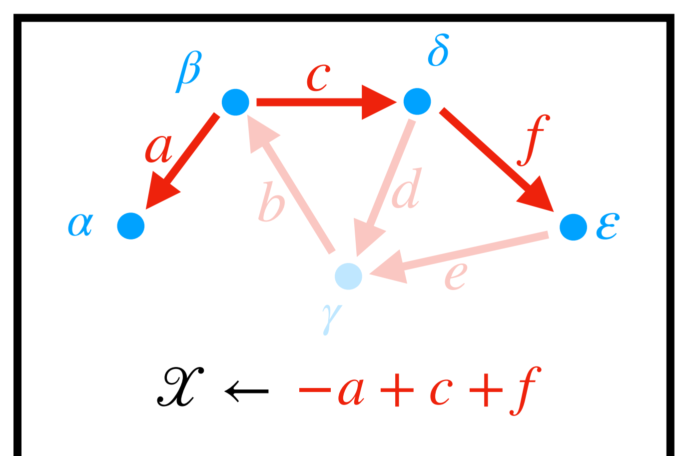
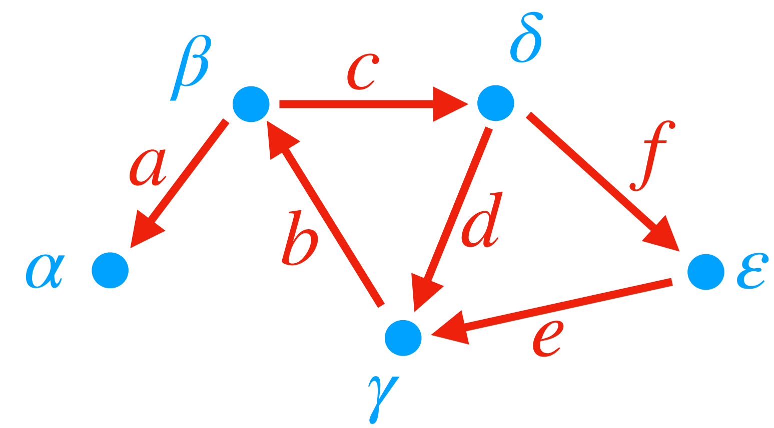


Minimal homology notions:

Algebraic formulation of topological properties

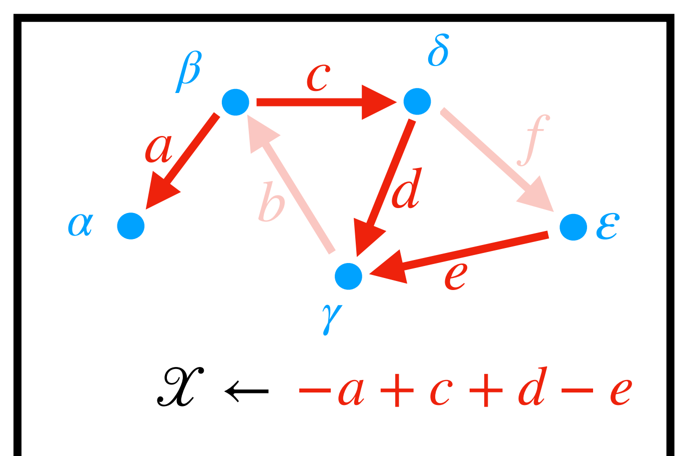
There a path between α and ε



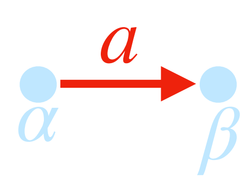
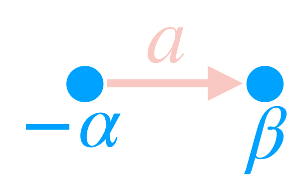
$$\iff \exists \mathscr{X} \in C_1 \mid \partial\mathscr{X} = \varepsilon - \alpha$$

$$\iff \exists \mathscr{X} \in C_1 \mid \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_a \\ X_b \\ X_c \\ X_d \\ X_e \\ X_f \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

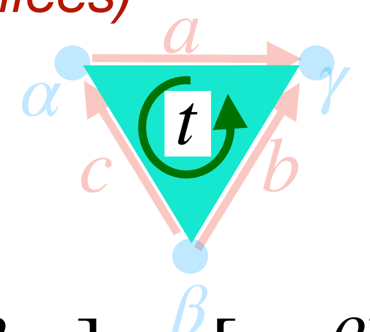
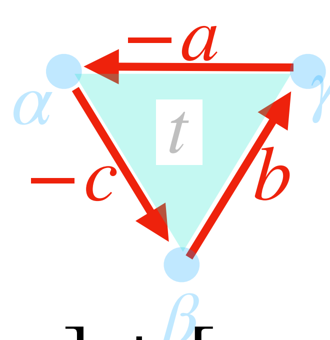


Minimal homology notions:
Algebraic formulation of topological properties

$\partial_1 : C_1 \to C_0 : \partial_1($  $) =$ 

$$\partial_1 a = \partial_1[\alpha, \beta] = \beta - \alpha$$

(C_2 : basis = ``oriented" 2-simplices)

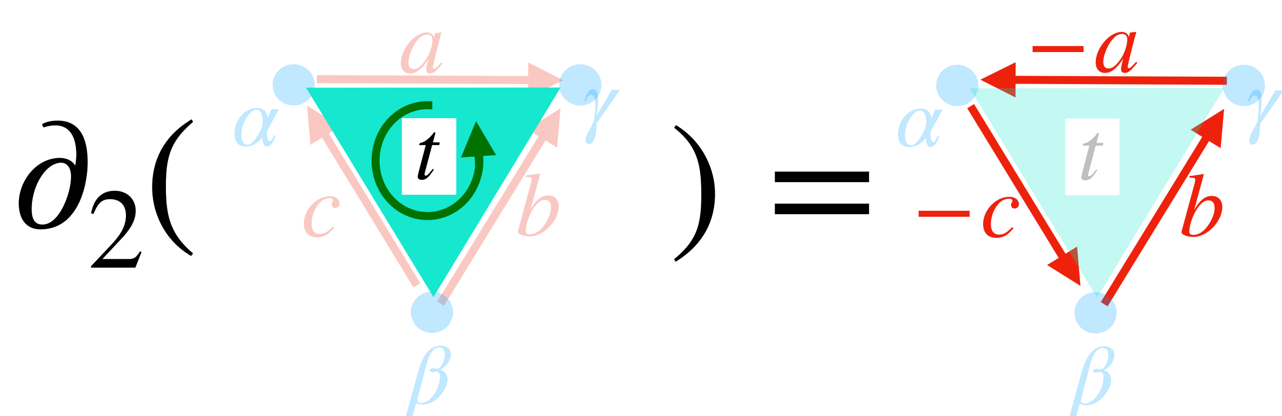
$\partial_2 : C_2 \to C_1 : \partial_2($  $) =$ 

$$\partial_2 t = \partial_2[\alpha, \beta, \gamma] = [\alpha, \beta] + [\beta, \gamma] + [\gamma, \alpha] = -c + b - a$$

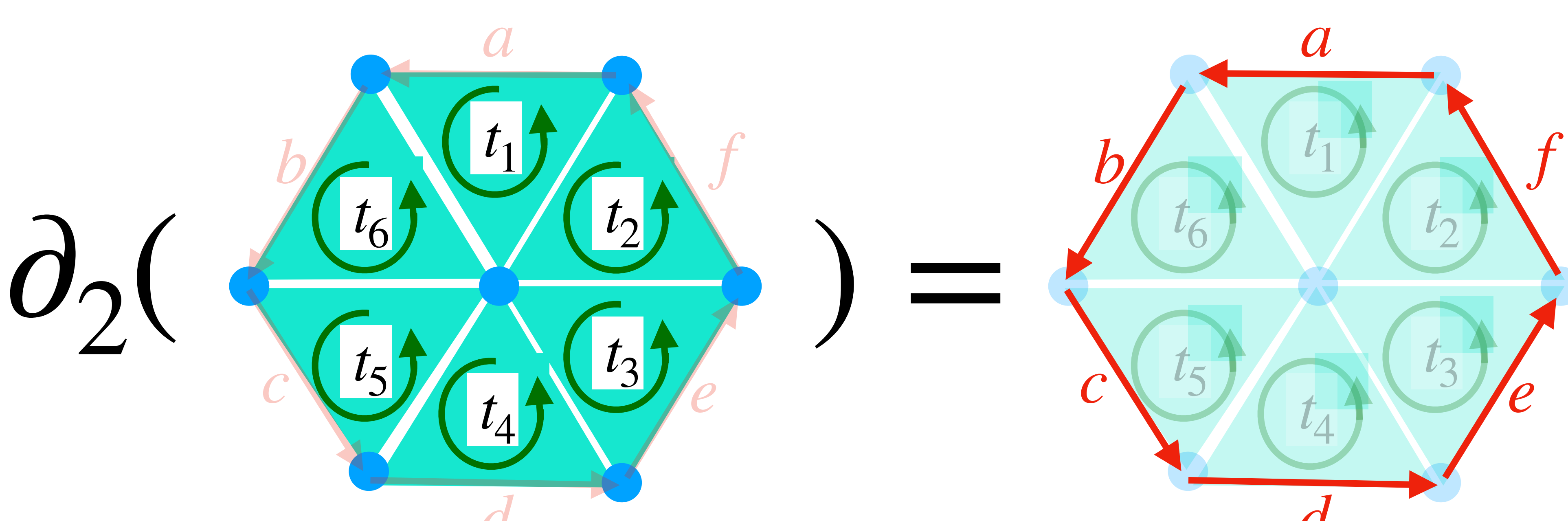
$$\boxed{\partial_1 \circ \partial_2 = 0}$$

Minimal homology notions:

Algebraic formulation of topological properties



$$\partial_2\, t = -\,c + b - a$$



$$\boxed{\partial_1 \circ \partial_2 = 0}$$

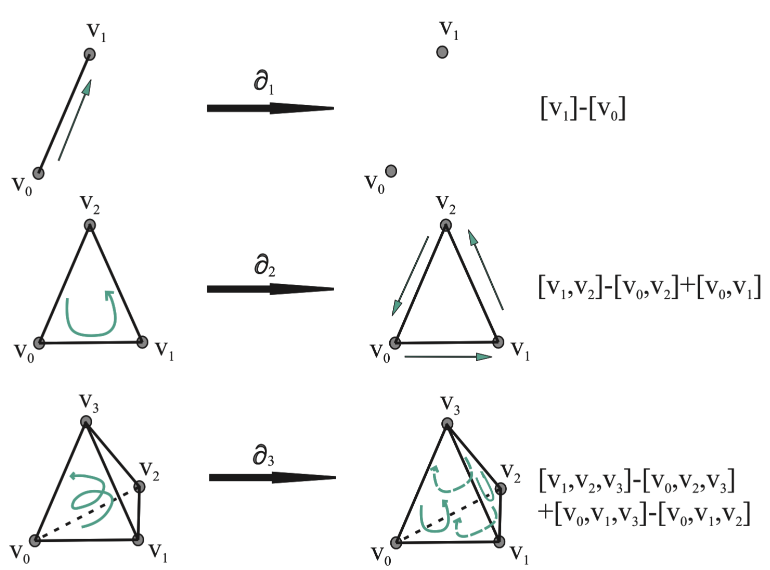
$$\partial_2\,(t_1 + t_2 + t_3 + t_4 + t_5 + t_6) = a + b + c + d + e + f$$

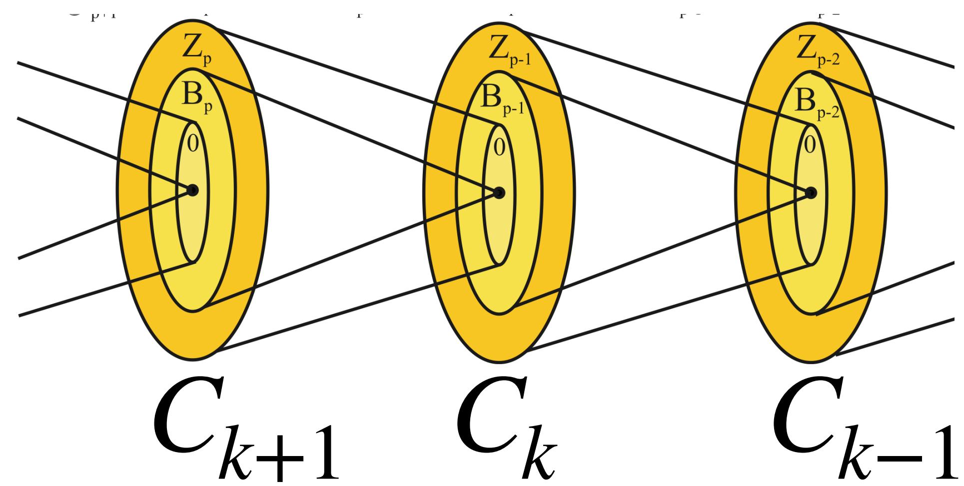
Minimal homology notions:

Simplicial homology in a single slide !

$$\partial_k \circ \partial_{k+1} = 0 \iff \operatorname{Im} \partial_{k+1} \subset \ker \partial_k$$

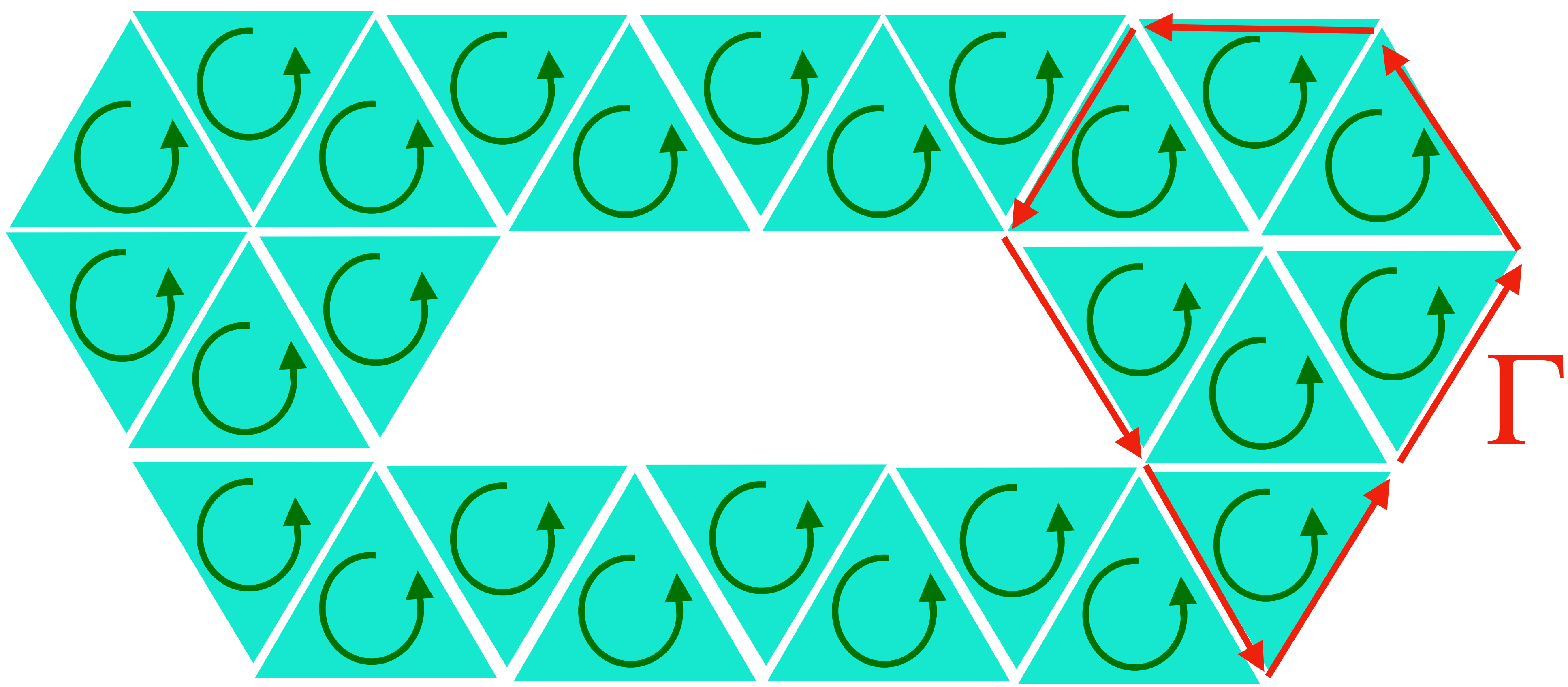
$$\partial_k \colon C_k \to C_{k-1}$$





$$H_k = \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}}$$

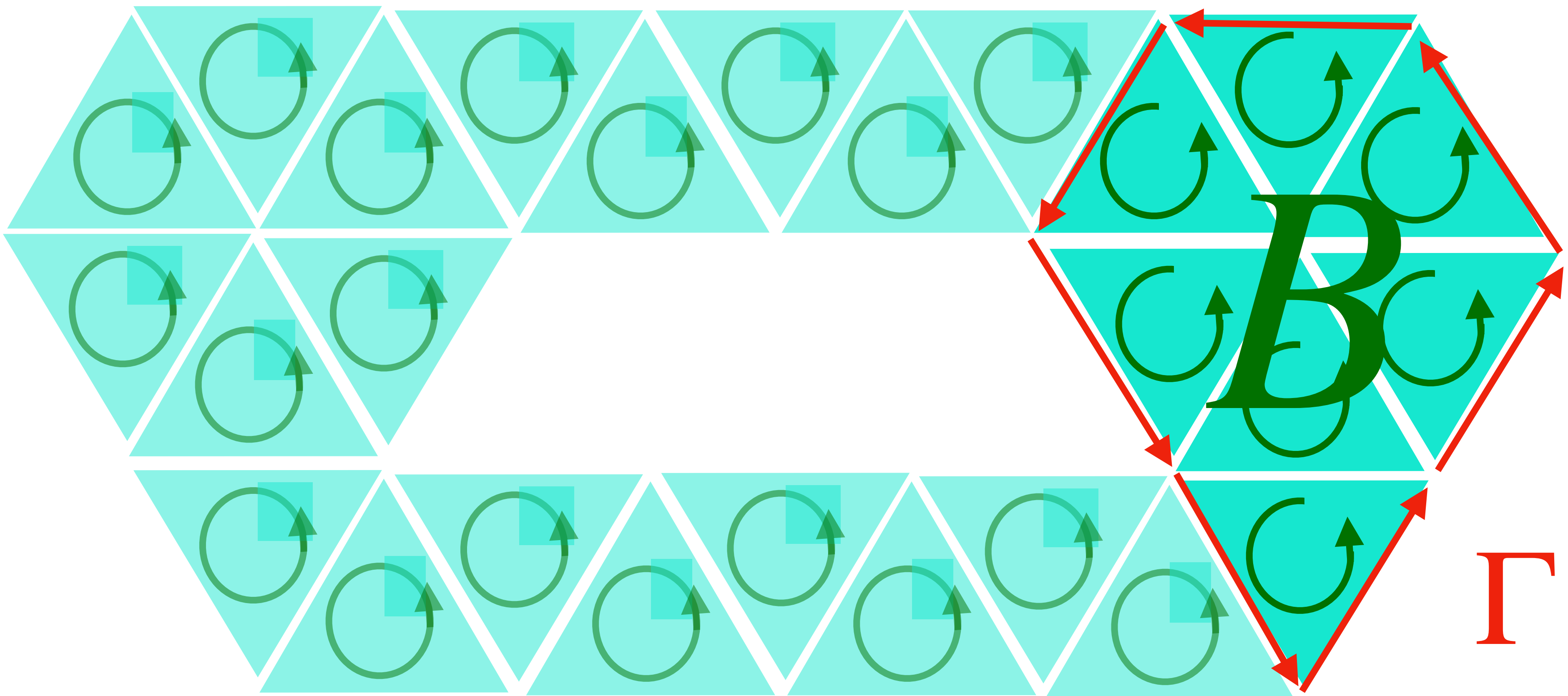
Minimal homology notions:



$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

$$\partial_1 \Gamma = 0 \qquad \Rightarrow \Gamma \in \ker \partial_1$$

Minimal homology notions:



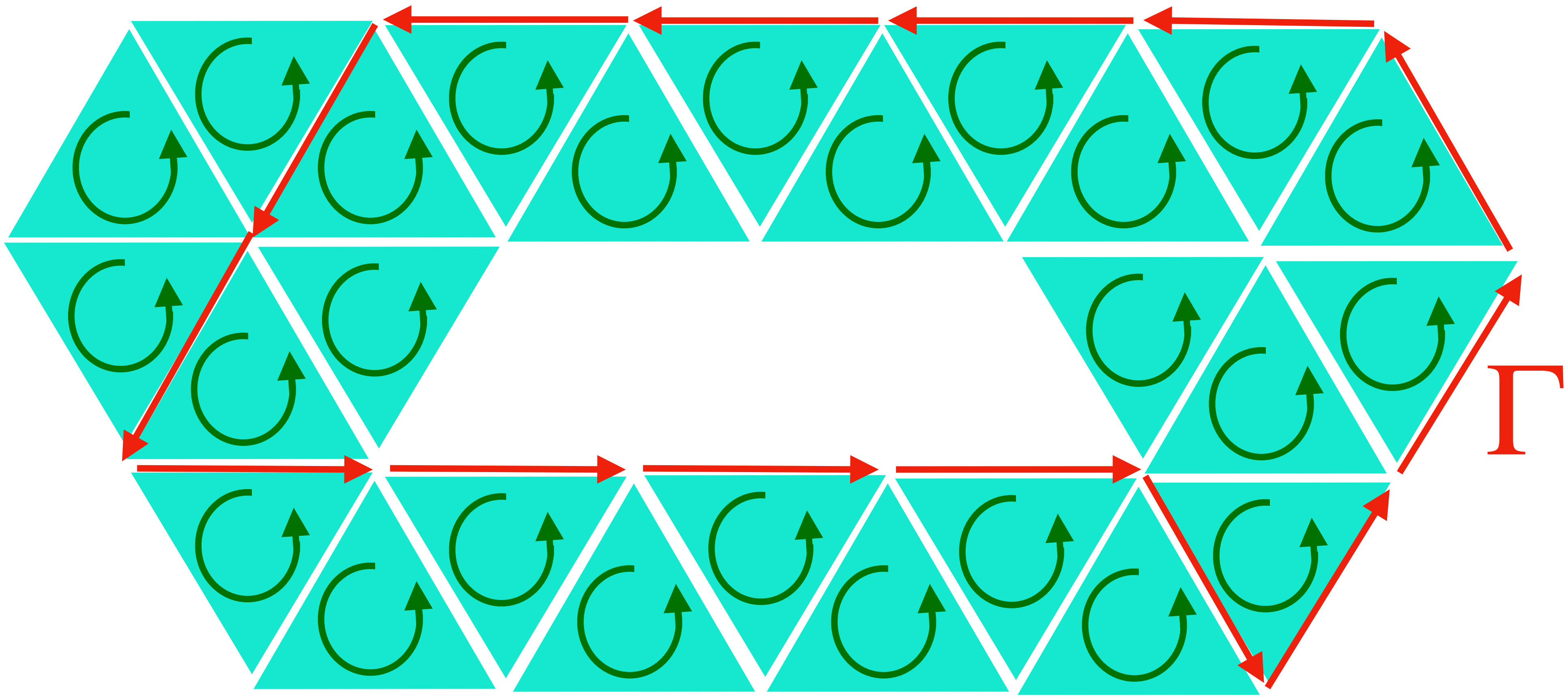
$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

$$\Gamma = \partial_2 B$$

$$\partial_1 \Gamma = 0 \qquad \Rightarrow \Gamma \in \ker \partial_1$$

But... $\Gamma \in \operatorname{Im} \partial_2 \;\Rightarrow\; \left[\Gamma\right]_{\operatorname{Im} \partial_2} = 0$

Minimal homology notions:

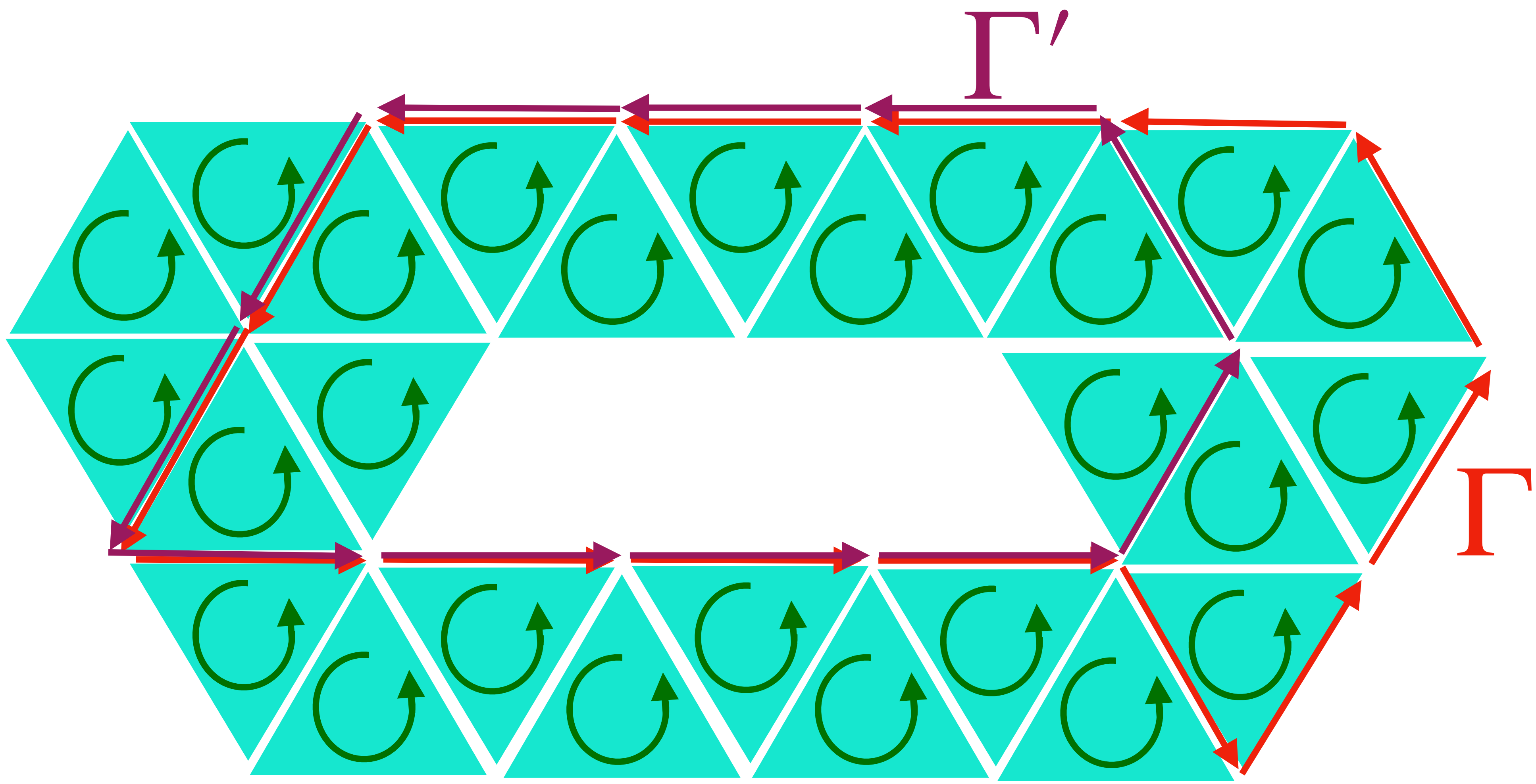


$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

$$\partial_1 \Gamma = 0 \quad \Rightarrow \Gamma \in \ker \partial_1$$

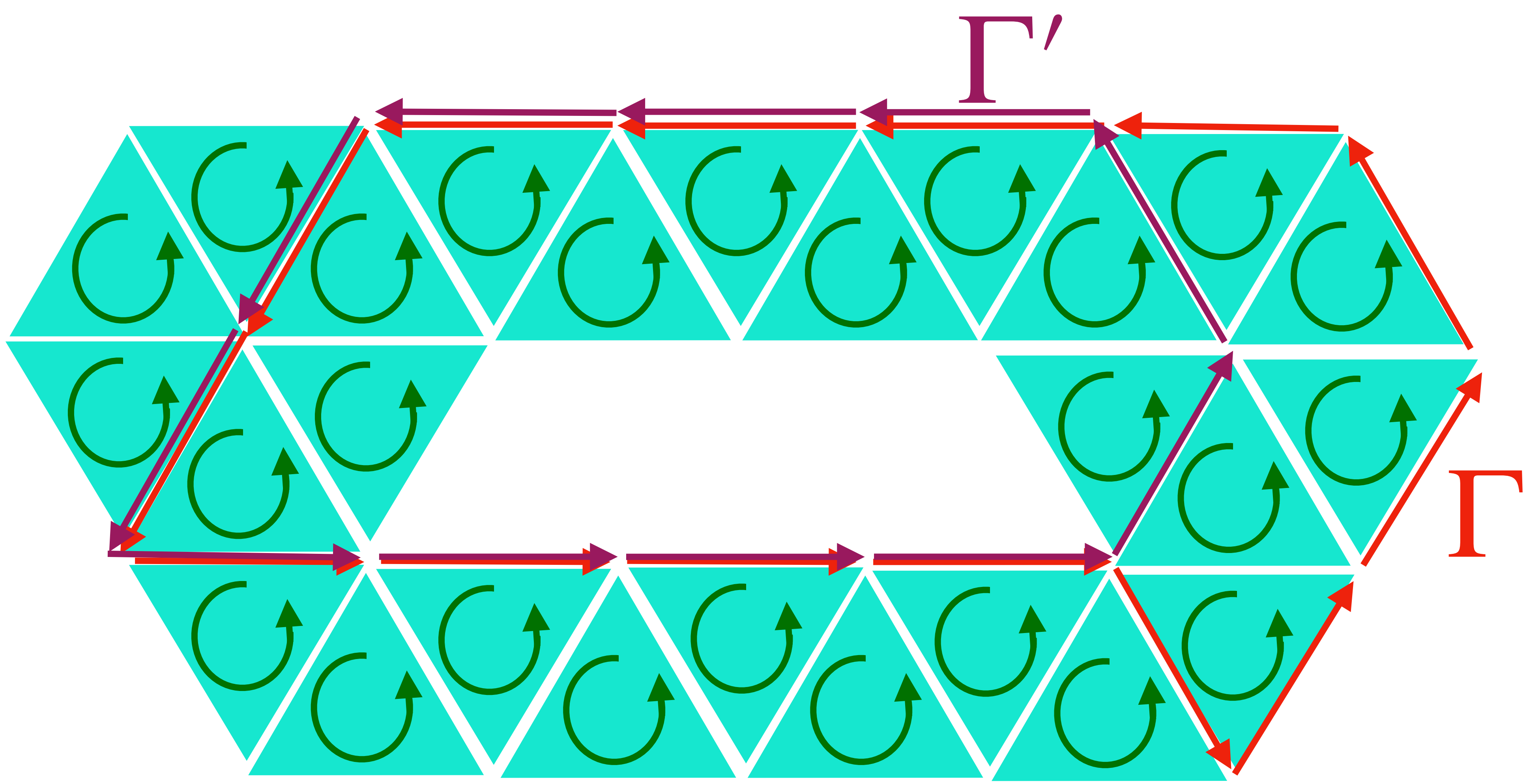
$$\Gamma \notin \operatorname{Im} \partial_2 \Rightarrow [\Gamma]_{\operatorname{Im} \partial_2} \neq 0$$

Minimal homology notions:



$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

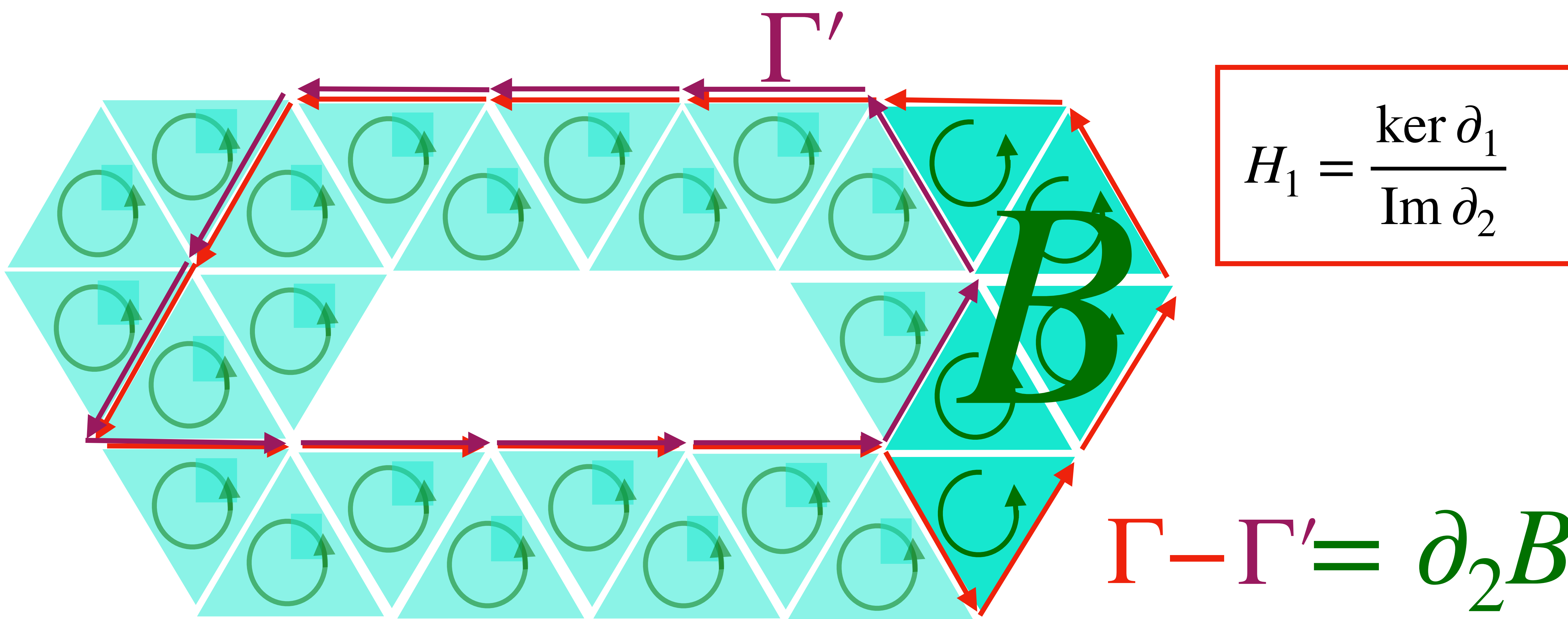
Minimal homology notions:



$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

$$\partial_1 \Gamma = 0 \quad \partial_1 \Gamma' = 0 \quad \Rightarrow \Gamma, \Gamma' \in \ker \partial_1$$

Minimal homology notions:



$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

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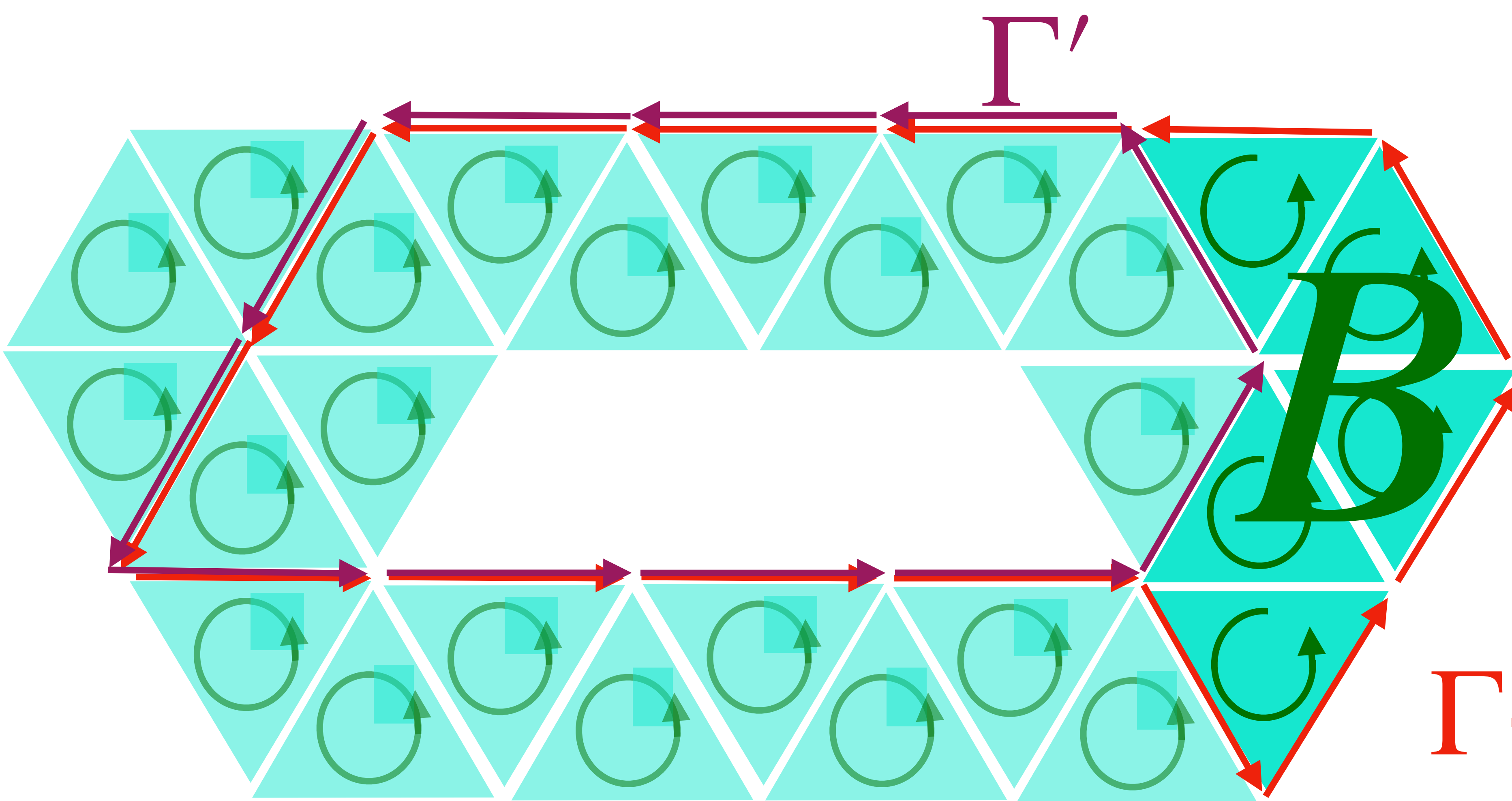
Minimal homology notions:



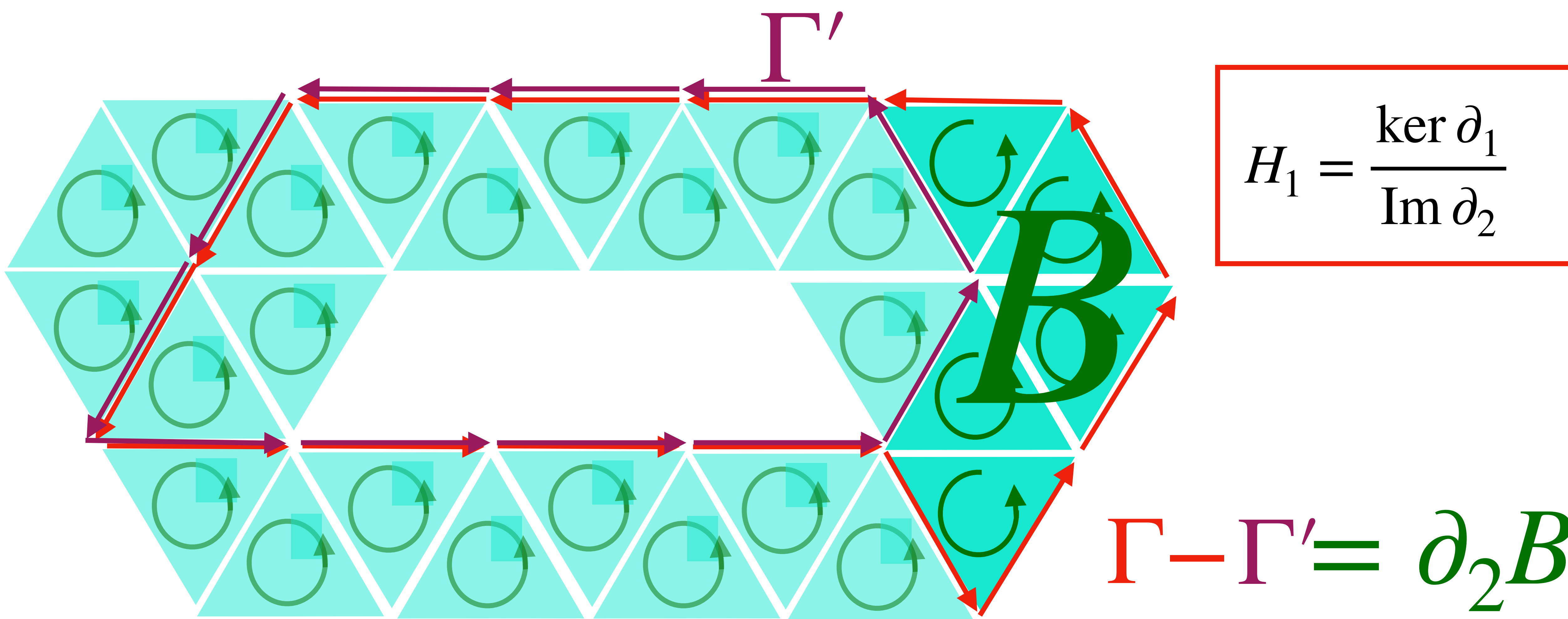
$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

$$\Gamma - \Gamma' = \partial_2 B$$

$$\partial_1 \Gamma = 0 \quad \partial_1 \Gamma' = 0 \qquad \Rightarrow \Gamma, \Gamma' \in \ker \partial_1$$

$$\Gamma - \Gamma' \in \operatorname{Im} \partial_2 \Longleftrightarrow [\Gamma]_{\operatorname{Im} \partial_2} = [\Gamma']_{\operatorname{Im} \partial_2}$$

Minimal homology notions:



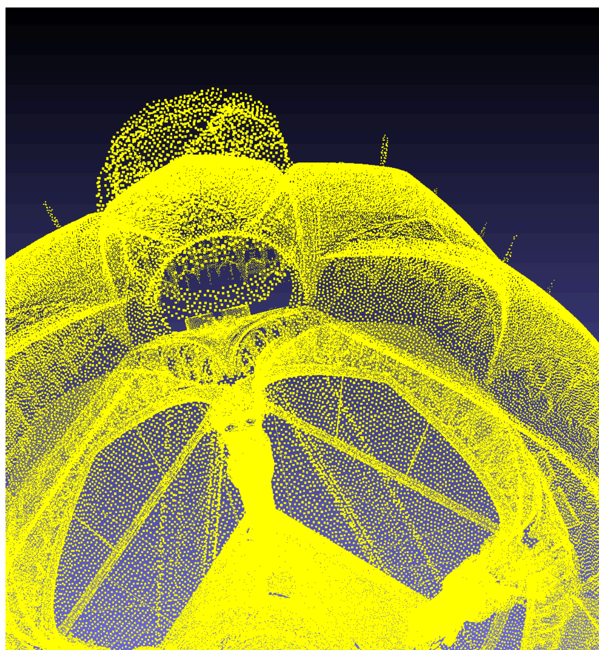
$$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2}$$

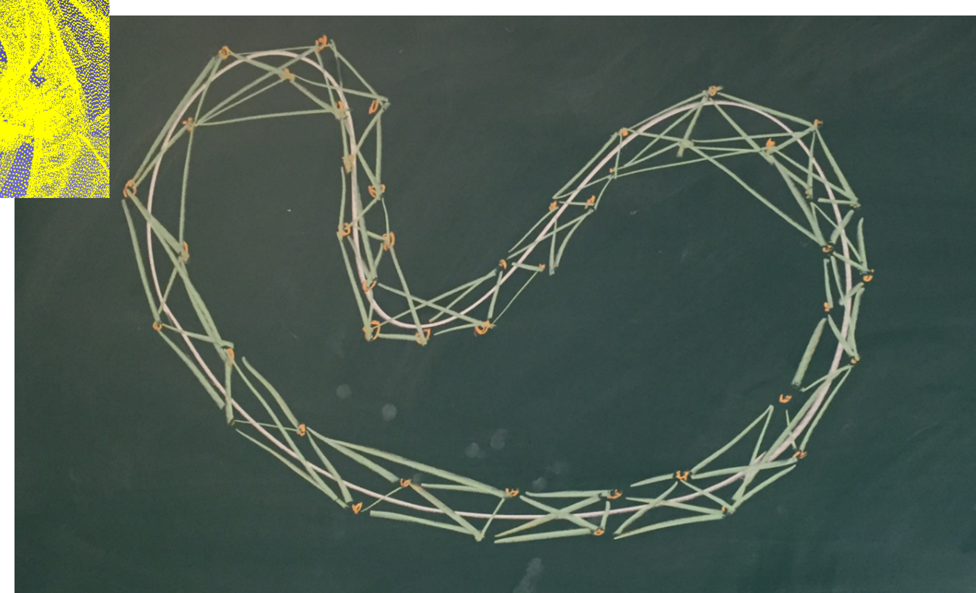
$$\Gamma - \Gamma' \in \operatorname{Im} \partial_2 \iff [\Gamma]_{\operatorname{Im} \partial_2} = [\Gamma']_{\operatorname{Im} \partial_2}$$

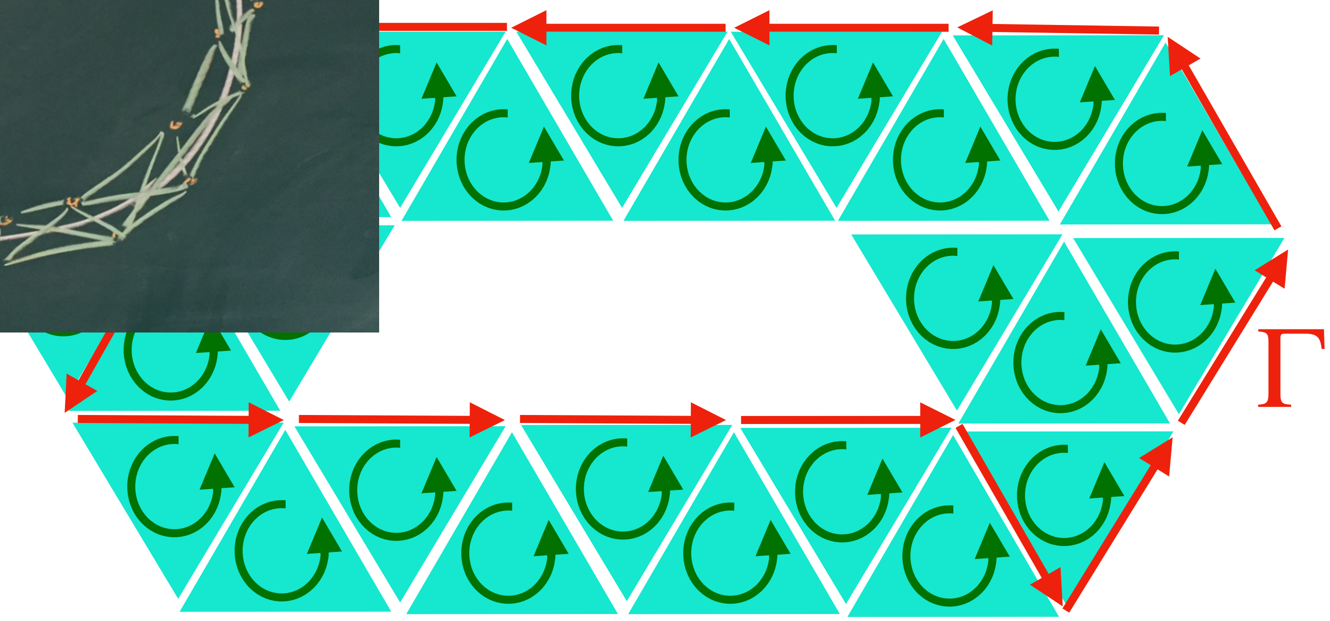
$\iff$ Γ and Γ' **are homologous cycles**

Minimal homology notions:
back to algorithms

$$H_k = \frac{\ker \partial_k}{\mathrm{Im}\, \partial_{k+1}}$$



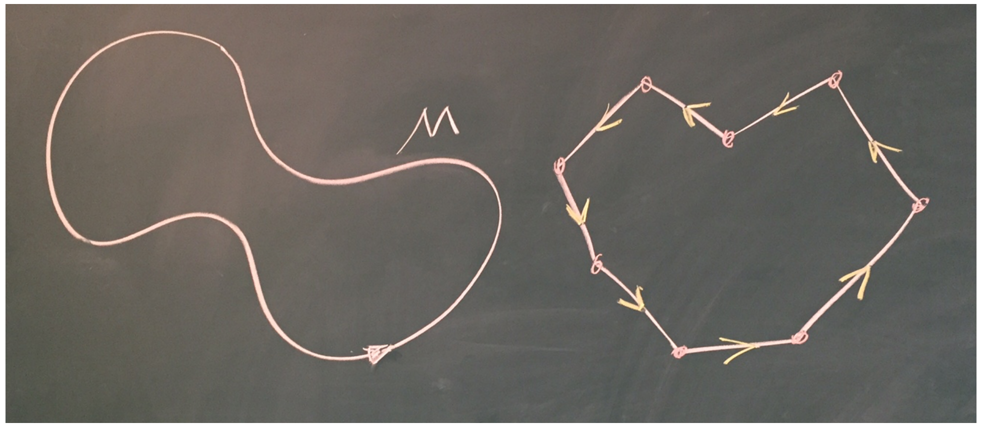




Fundamental class

(orientable and non-orientable, with/without boundary)

If M is a **connected compact orientable** d-manifold, its d-homology group is one dimensional and a generator of it is called the **Fundamental class.**



$$\dim \mathbf{H}_d(M^d) = 1$$

Fundamental class

(orientable and non-orientable, with/without boundary)

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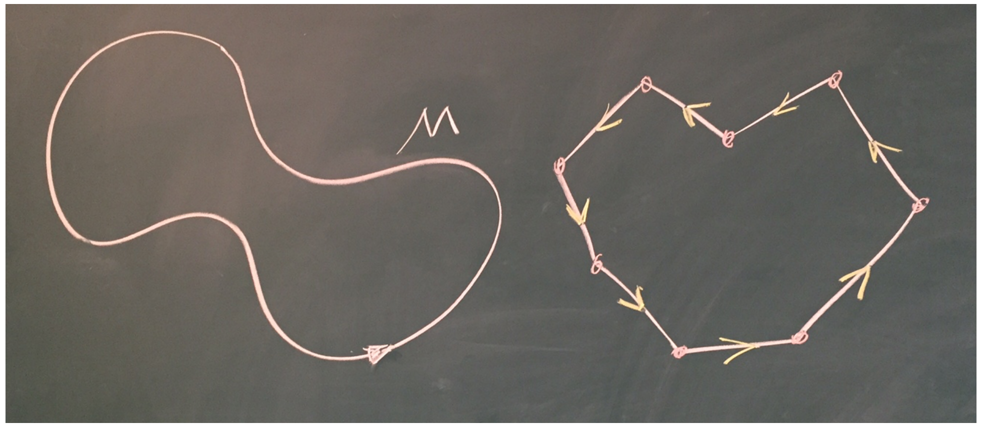
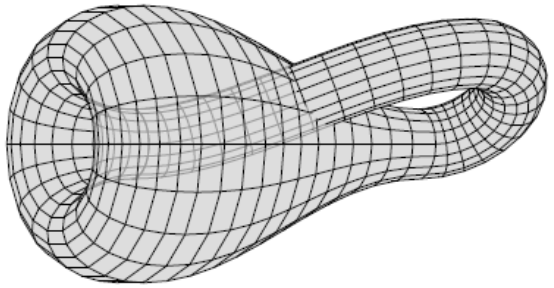


$$\dim \mathbf{H}_d(M^d, \mathbb{Z}_2) = 1$$

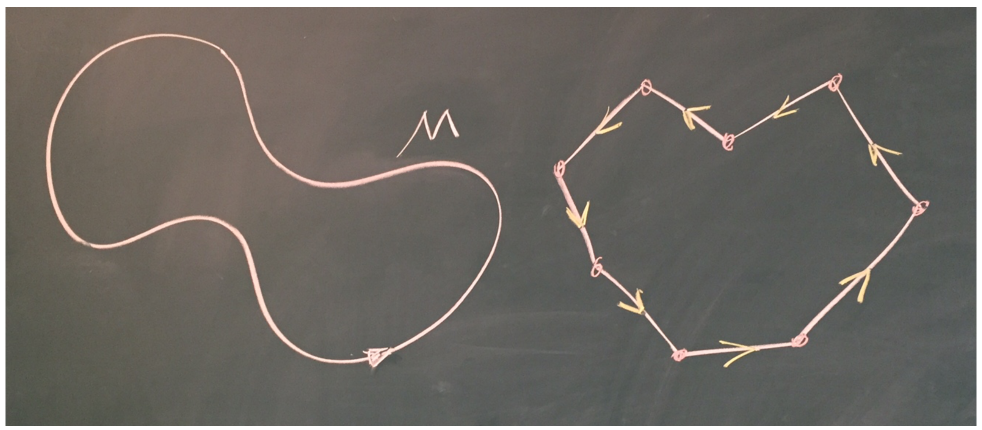
If the coefficients field is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, this is also true for non-orientable (compact, connected) manifolds.



Fundamental class

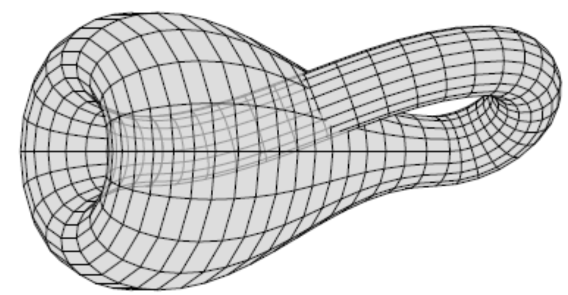
(orientable and non-orientable, with/without boundary)

If M is a **connected compact orientable** d-manifold, its d-homology group is one dimensional and a generator of it is called the **Fundamental class**.



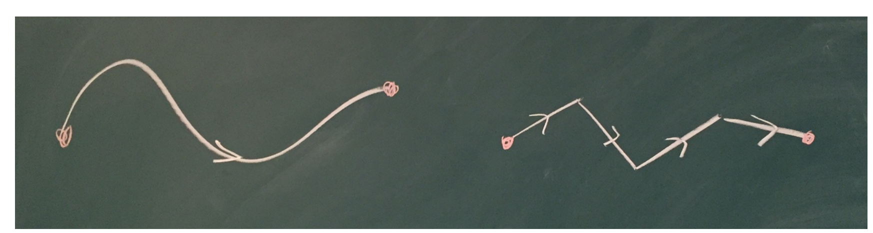
$$\dim \mathbf{H}_d(M^d, \mathbb{Z}_2) = 1$$

If the coefficients field is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, this is also true for non-orientable (compact, connected) manifolds.



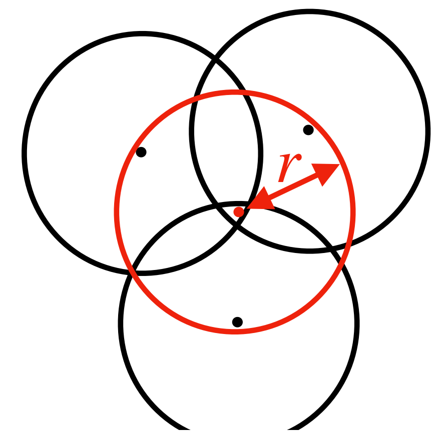
For manifolds with boundaries, this generalizes with relative homology:

$$\dim \mathbf{H}_d(M, \partial M, \mathbb{Z}_2) = 1$$

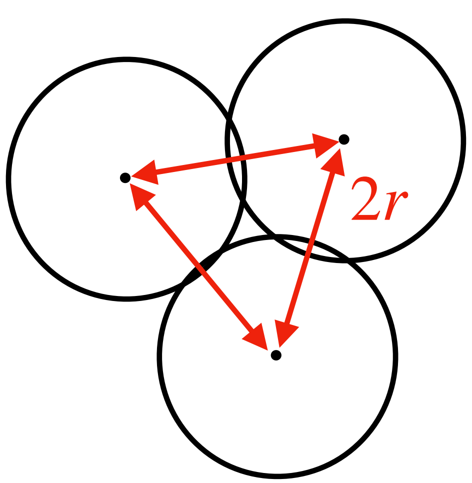


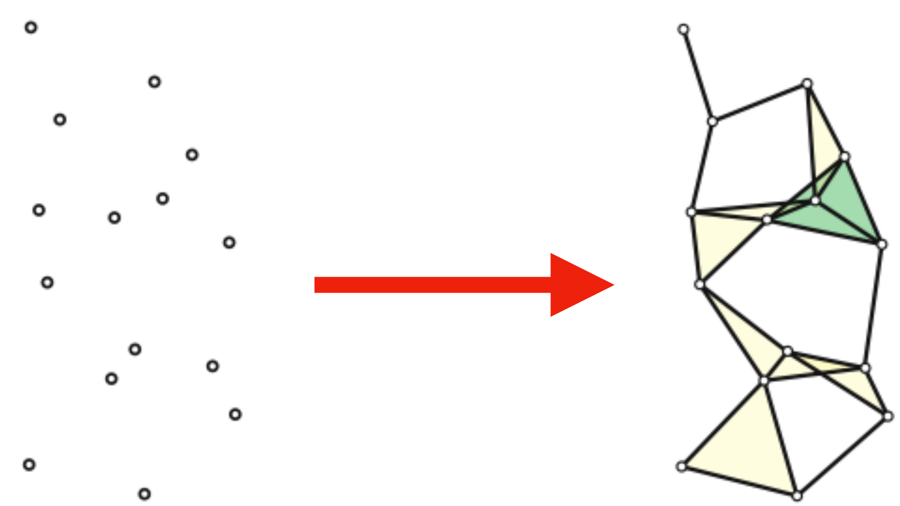
Remember:

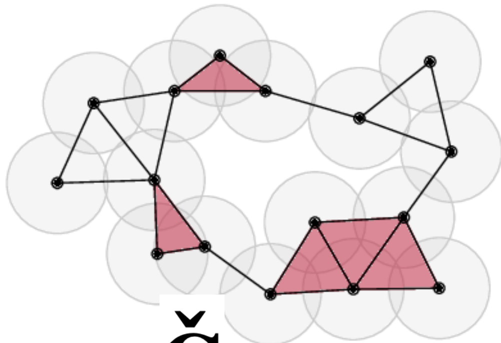
Given a finite set $\mathcal{P}$ and a radius $r > 0$, the **Čech complex** $\mathrm{C}_r(\mathcal{P})$ is the set of simplices in $\mathcal{P}$ enclosed in ball of radius r.



Given a finite set $\mathcal{P}$ and a parameter $r > 0$, the **Vietoris-Rips complex** $\mathrm{R}_r(\mathcal{P})$ is the set of simplices in $\mathcal{P}$ with diameter at most $2r$.

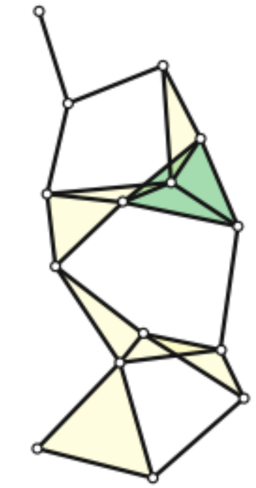




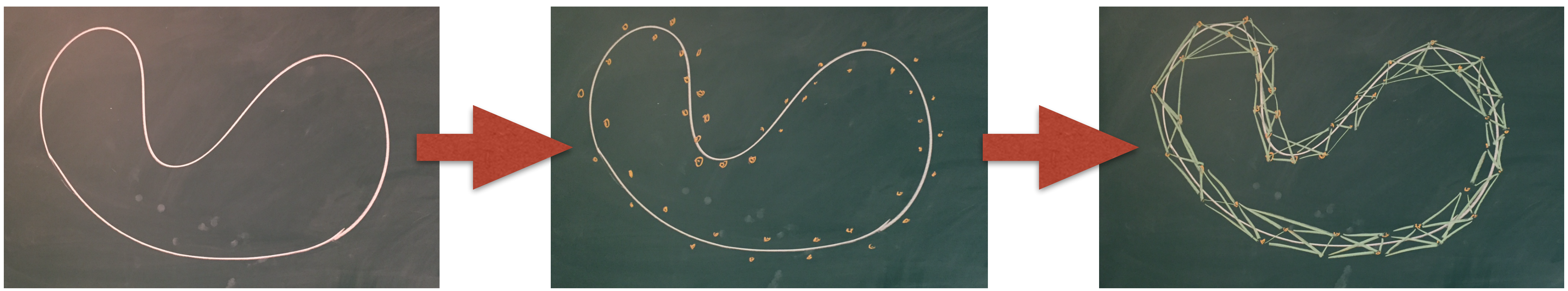


Čech

Fundamental class



Rips

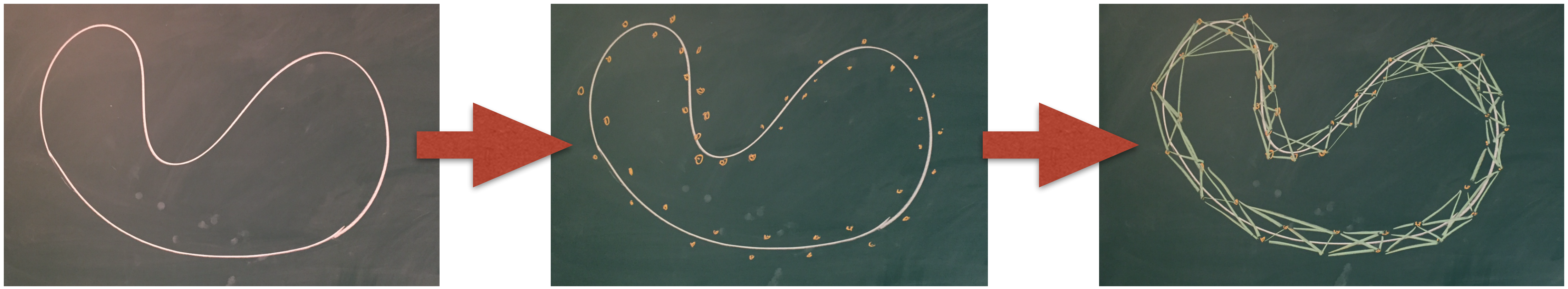


In particular, under adequate sampling conditions and parameters, **Čech** or **Vietoris-Rips** complexes ***K*** share **the homotopy type** and therefore the **d-homology of the complex.**
Which is then is one dimensional and **reproduces the fundamental class of the manifold**.

$$\Rightarrow \quad \boxed{\mathbf{H}_d(K, \mathbb{Z}_2) \simeq \mathbb{Z}_2}$$

$\Rightarrow \mathbf{H}_d(K)$ contains a single non zero element.

Fundamental class



$\Rightarrow$ $$\mathbf{H}_d(K, \mathbb{Z}_2) \simeq \mathbb{Z}_2$$ $\Rightarrow \mathbf{H}_d(K)$ contains a single non zero element.

But **Homology classes are not geometric**: we look for **a particular simplicial chain representative of the homology class** whose **support** could be **homeomorphic** to the sampled manifold:

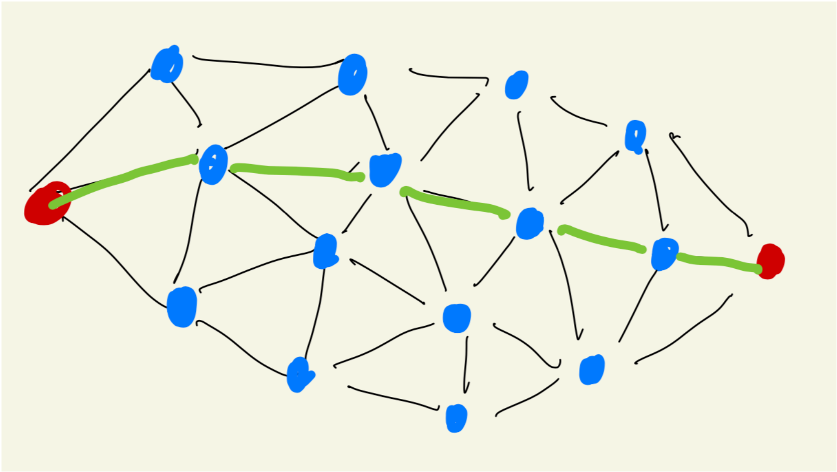
We search for it as the **minimum representative** chain in the fundamental class

Two canonical problems

Minimal chain for a given boundary β

Given $\beta \in C_{d-1}(K, \mathbb{F})$ **find:**

$$\Gamma_{\min} = \min\{\Gamma \in C_d(K, \mathbb{F}), \partial\Gamma = \beta\}$$



$\mathbf{dim}(K) = 1$

Minimal chain homologous to α

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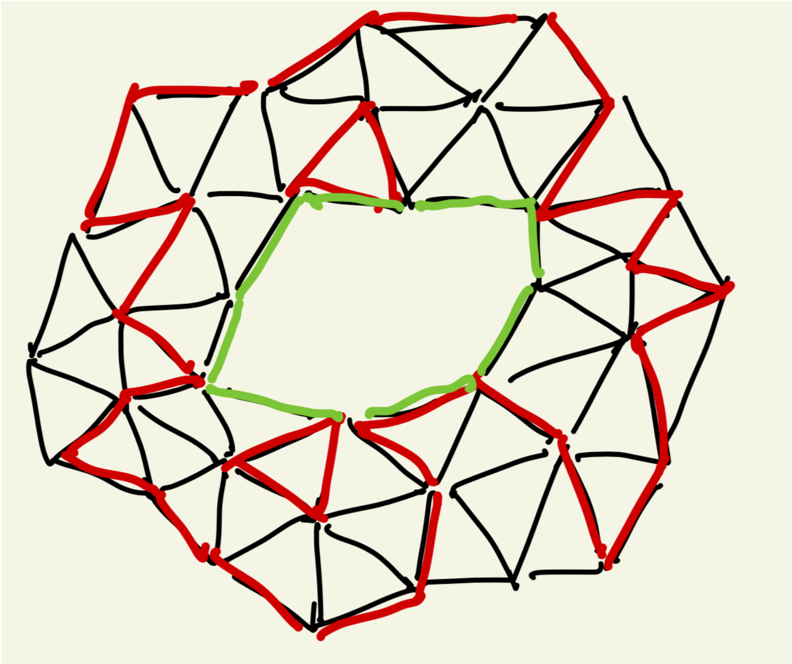
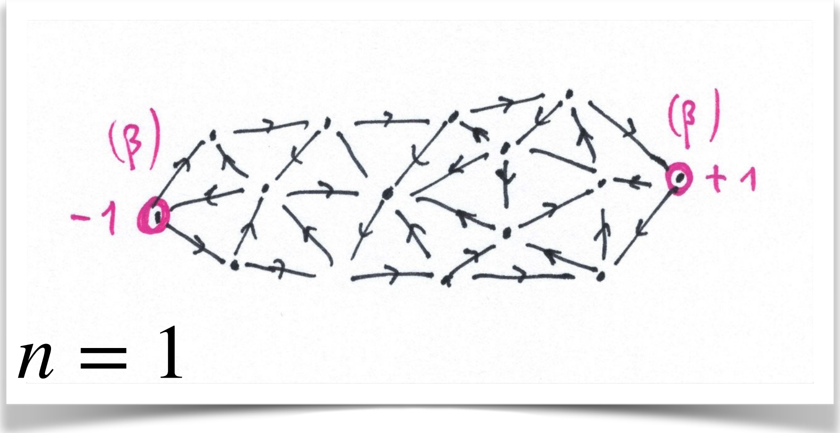


$\mathbf{dim}(K) = 2$

Minimal homology representative cycle
(real coefficients)

Minimal chain for a given boundary β

$$\underset{x,\ \partial x = \beta}{\arg\min} \|x\|_2$$

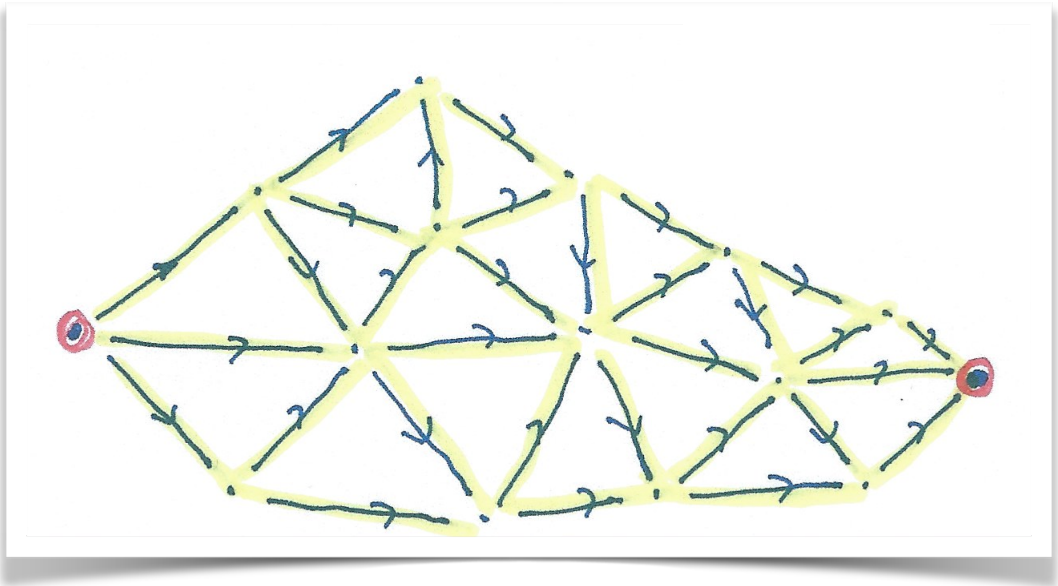


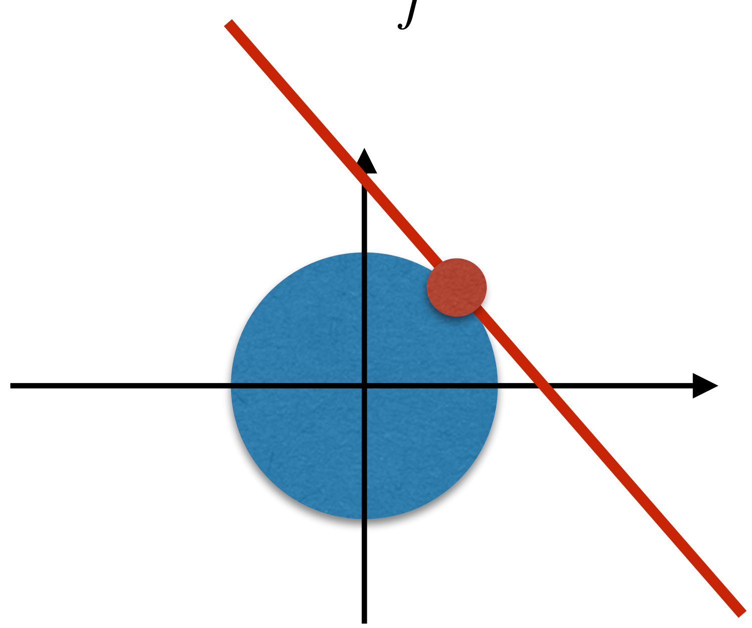
L² minima are not sparse

Minimizing L² norm
=> **harmonic form**:

$$\sum_j \frac{1}{R_j} I_j^2$$ (= electrical power)

R_j = electrical resistance

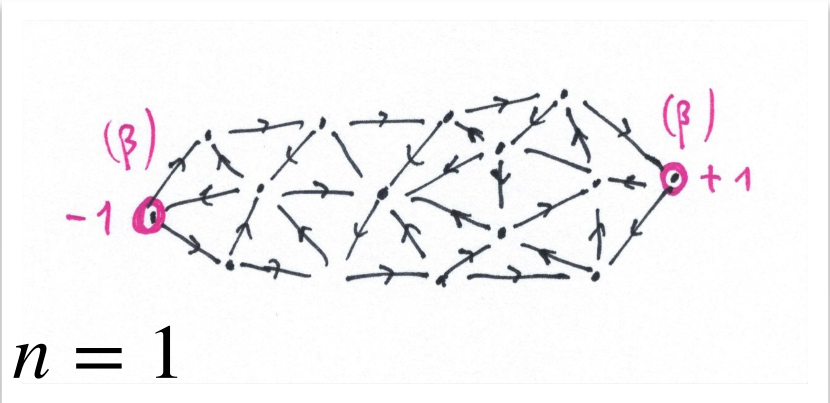




Minimal homology representative cycle
(real coefficients)

Minimal chain for a given boundary β

$$\underset{x,\ \partial x=\beta}{\arg\min} \|x\|_1$$

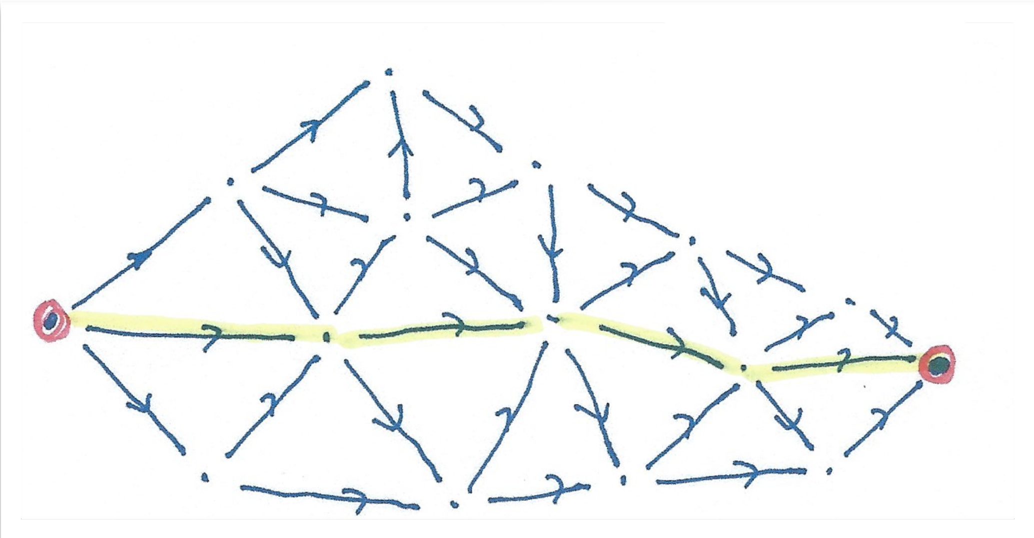


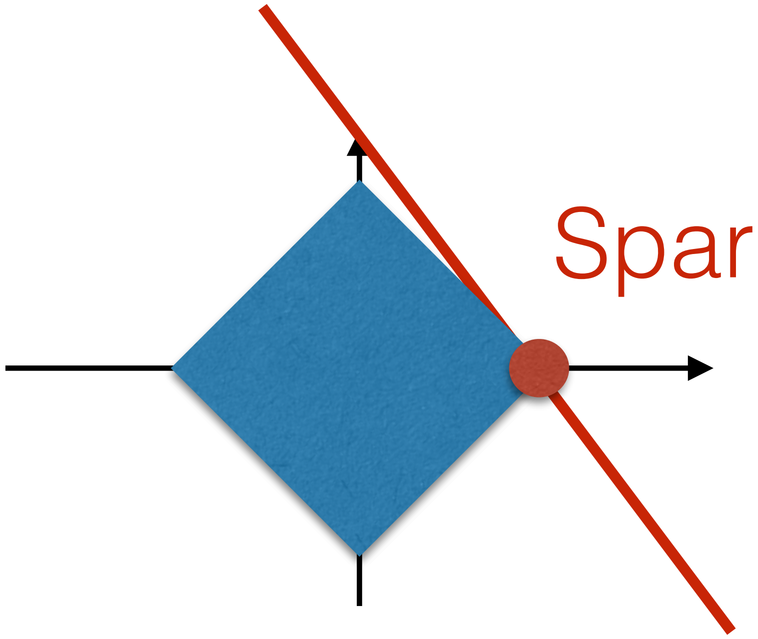
L¹ minima are sparse

Minimizing L¹ norm :
=> **shortest path**

$$\sum_j l_j |I_j|$$

(Path length)





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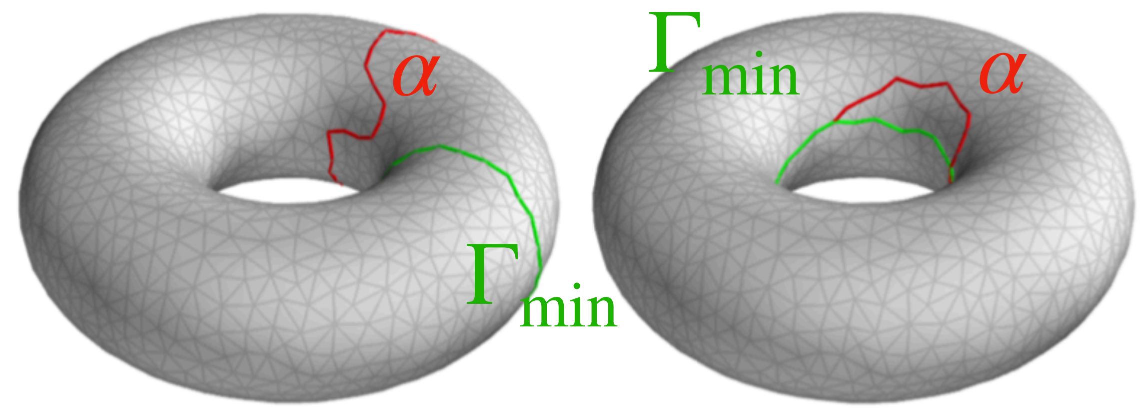
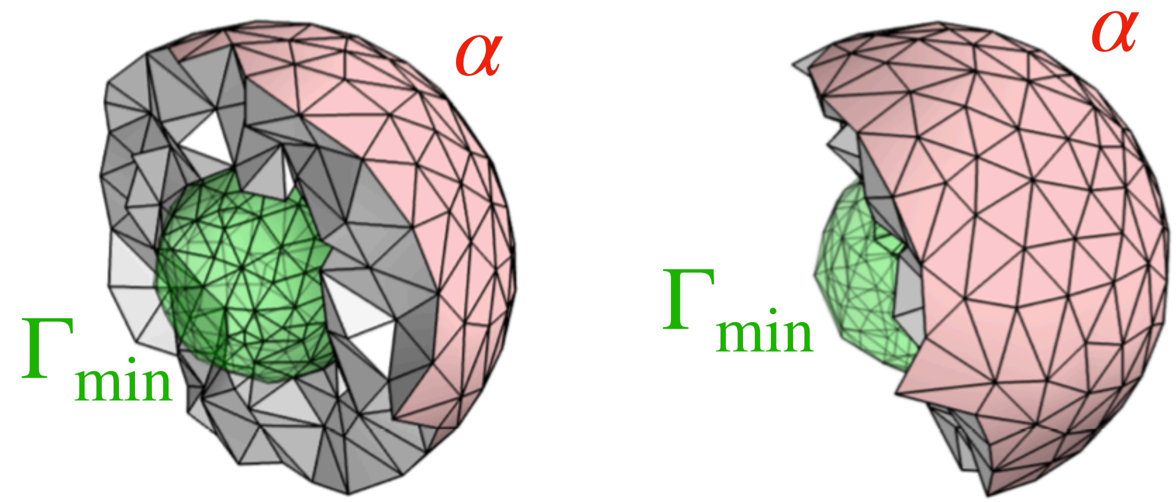


$$\|\Gamma\|_1 = \textbf{length}(\Gamma) = \sum |\Gamma(\tau)| \textbf{length}(\tau)$$

Minimality for L^1 norm, typically « **volumes**»:

$$\|\Gamma\|_1 = \textbf{Vol}(\Gamma) = \sum |\Gamma(\tau)| \textbf{Vol}(\tau)$$



$$\|\Gamma\|_1 = \textbf{area}(\Gamma) = \sum |\Gamma(\tau)| \textbf{area}(\tau)$$

(Thanks to T. Dey et Al. for the figures)

Minimal homology representative cycle

Some related works on L^1minimal homologous chain…

Erin W Chambers, Jeff Erickson, and Amir Nayyeri. Minimum cuts and shortest homologous cycles. In *Proceedings of the twenty-fifth annual symposium on Computational geometry*, pages 377–385. ACM, 2009.

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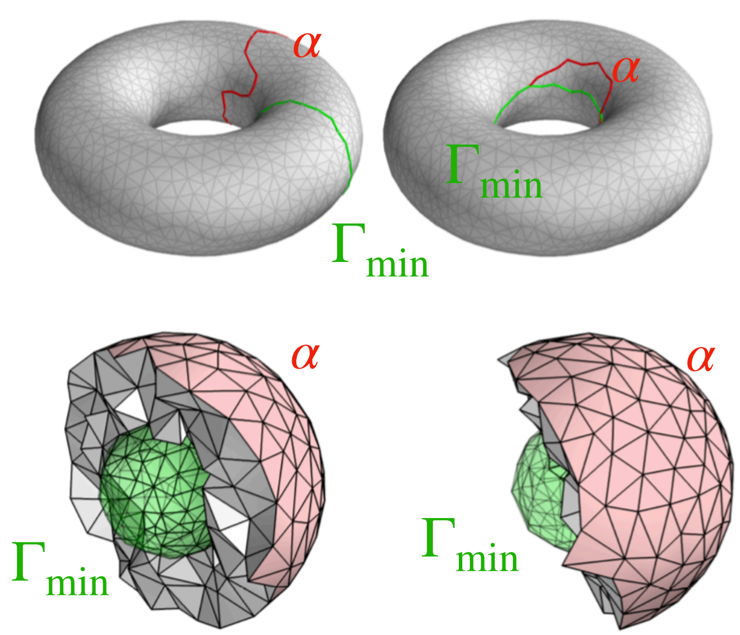
Tamal K Dey, Tao Hou, and Sayan Mandal. Computing minimal persistent cycles: Polynomial and hard cases. *arXiv preprint arXiv:1907.04889*, 2019.

Hardness results
(linear programming):

NP-Hard in general
for coefficients in $\mathbb{Z}_2$



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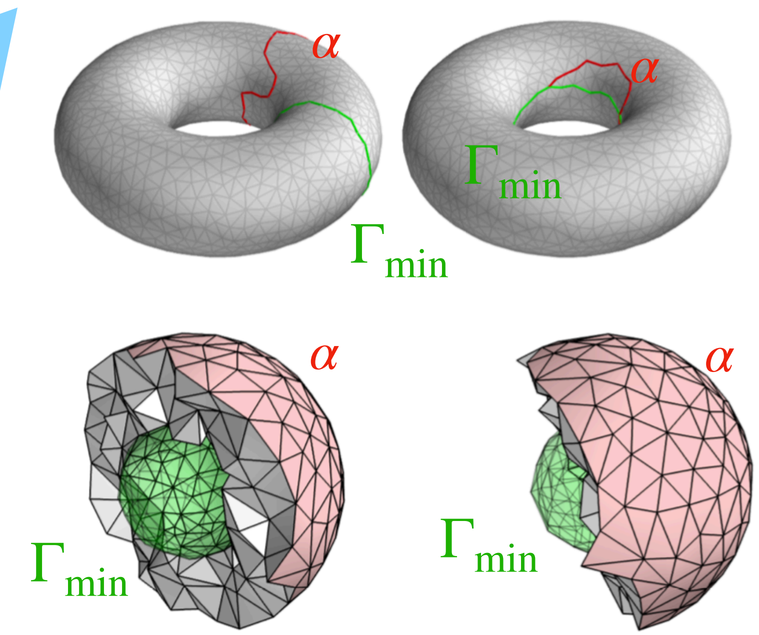
Hardness results (linear programming):

polynomial algorithm when total unimodularity of boundary operator



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Our two canonical problems

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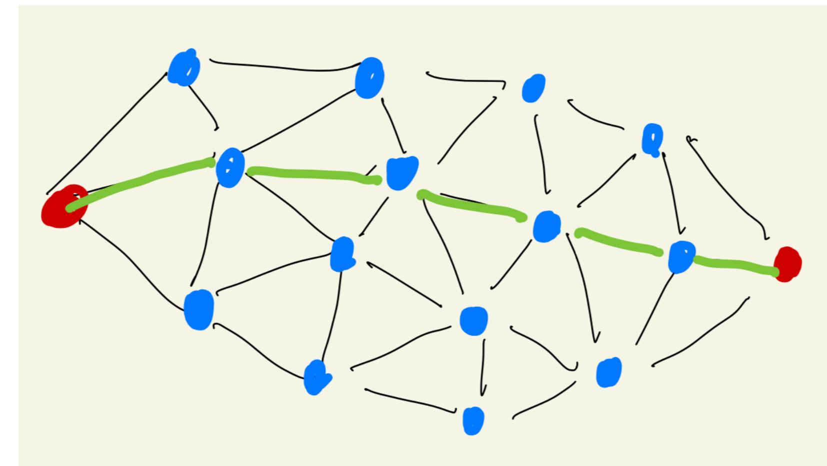

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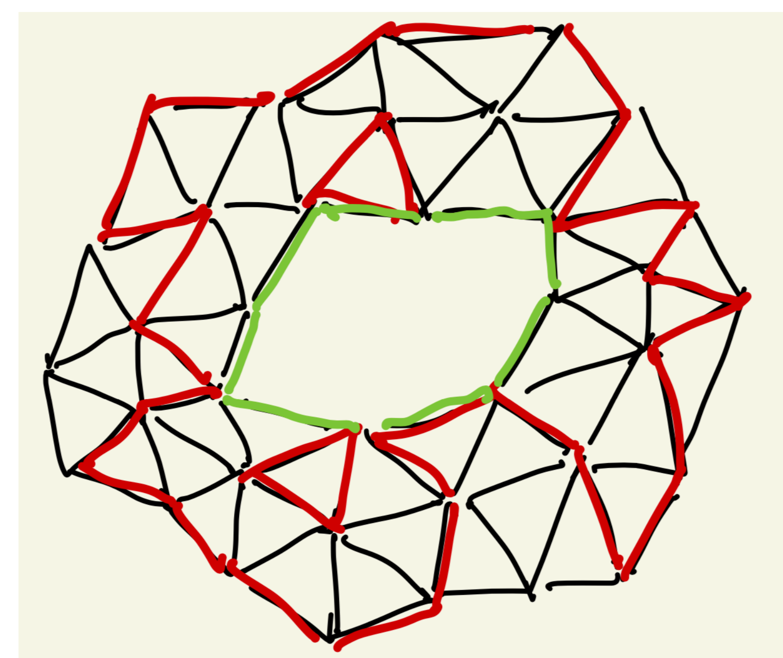

$\mathbf{dim}(K) = 2$

$\min$ according to:

- * L^1 **norm,**
- * **lexicographic order.**

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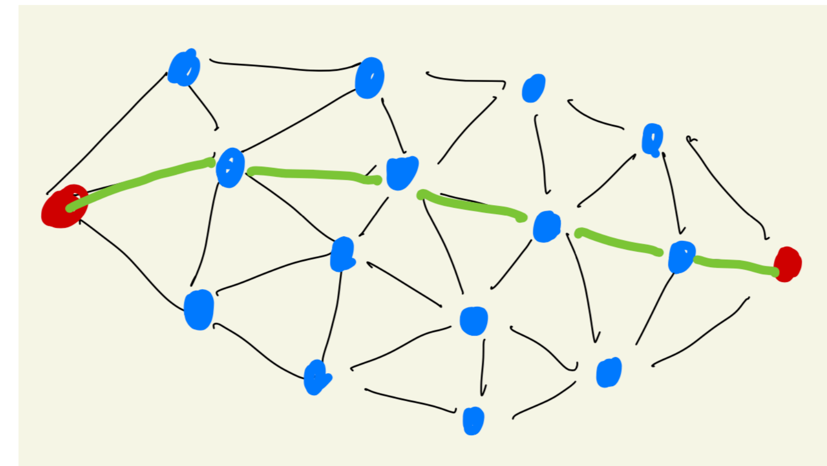

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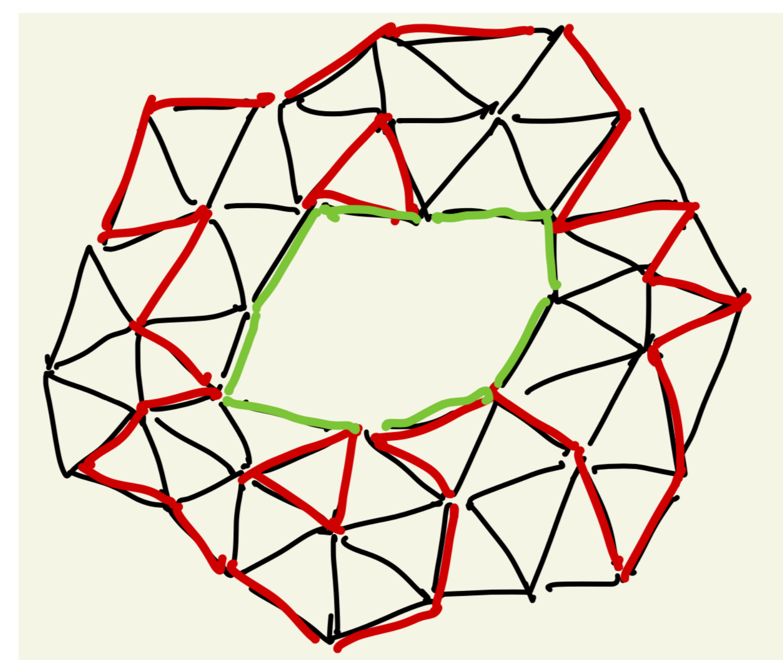

$\mathbf{dim}(K) = 2$

$\min$ according to:

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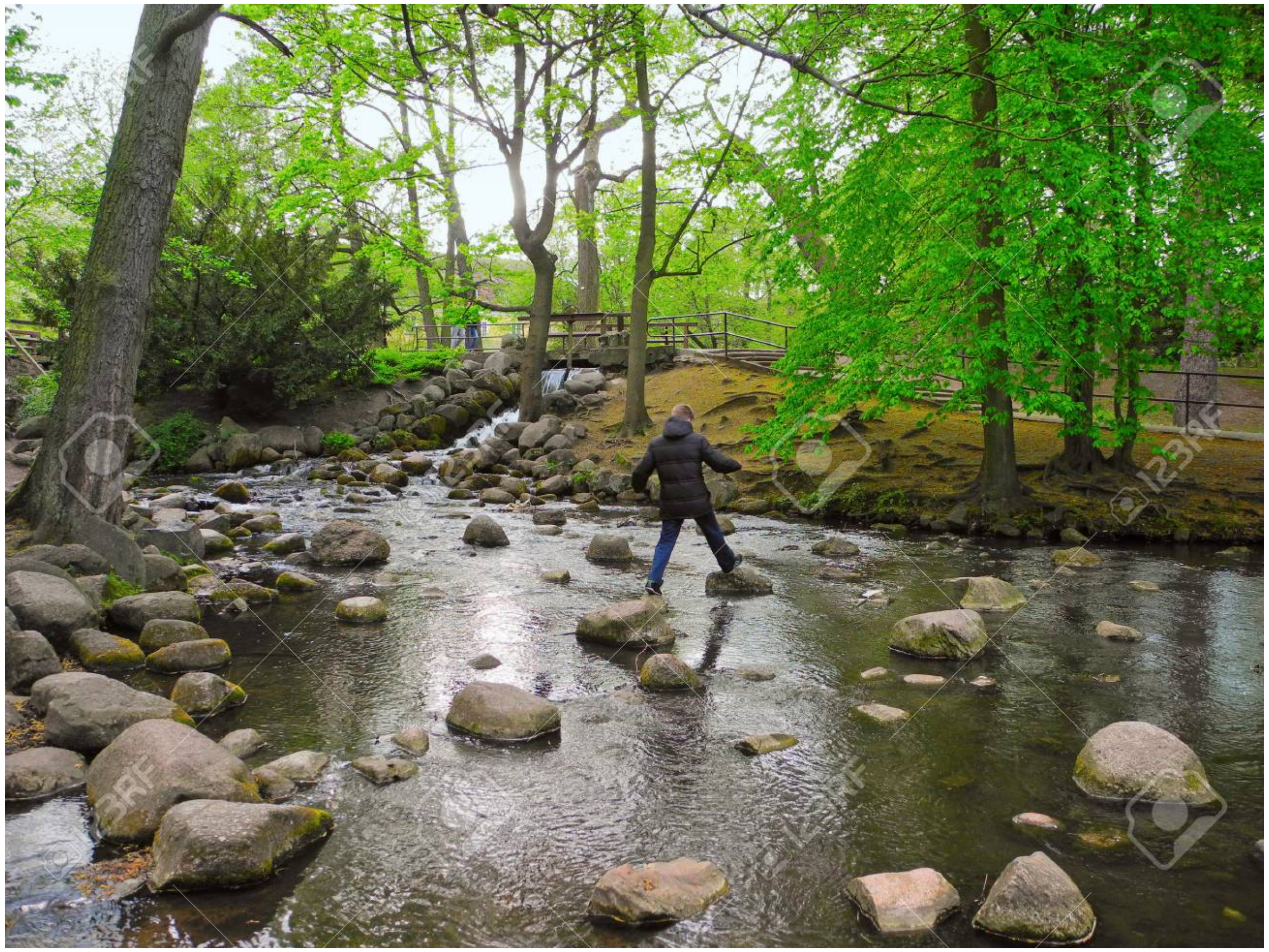
- **lexicographic order.**

NP-hard in general *(Chen, Freedman, 2011)*

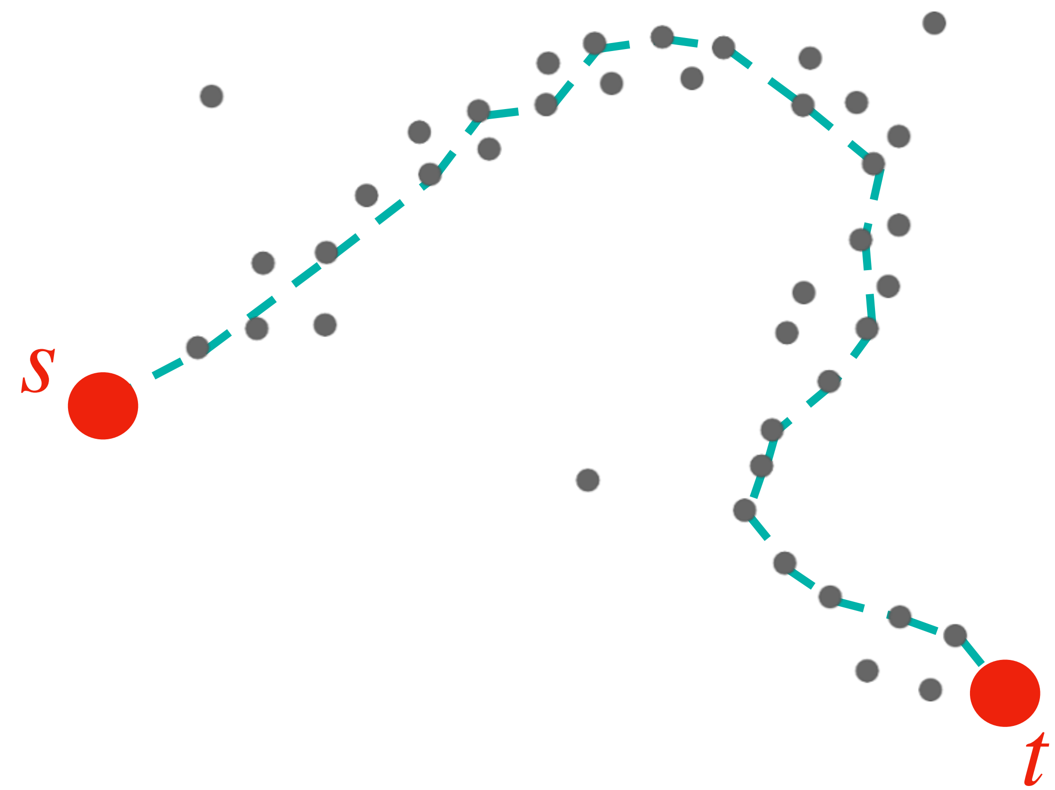
$\mathcal{O}(n^3)$ *(Cohen-Steiner, L, Vuillamy, 2019)*

Lexicographic order



Lexicographic order minimal 1-chain

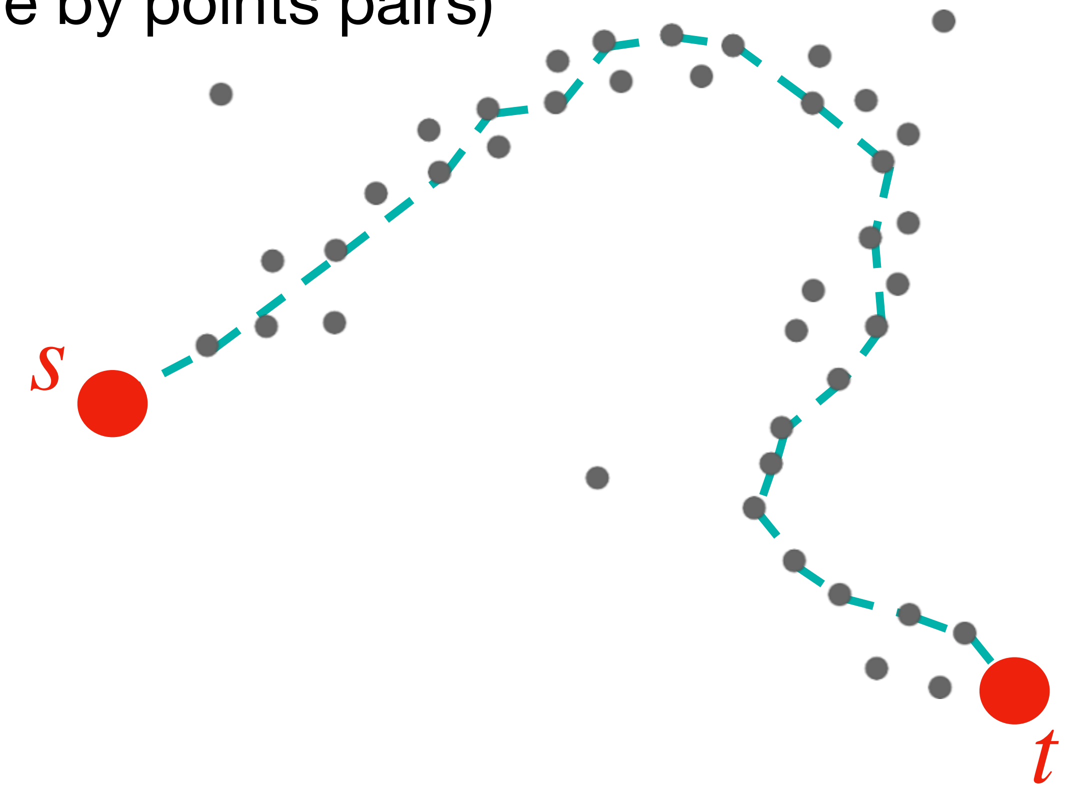
Connect ~~the~~ some dots to form a path between s and t



Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through “**densest**” parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)



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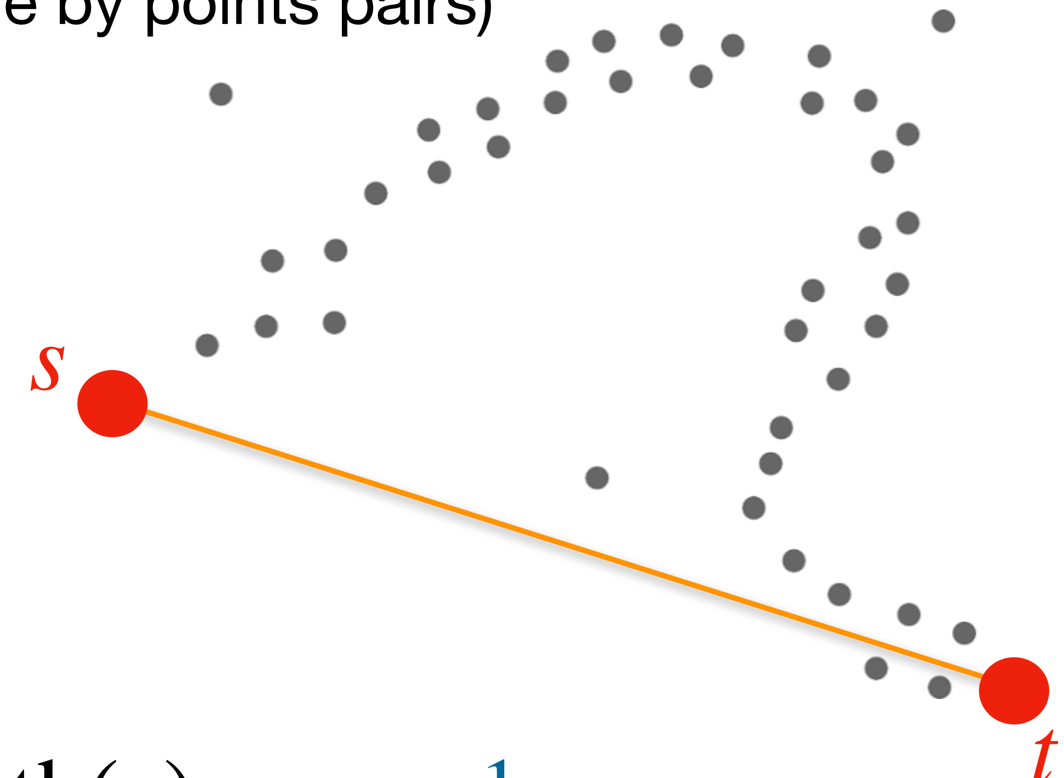
Classic graph problem:
Find minimal path for given edge weights (***Dijkstra****'s algorithm)*



$$\arg\min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e) \qquad p = 1$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

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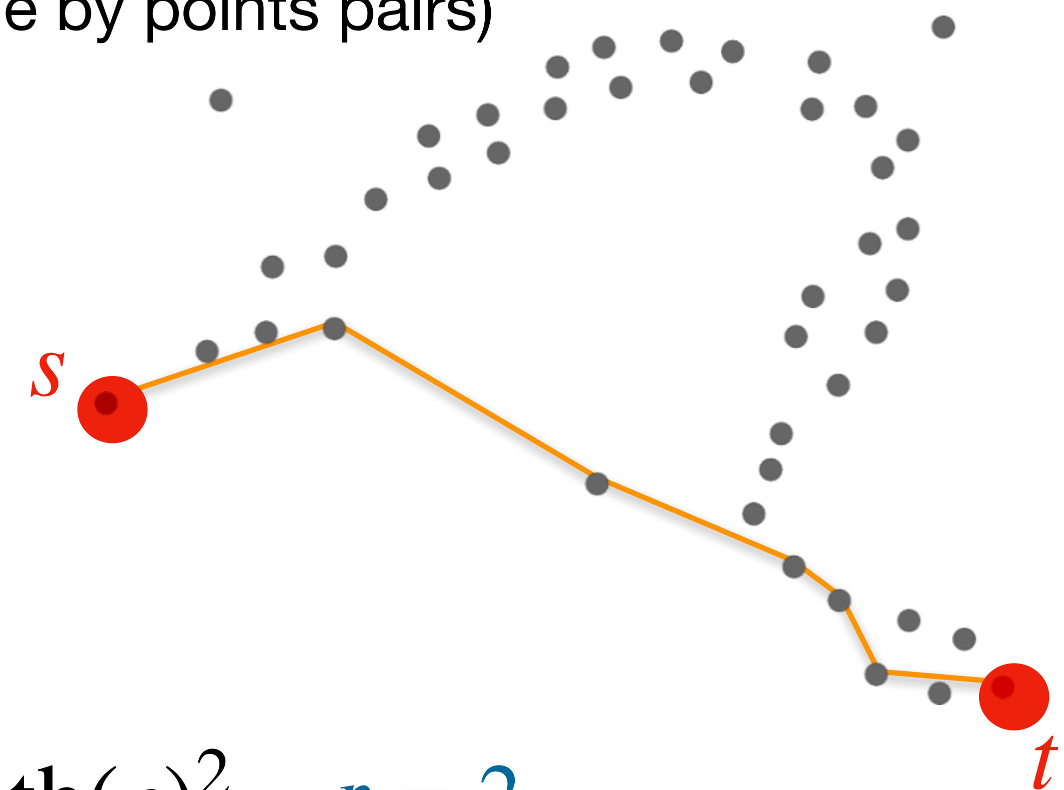


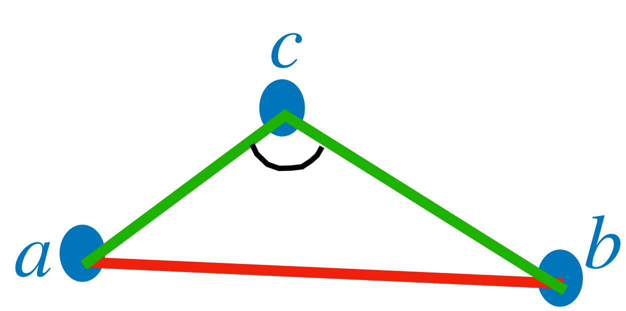
$$\arg \min_{\partial \Gamma = s+t} \sum_{e \in \Gamma} \text{length}(e)^2 \quad p = 2$$

Lexicographic order minimal 1-chain

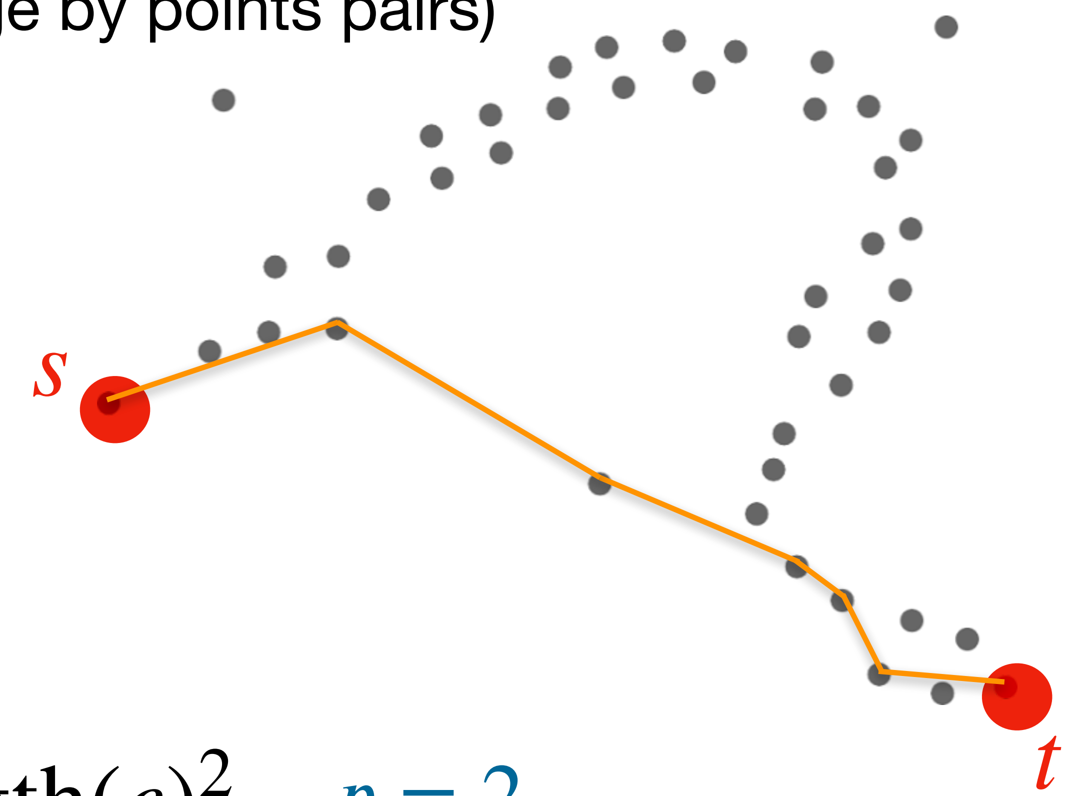
Connect ~~the~~ some dots to form a path between s and t

Objective: find path going through “**densest**” parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)

Pythagoras:



$$\angle acb > \frac{\pi}{2} \Rightarrow ac^2 + cb^2 < ab^2$$

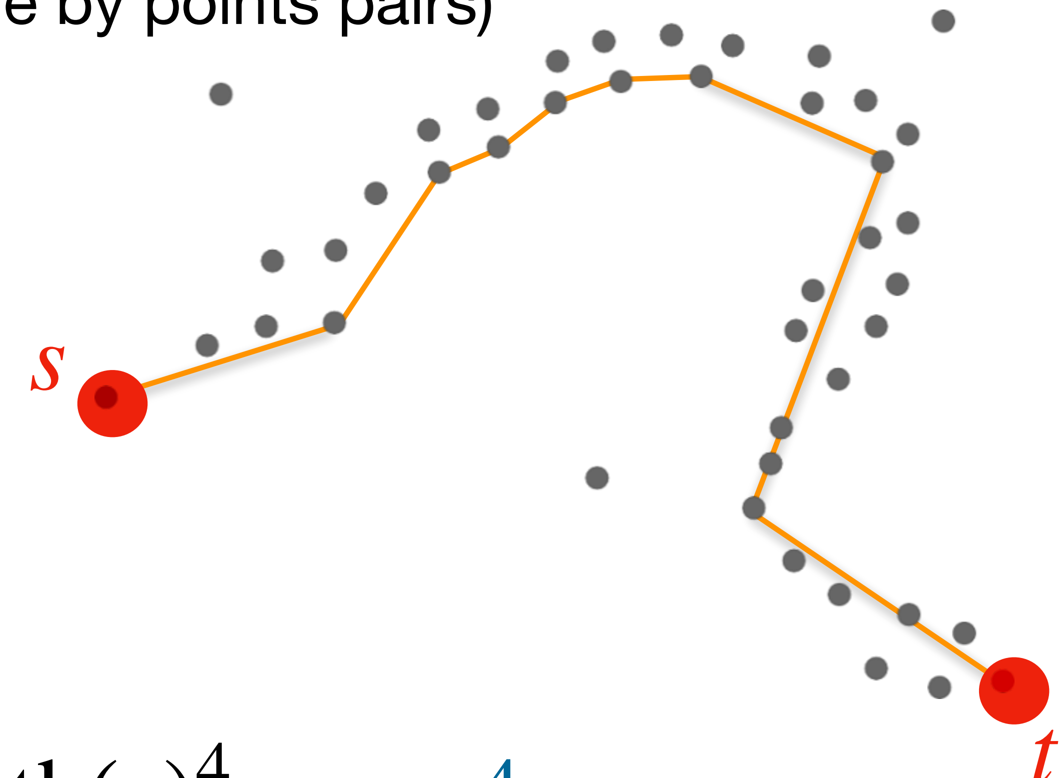


$$\arg \min_{\partial\Gamma = s+t} \sum_{e \in \Gamma} \text{length}(e)^2 \qquad p = 2$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through "**densest**" parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)

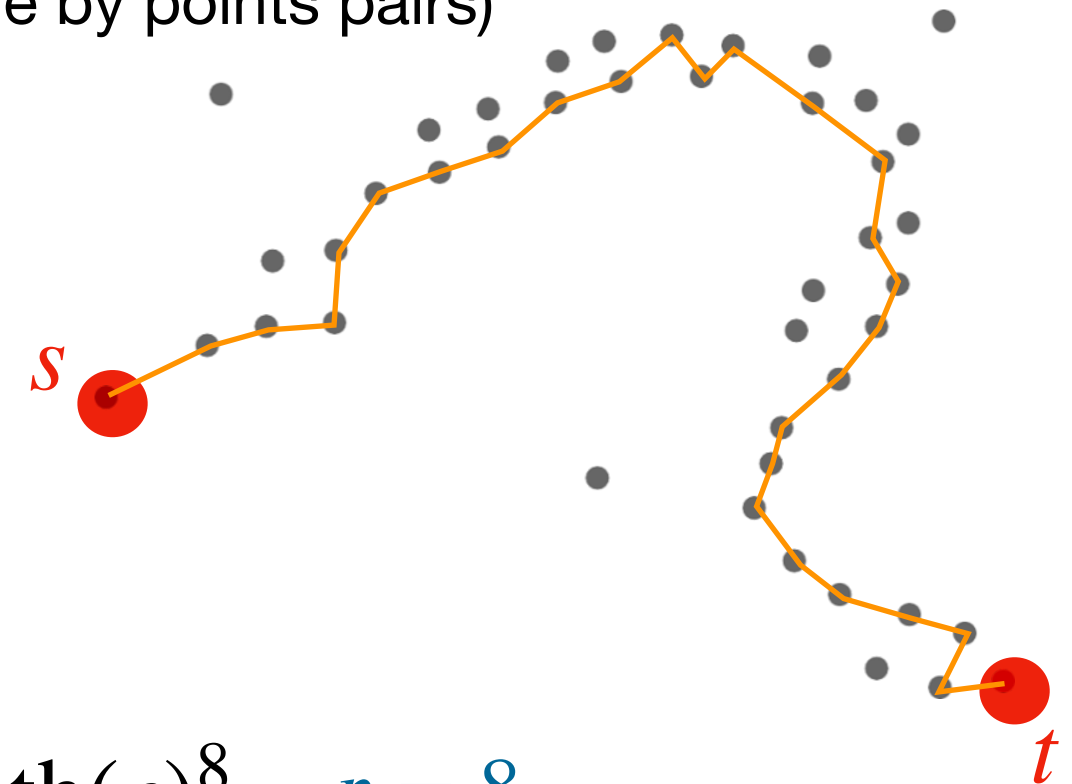


$$\arg\min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e)^4 \qquad p = 4$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

<u>Objective</u>: find path going through "**densest**" parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)

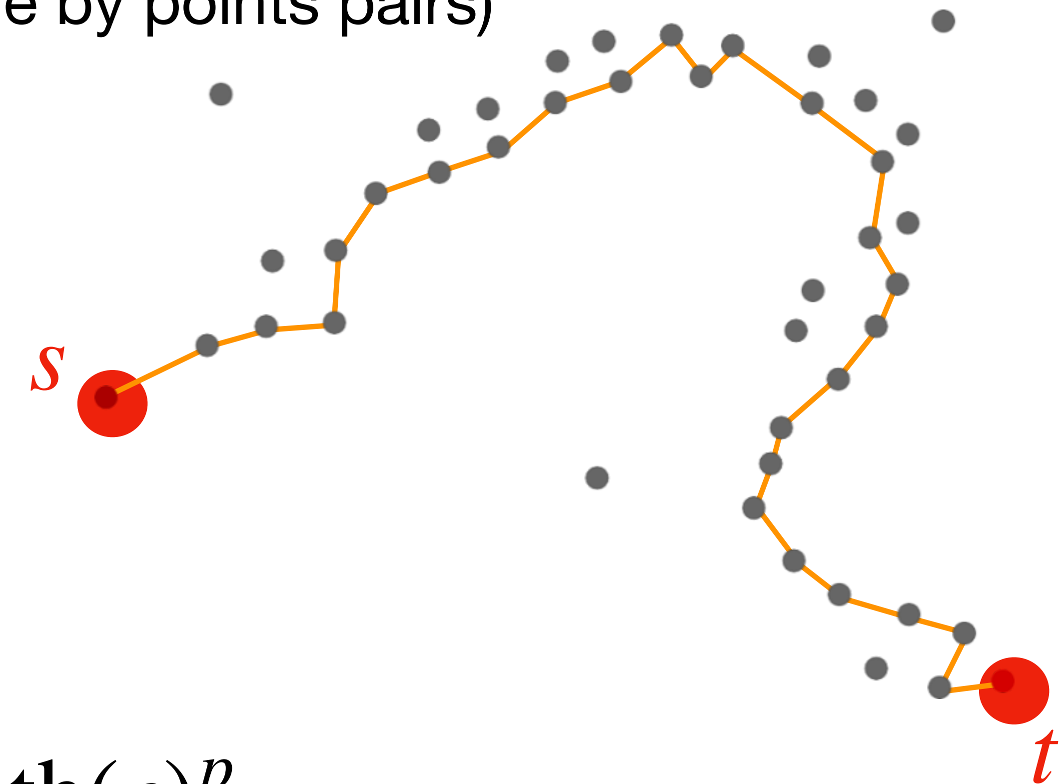


$$\arg\min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e)^8 \qquad p = 8$$

Lexicographic order minimal 1-chain

Connect ~~the~~ some dots to form a path between s and t

Objective: find path going through "**densest**" parts of the point cloud.
1D simplicial complex = **Complete graph** (= one edge by points pairs)



$$\arg\min_{\partial\Gamma = s+t} \sum_{e\in\Gamma} \text{length}(e)^p$$

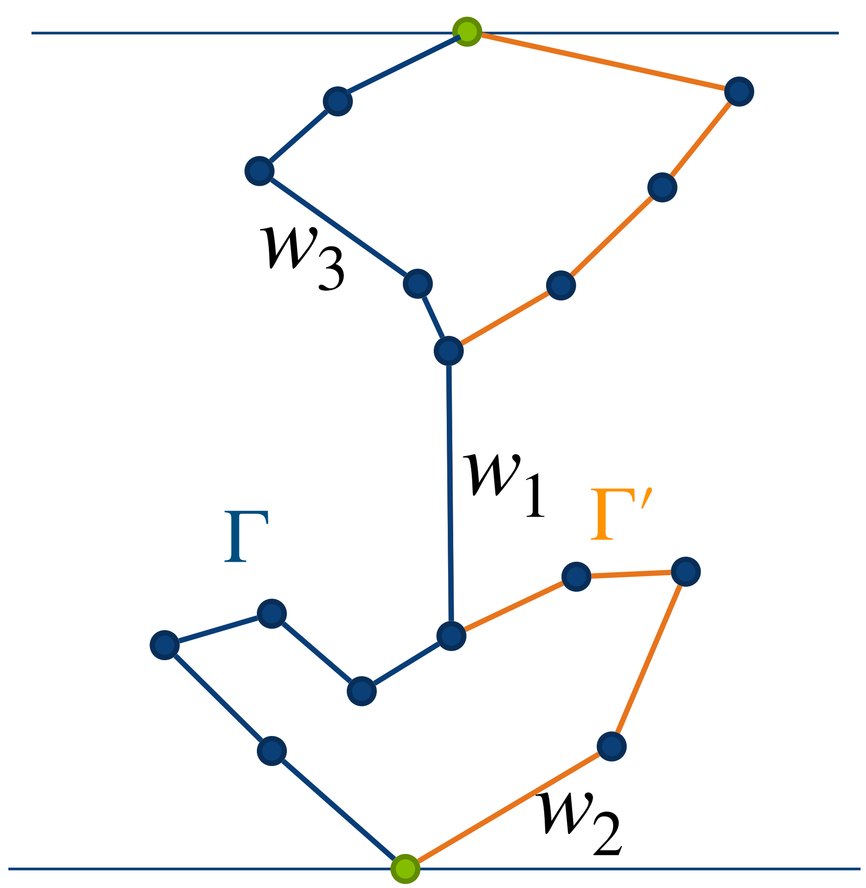
Behavior as $p \to \infty$?

Lexicographic order minimal 1-chain

Limit behavior as $p \to \infty$? : **lexicographic order**

Assume no two edges have same length (generic condition):
Sort edges along decreasing length:

$w_1 > w_2 > \ldots > w_N$, where $w_i = \text{length}(\tau_i)$



$$\exists p \in \mathbb{N}, \forall i, w_i^p > \sum_{j>i} w_j^p$$

$\Gamma = \tau_1 + \tau_3 + \ldots$

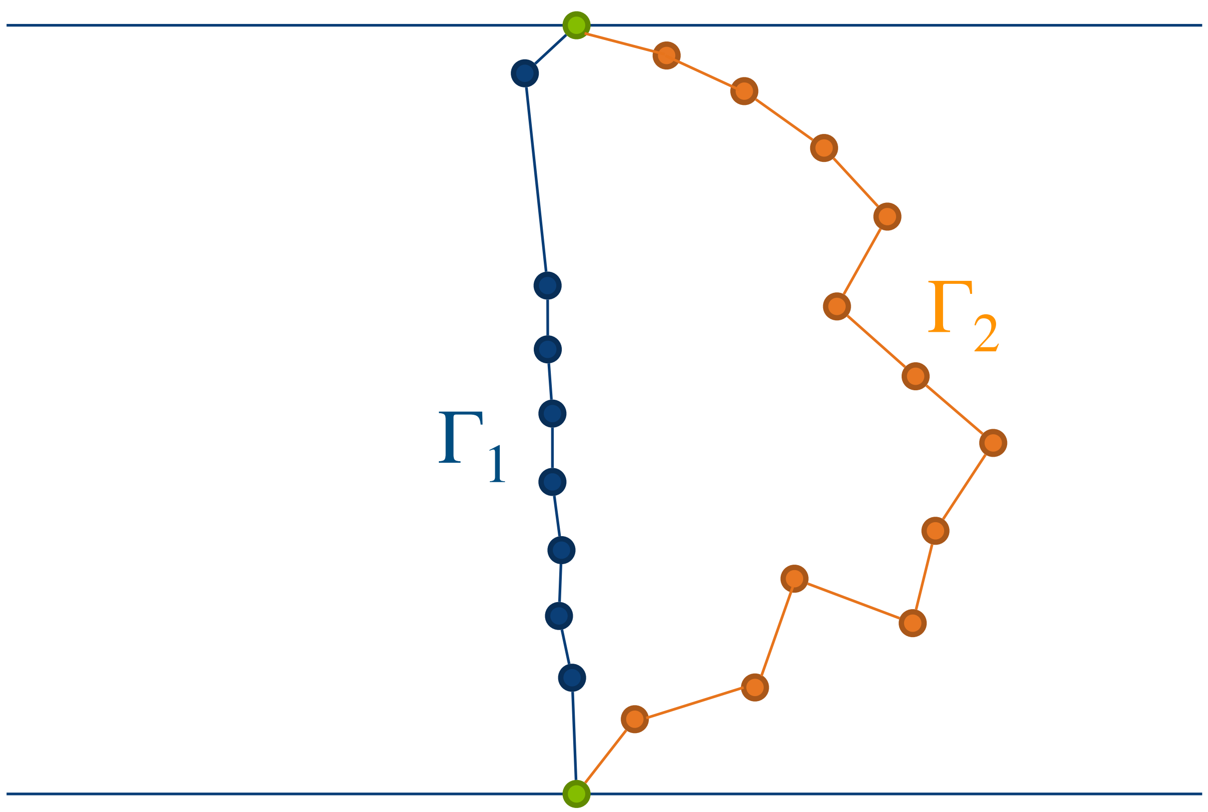
$\Gamma' = \tau_1 + \tau_2 + \ldots$

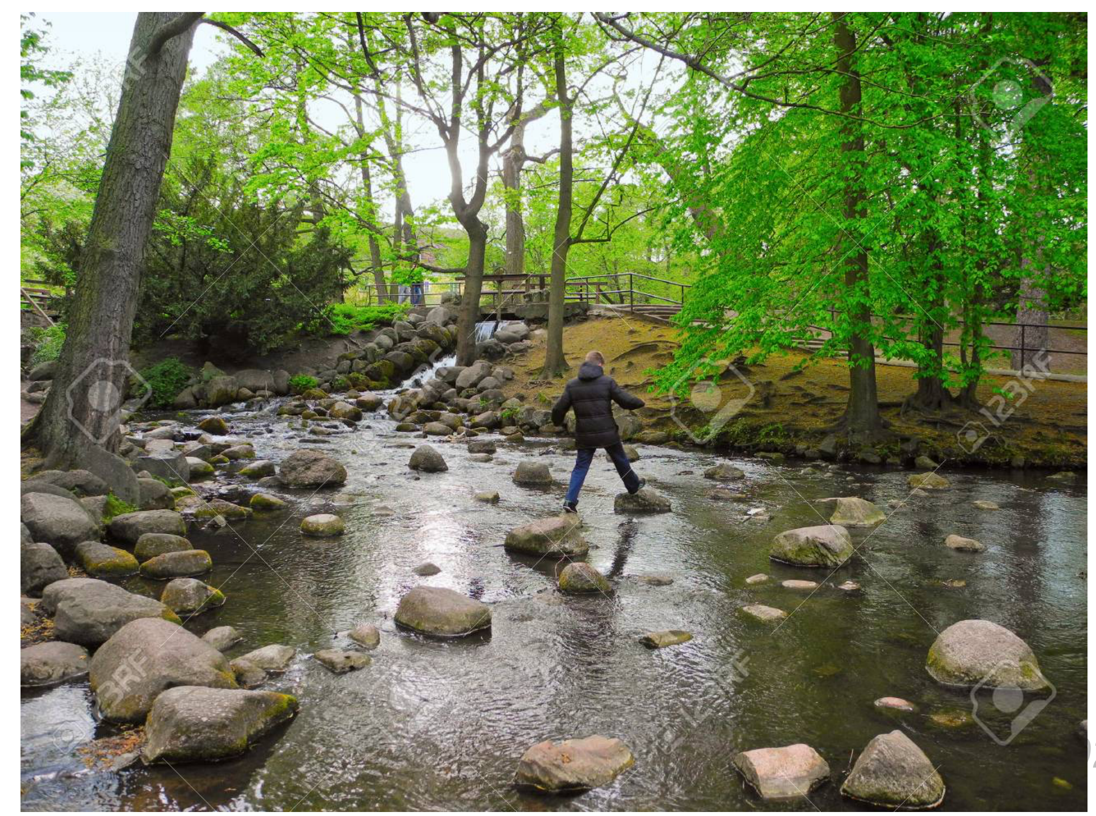
$$\Gamma \sqsubseteq_{lex} \Gamma'$$

Lexicographic order minimal 1-chain

Analogy for lexicographic order: “Rock hopping"

Which path is smaller in the lexicographic order ?





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Lexicographic order minimal 1-chain

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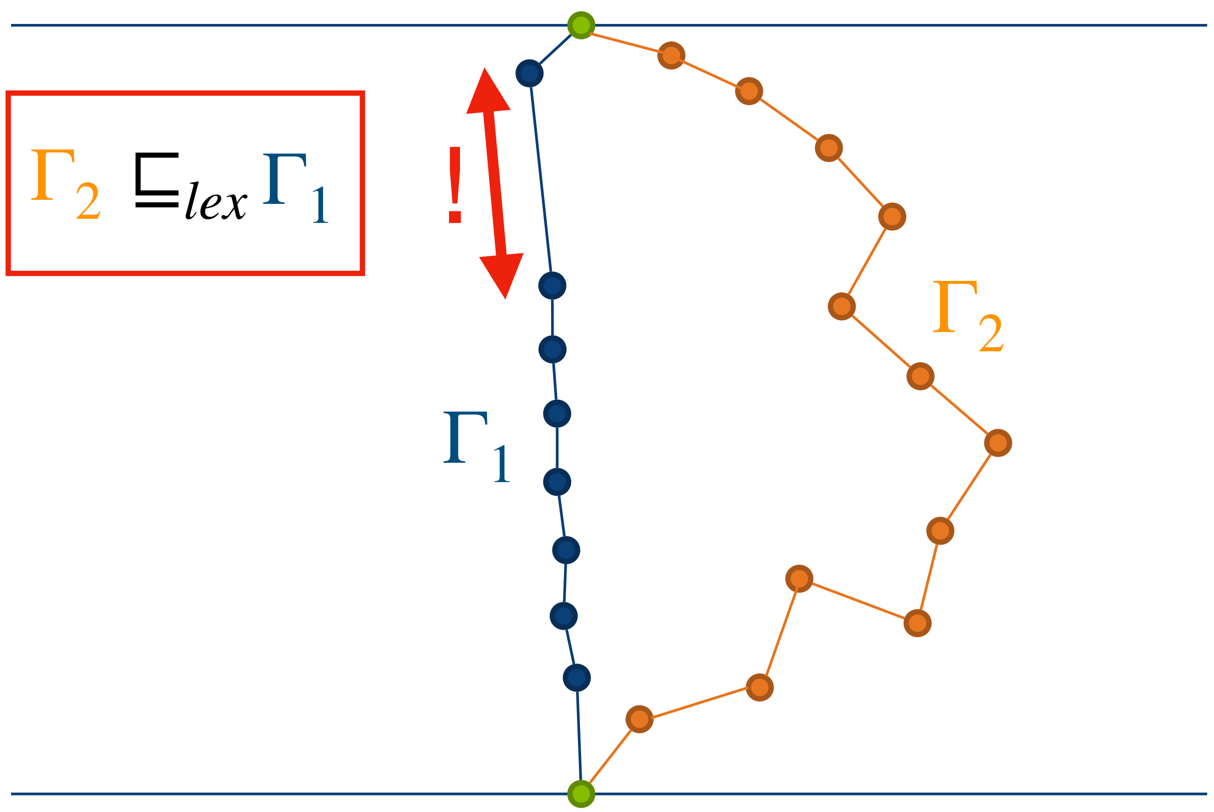

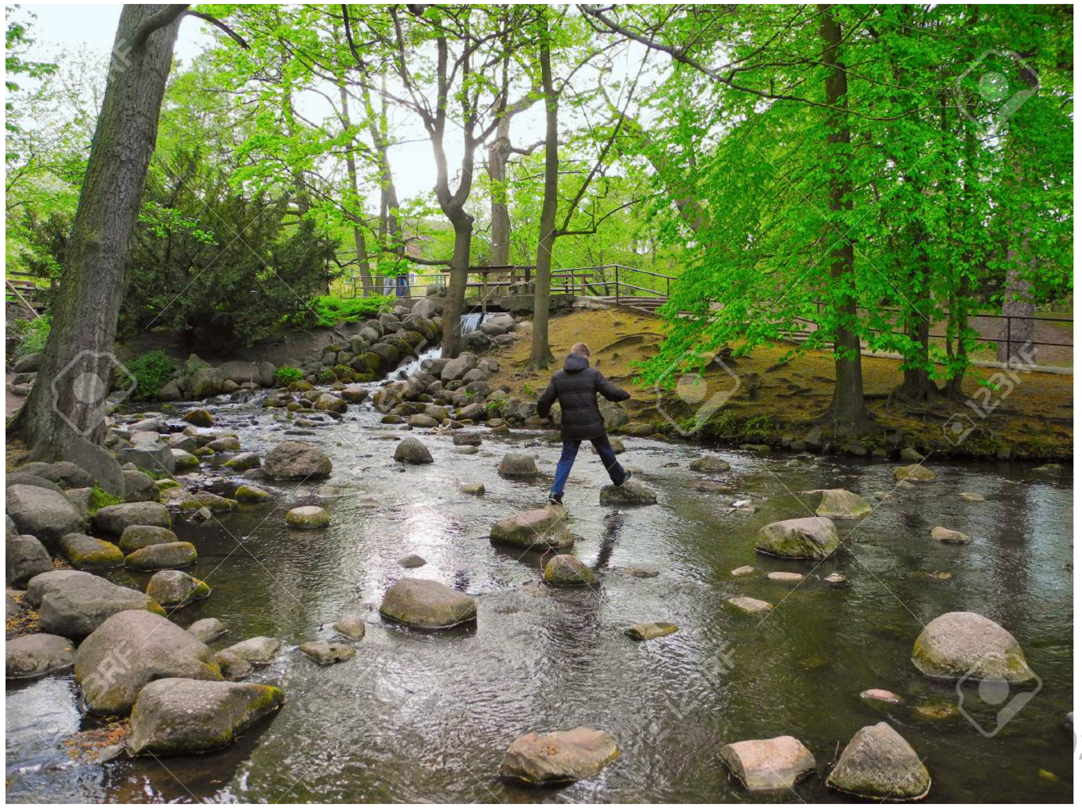

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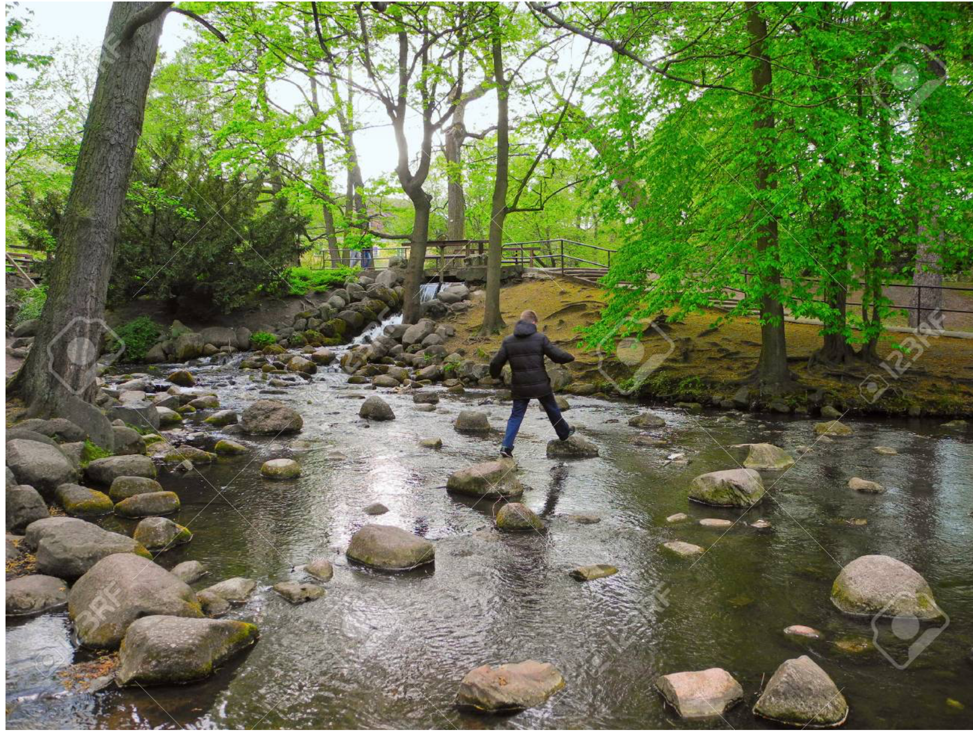
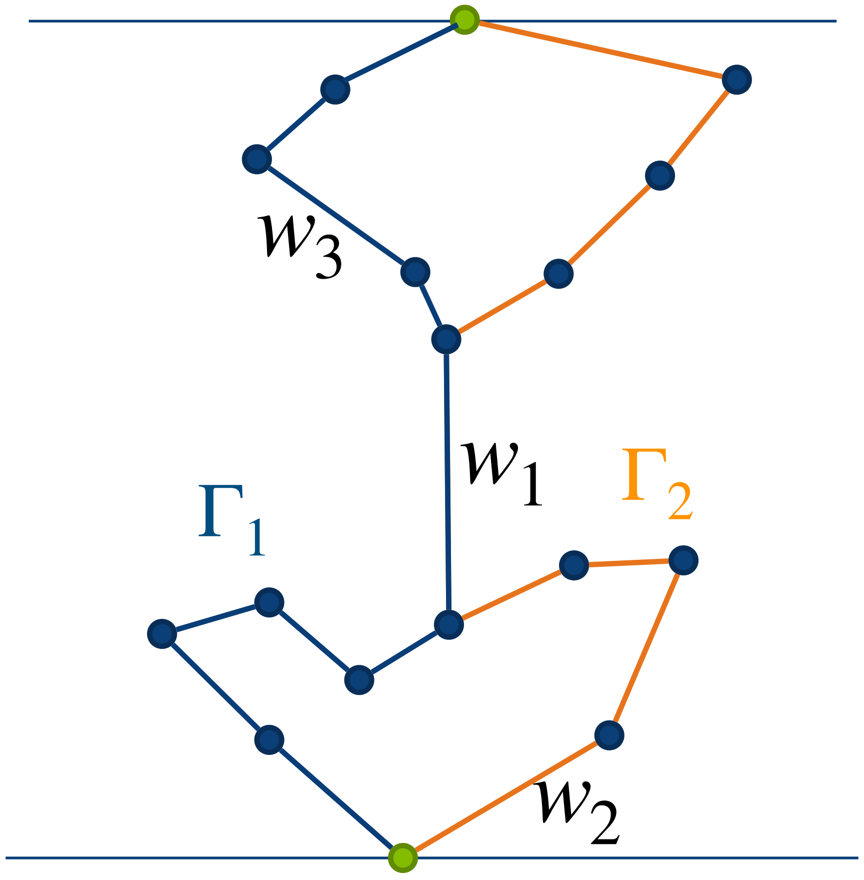
Lexicographic order



$\leq$ defines a **lexicographic order** $\sqsubseteq_{lex}$ on chains:

$$\Gamma_1 \sqsubseteq_{lex} \Gamma_2 \underset{def.}{\Longleftrightarrow} \begin{cases} \Gamma_1 = \Gamma_2 \\ \text{or} \\ \sigma_{\max} = \max\left\{\sigma \in \Gamma_1 - \Gamma_2\right\} \in \Gamma_2 \end{cases}$$

(With coefficients in $\mathbb{Z}_2$, $\Gamma_1 - \Gamma_2$ is the symmetric difference between Γ_1 and Γ_2)

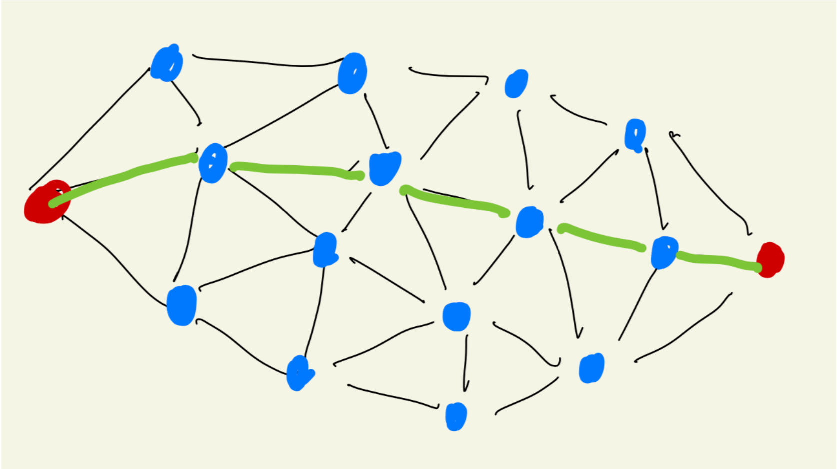


Our two canonical problems again

Lexicographic-minimal chain for a given boundary

Given $\beta \in C_{d-1}(K, \mathbb{Z}_2)$ **find:**

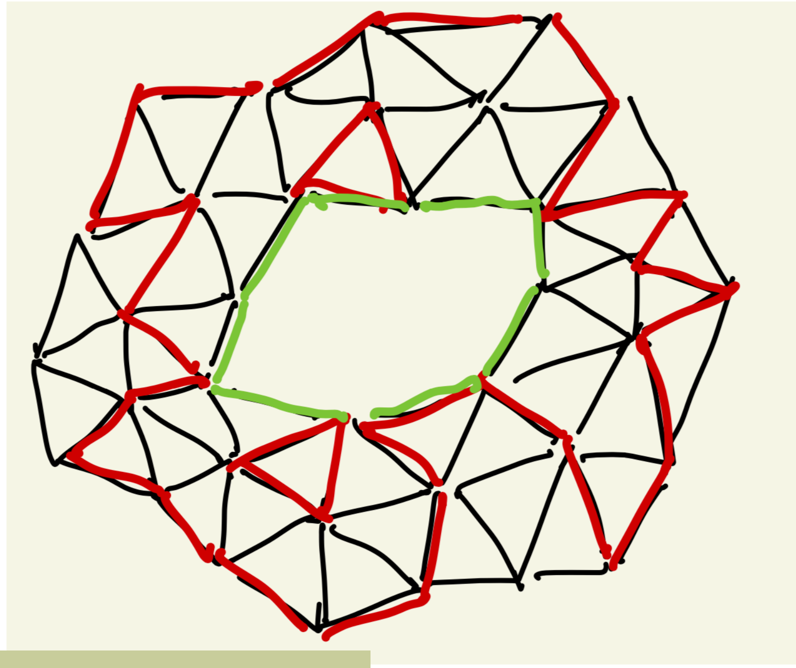
$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\Gamma \in C_d(K, \mathbb{Z}_2), \partial\Gamma = \beta\}$$



Lexicographic-minimal homologous chain:

Given $\alpha \in C_d(K, \mathbb{Z}_2)$ **find:**

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\alpha + \partial\omega, \omega \in C_{d+1}(K, \mathbb{Z}_2)\}$$



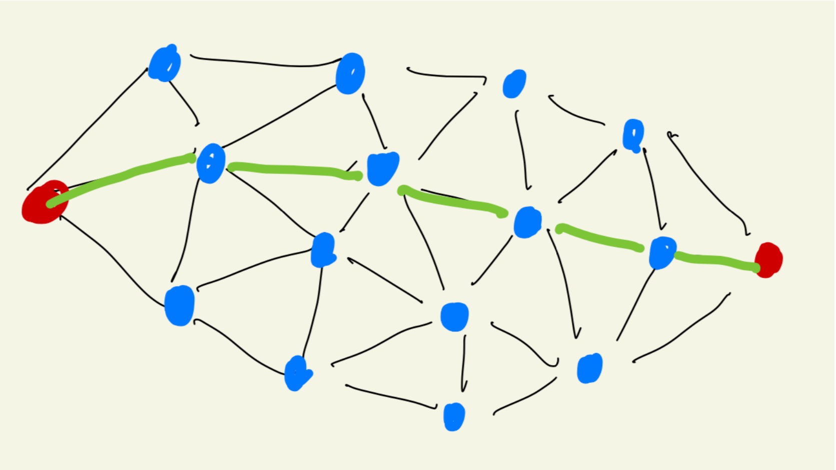
Both problem can be solved in less than $\mathcal{O}(n^3)$ time complexity

Our two canonical problems again

Lexicographic-minimal chain for a given boundary

Given $\beta \in C_{d-1}(K, \mathbb{Z}_2)$ **find:**

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}}\{\Gamma \in C_d(K, \mathbb{Z}_2), \partial\Gamma = \beta\}$$



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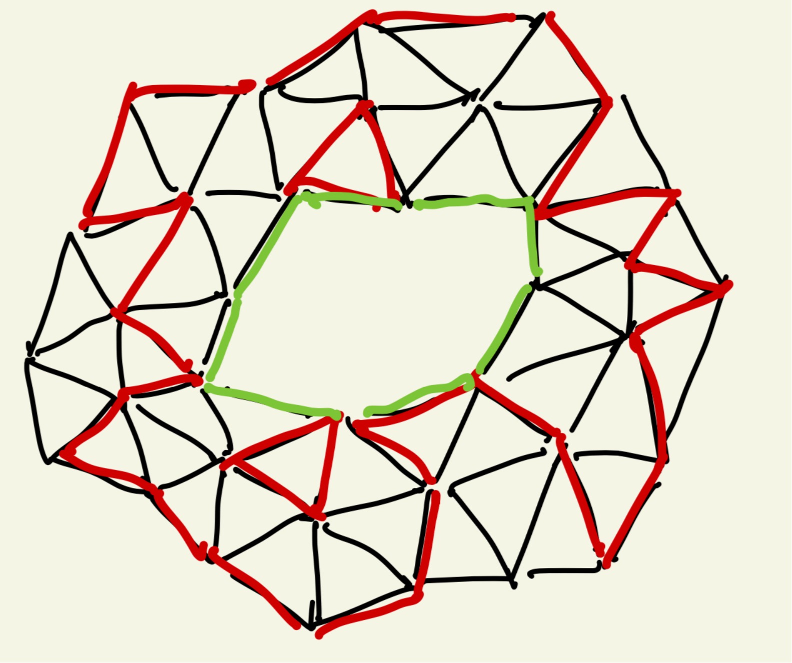

$\beta \mapsto \Gamma_{\min}$ and $\alpha \mapsto \Gamma_{\min}$ are **linear maps**, (as for L^2 minima) but minima are **sparse** (as for L^1 minima).

$\mathcal{O}(n^3)$ general algorithm

Lexicographic-minimal homologous chain:

Given $\alpha \in C_d(K, \mathbb{Z}_2)$ **find:**

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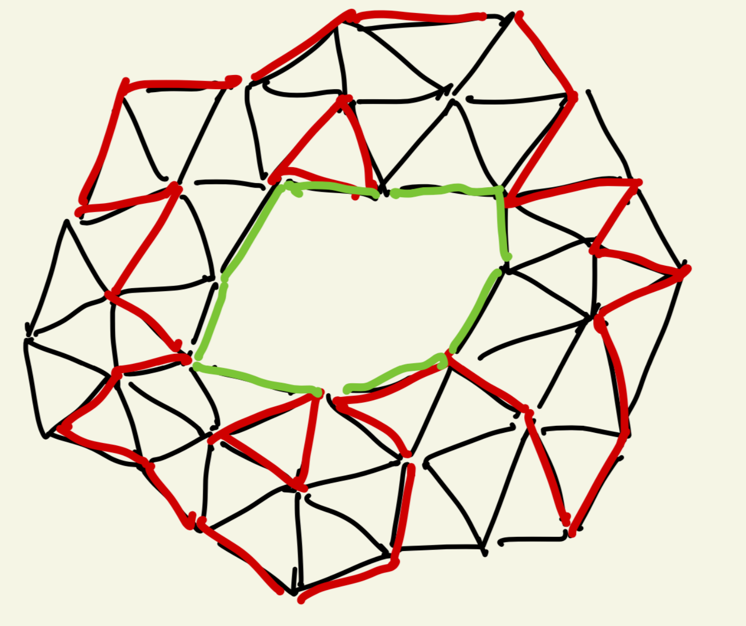

A chain Γ' is said to be a **reduction** of a chain Γ if:

Γ' **is homologous to** Γ **and** $\Gamma' <_{lex} \Gamma$

$\mathcal{O}(n^3)$ general algorithm

Lexicographic-minimal homologous chain:

Given $\alpha \in C_d(K, \mathbb{Z}_2)$ **find:**

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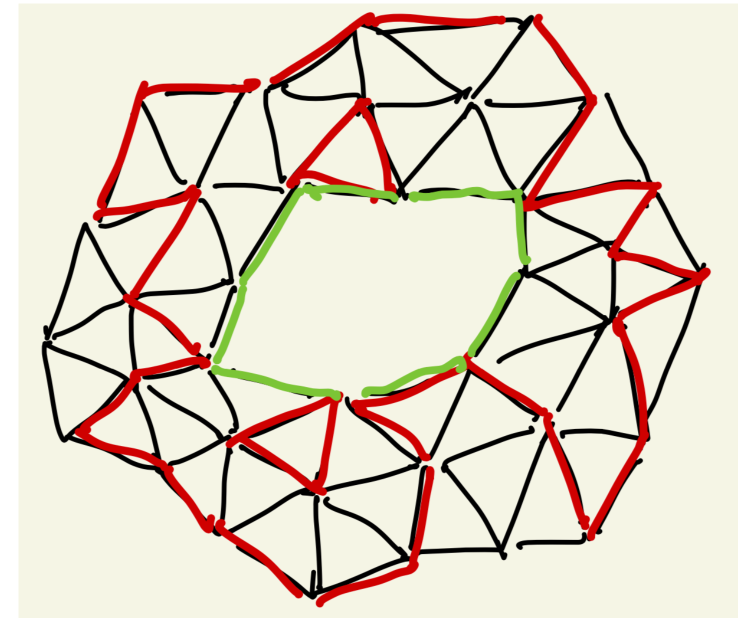

$(d+1)$-simplices

$$\partial_{d+1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

d-simplices ordered along increasing $\leq$

$\mathcal{O}(n^3)$ general algorithm

$$\partial_{d+1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} = \mathbf{R} \, . \, \mathbf{V}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In $\mathbf{R}$, there is exactly one column with a lowest 1 for each reducible simplex 1

Same as **Homological persistence**

Algorithm 1: Reduction algorithm for the ∂_{d+1} matrix

R $= \partial_{d+1}$
for $j \leftarrow 1$ **to** n **do**
  while $R_j \neq 0$ **and** $\exists j_0 < j$ *with* $\text{low}(j_0) = low(j)$ **do**
    $R_j \leftarrow R_j + R_{j_0}$
  end
end

$\mathcal{O}(n^3)$ general algorithm

$\partial_{d+1} = \mathbf{R} \,.\, \mathbf{V}$

$+$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \Gamma_0 = \alpha = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

In $\mathbf{R}$, there is exactly one column with a lowest 1 for each reducible simplex 1

Total reduction of Γ using the reduced boundary operator $\mathbf{R}$

Algorithm 2: Total reduction algorithm

Inputs: A d-chain Γ, the reduction matrix R from Algorithm 1

for $i \leftarrow m$ **to** 1 **do**
  if $\Gamma[i] \neq 0$ **and** $\exists j \in [1, n]$ *with* $\mathrm{low}(j) = i$ *in* R **then**
    $\Gamma \leftarrow \Gamma + R_j$
  end
end

$\mathcal{O}(n^3)$ general algorithm

$$\partial_{d+1} = \mathbf{R}\,.\,\mathbf{V}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \Gamma_0 = \alpha = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \Gamma_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$+$

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    $\Gamma \leftarrow \Gamma + R_j$
  end
end

$\mathcal{O}(n\alpha(n))$ algorithm in co-dimension 1

Lexicographic-minimal homologous chain:

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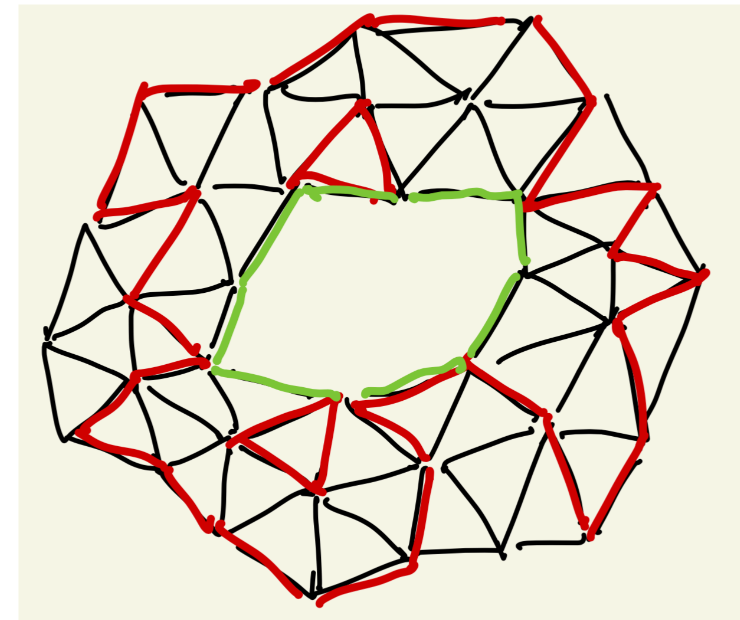
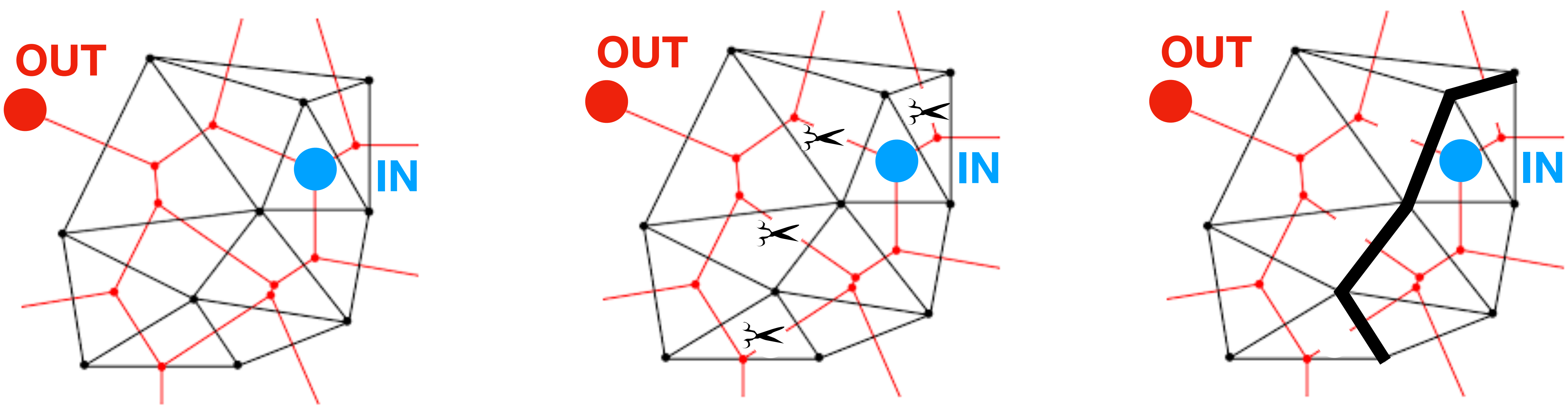


Once d-simplices are **sorted** (in time $\mathcal{O}(n \log n)$):

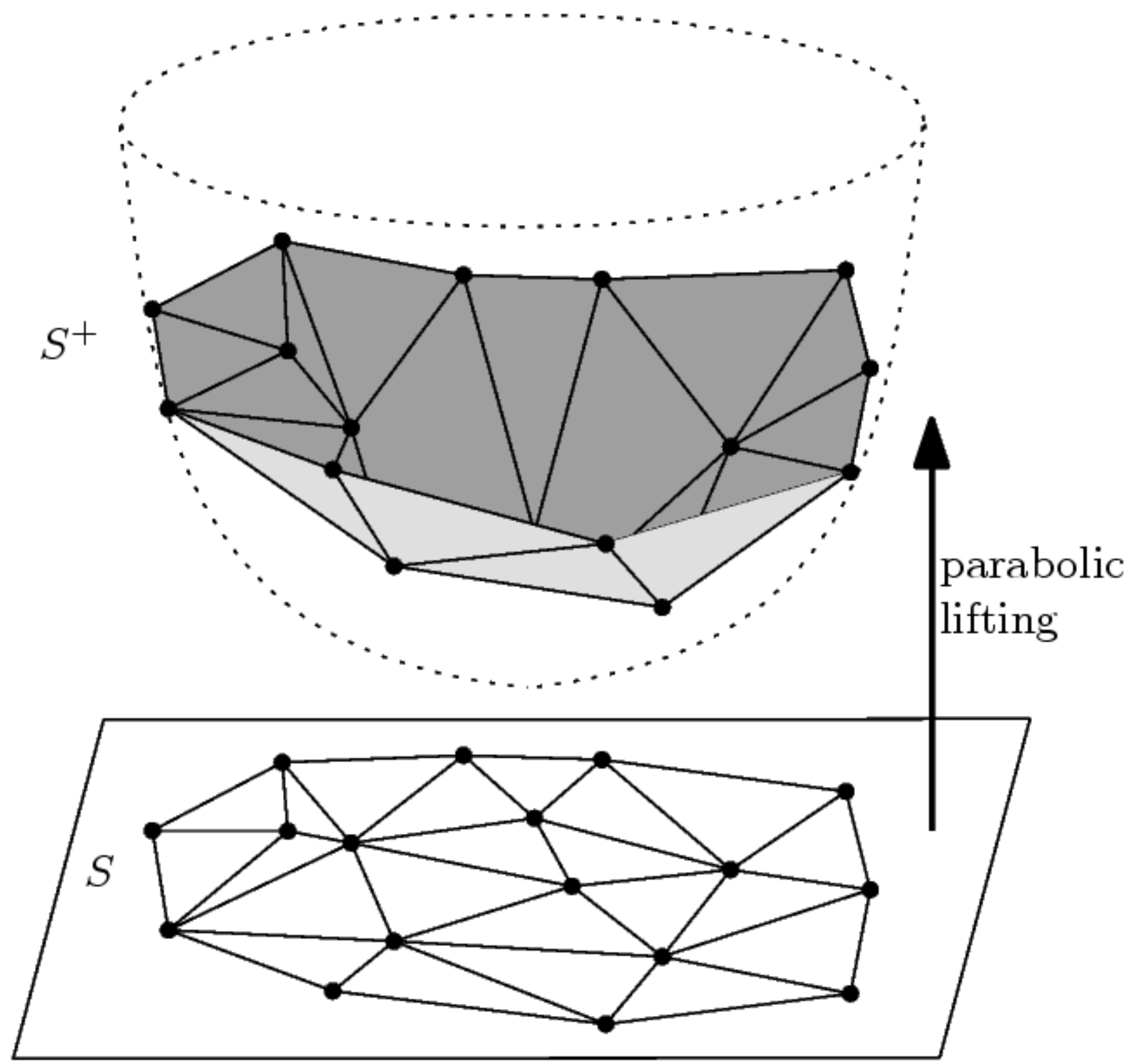
$\mathcal{O}(n\,\alpha(n))$ algorithm using **union-find** data structure on the **dual graph** to solve a **lexicographic MIN-CUT/MAX-FLOW** problem.



Delaunay as linear programming

When L¹ minimal chain is Delaunay

For each point $u \in \mathbb{R}^n$, we consider its *lifted image* $\hat{u} = (u, \|u\|^2) \in \mathbb{R}^{n+1}$. A classical result says that σ is a Delaunay n-simplex of P if and only if $\hat{\sigma}$ spans an n-face of the lower convex hull of $\hat{P}$.



Delaunay as linear programming

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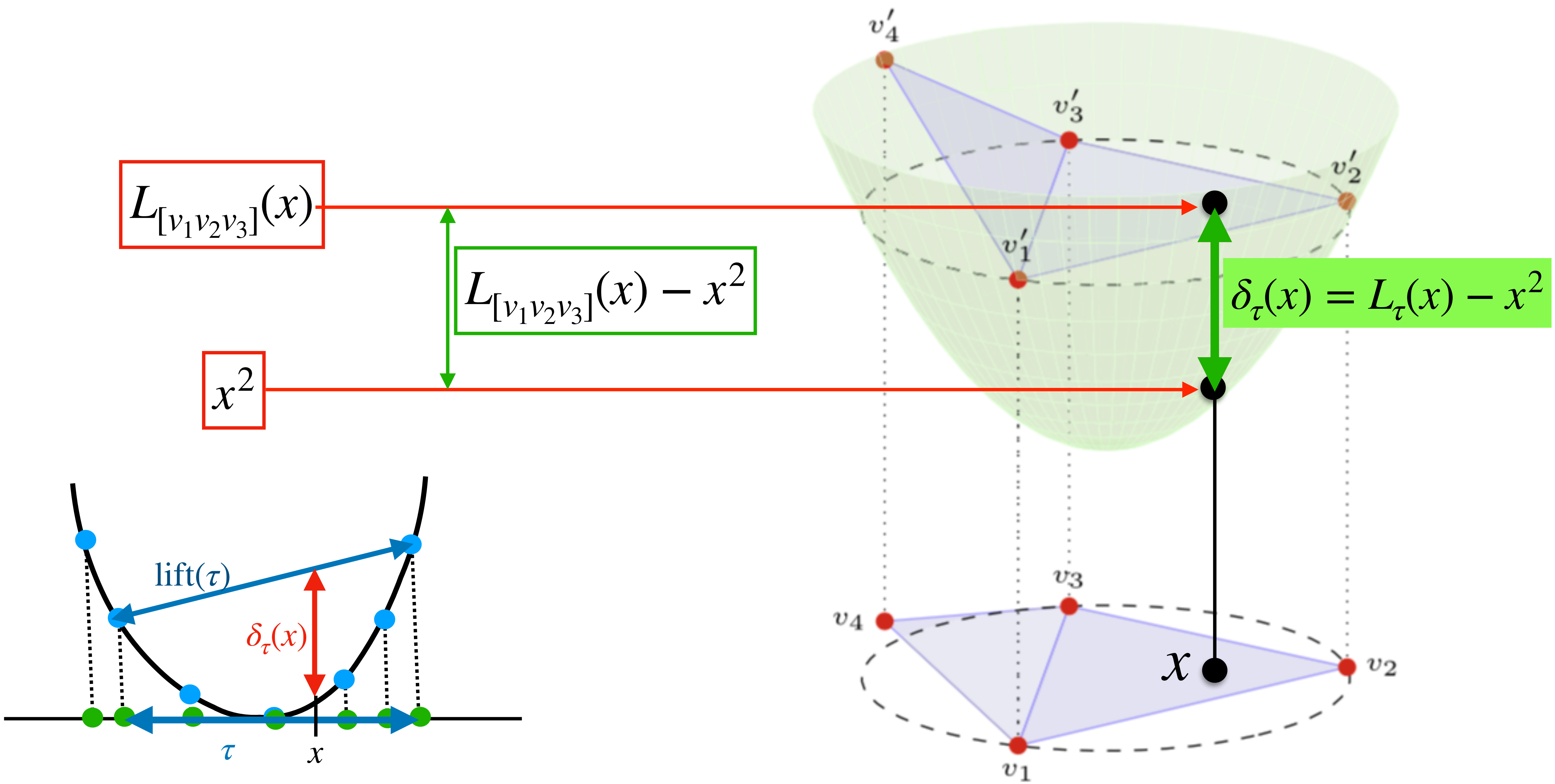


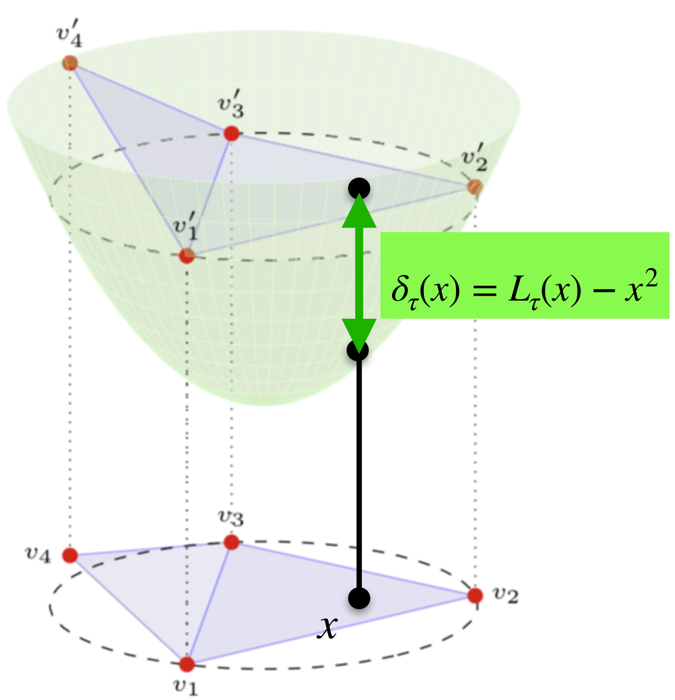
Delaunay as linear programming

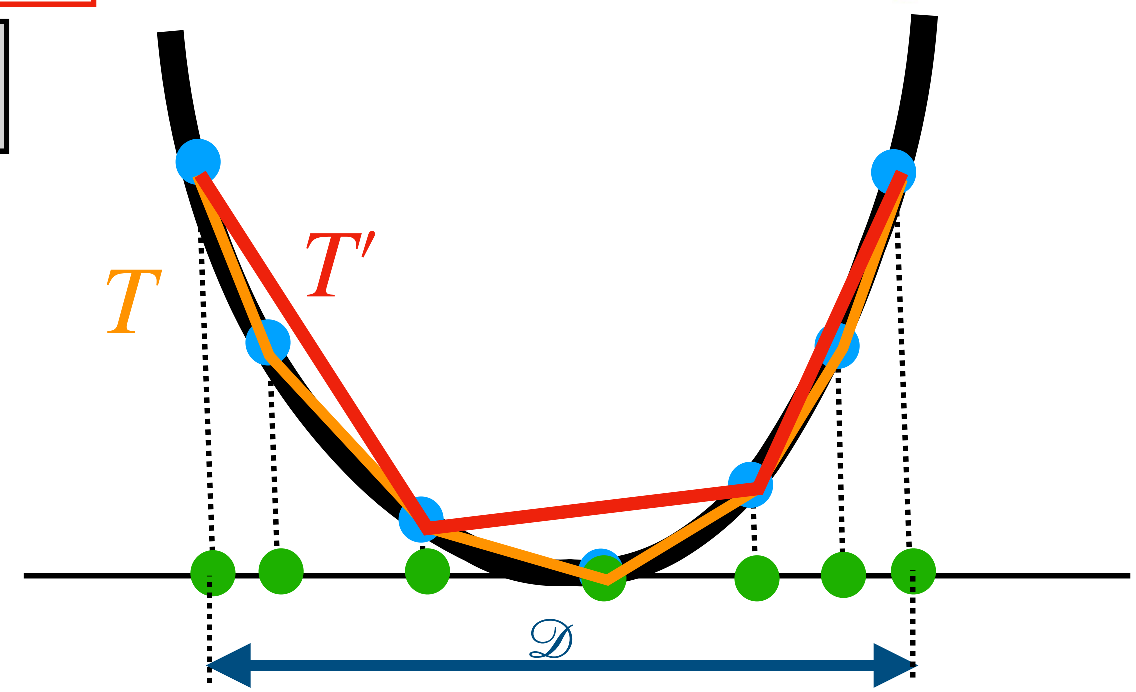
When L¹ minimal chain is Delaunay

Triangulation T is Delaunay iff.:

$$\forall T', \left(\int_{\mathscr{D}} \delta_T(x)^p dx\right)^{1/p} \leq \left(\int_{\mathscr{D}} \delta_{T'}(x)^p dx\right)^{1/p}$$

Long Chen and Jin-chao Xu. Optimal delaunay triangulations. *Journal of Computational Mathematics*, pages 299–308, 2004.





T minimum along the T' that triangulates $\mathscr{D}$

Delaunay as linear programming

When L^1 minimal chain is Delaunay

Triangulation T is Delaunay iff.:

$$\forall T', \sum_{\tau \in T} w_p(\tau)^p \leq \sum_{\tau \in T'} w_p(\tau)^p$$

$$w_p(\tau) = \left(\int_{|\tau|} \delta_\tau(x)^p dx \right)^{\frac{1}{p}}$$

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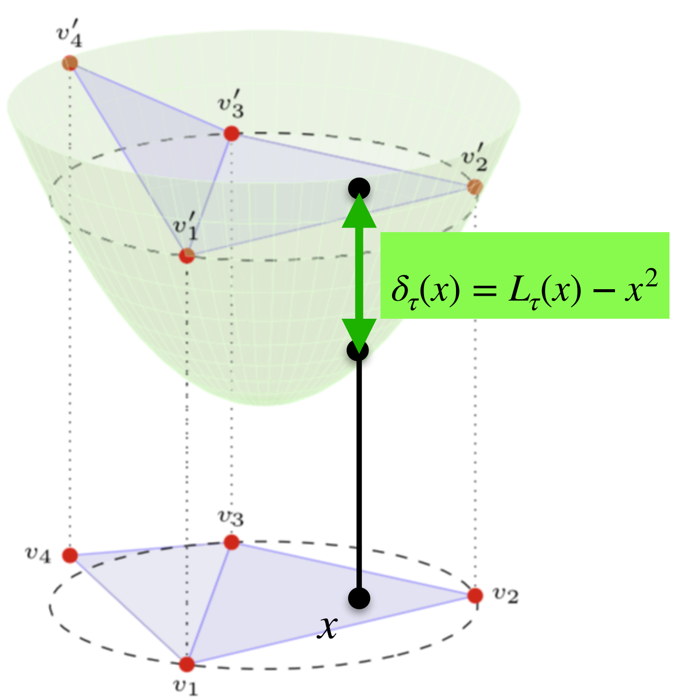

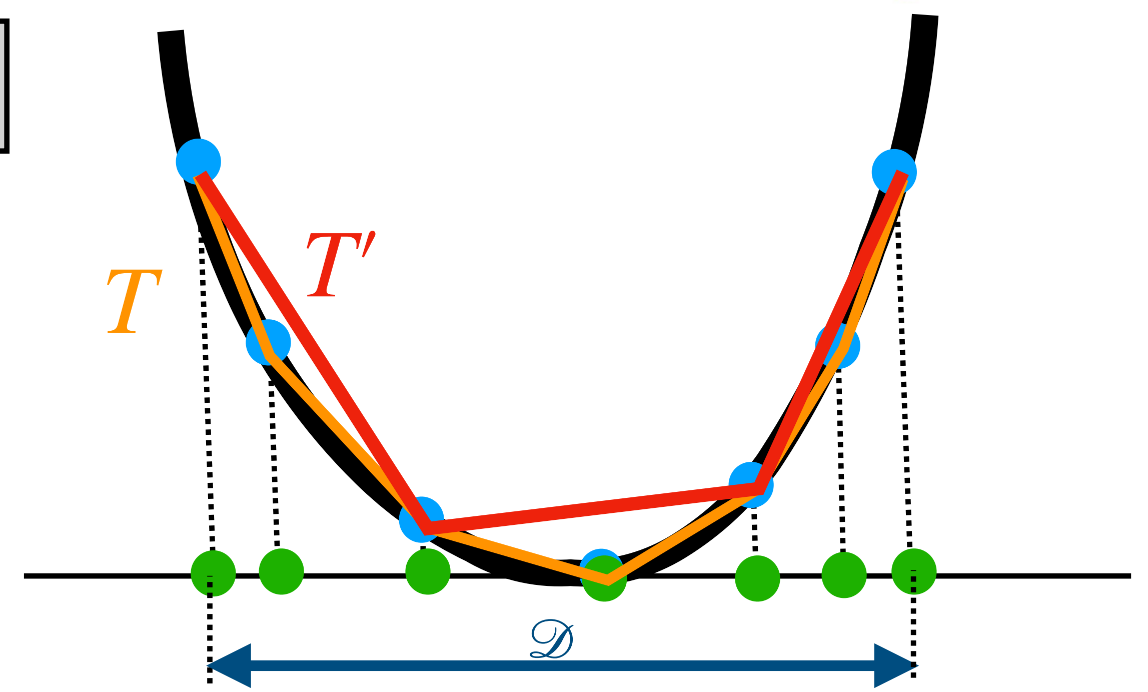

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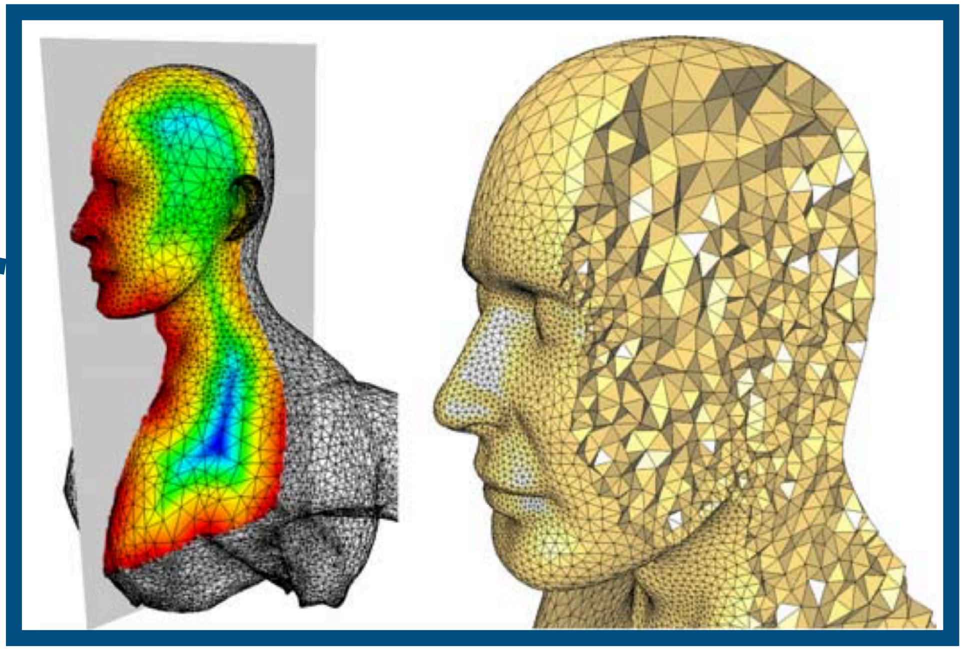
Variational definition of Delaunay
=> triangulation optimization :

Pierre Alliez, David Cohen-Steiner, Mariette Yvinec, and Mathieu Desbrun. Variational tetrahedral meshing. *ACM Transactions on Graphics (TOG)*, 24(3):617–625, 2005.

L. Chen and M. Holst. Efficient mesh optimization schemes based on optimal delaunay triangulations. *Computer Methods in Applied Mechanics and Engineering*, 200(9):967–984, 2011.



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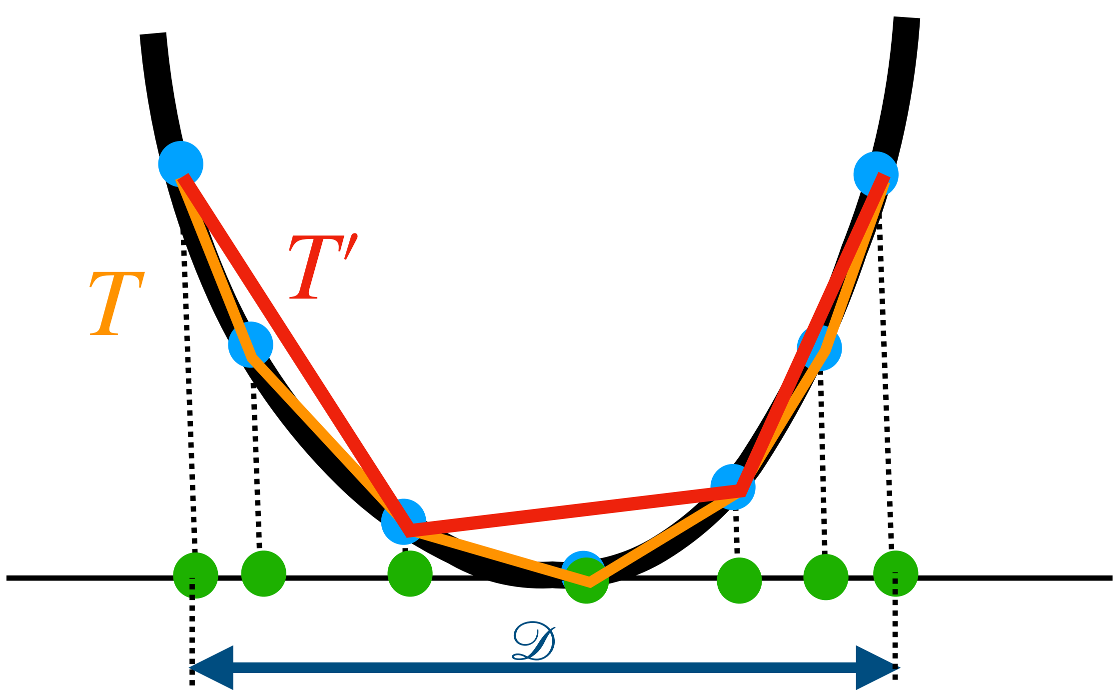

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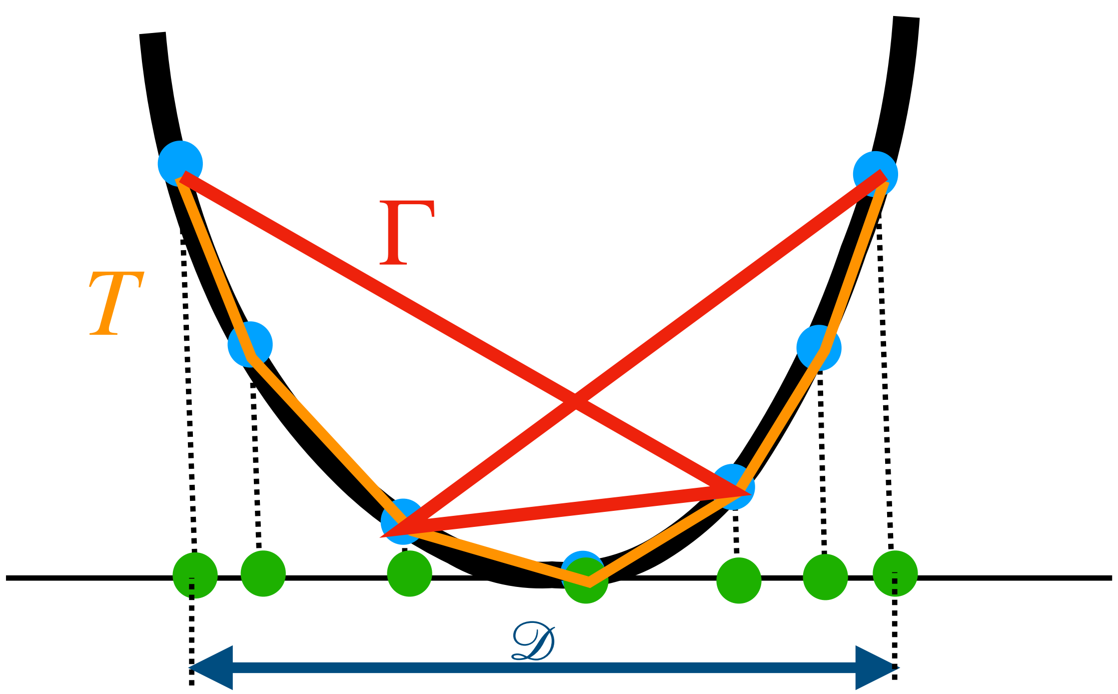

T minimum along the chains Γ such that $\partial\Gamma = \partial\mathscr{D}$

Delaunay as linear programming

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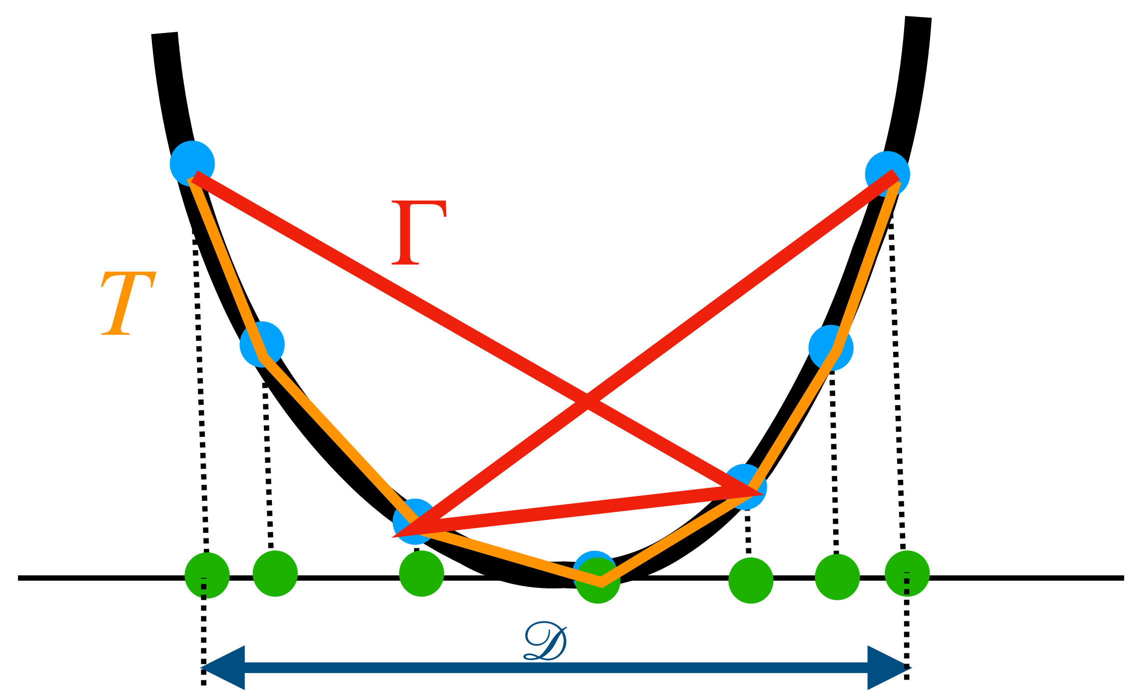
Define the following norm on chains:

$$\|\Gamma\|_p = \sum_{\sigma \in K_d} w_p(\tau)^p \, |\, \Gamma(\tau) \,|$$

Still a L^1 norm : exponent p is on the weight, not on the coordinate.



T minimum along the chains Γ such that $\partial\Gamma = \partial\mathscr{D}$

Delaunay as linear programming

Delaunay triangulation

$$\|\Gamma\|_p = \sum_{\sigma \in K_d} w_p(\tau)^p \, |\Gamma(\tau)|$$

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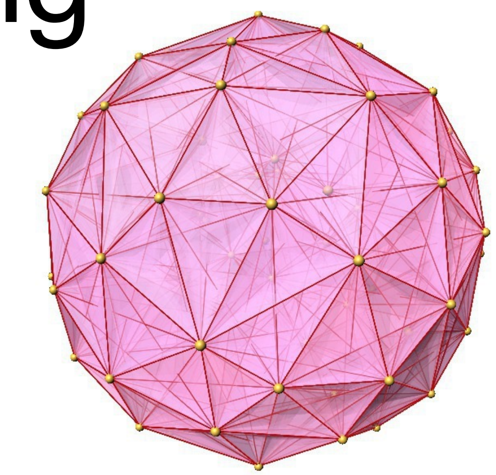
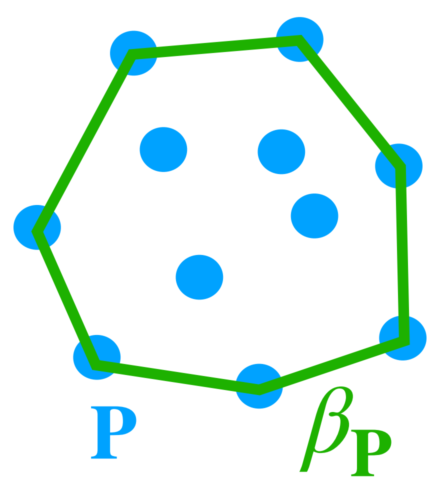


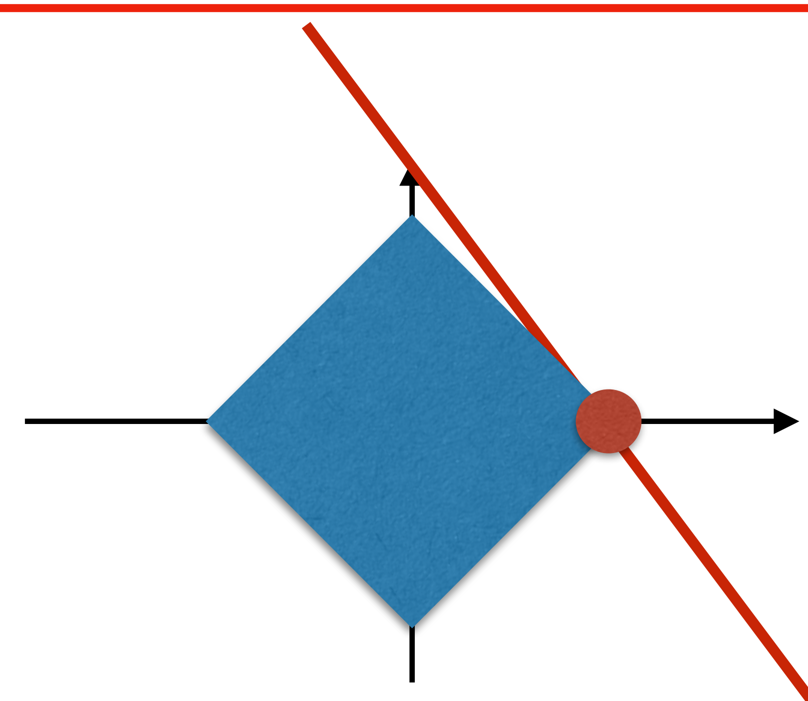
(Attali, L., 2016)

Let $\mathbf{P} \subset \mathbb{R}^d$ *be a finite set of points.*

Let $\beta_{\mathbf{P}}$ *be a* ***cycle*** *whose* ***support*** *is the boundary of the convex hull of* $\mathbf{P}$

The support of the chain that minimizes $\Gamma \mapsto \|\Gamma\|_p$ *under constraint* $\partial\Gamma = \beta_{\mathbf{P}}$ *is the* ***Delaunay triangulation*** *of* $\mathbf{P}$





Delaunay as linear programming

2-manifolds and ***perturbed*** *d*-manifolds:

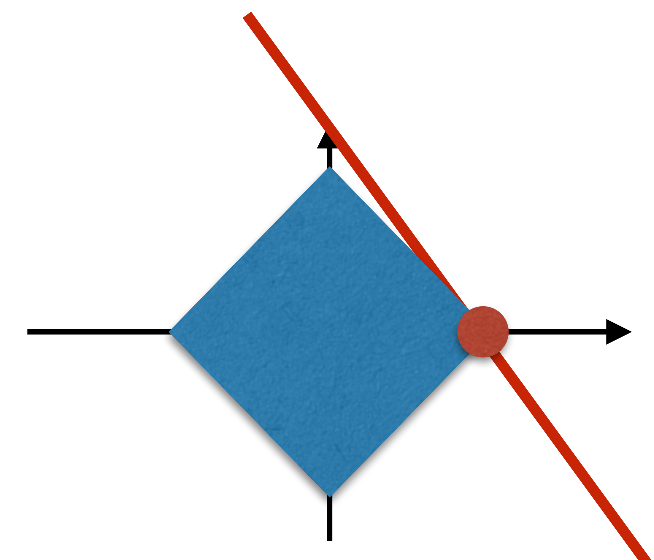
$$\|\Gamma\|_p = \sum_{\sigma \in K_d} w_p(\tau)^p \, |\Gamma(\tau)|$$

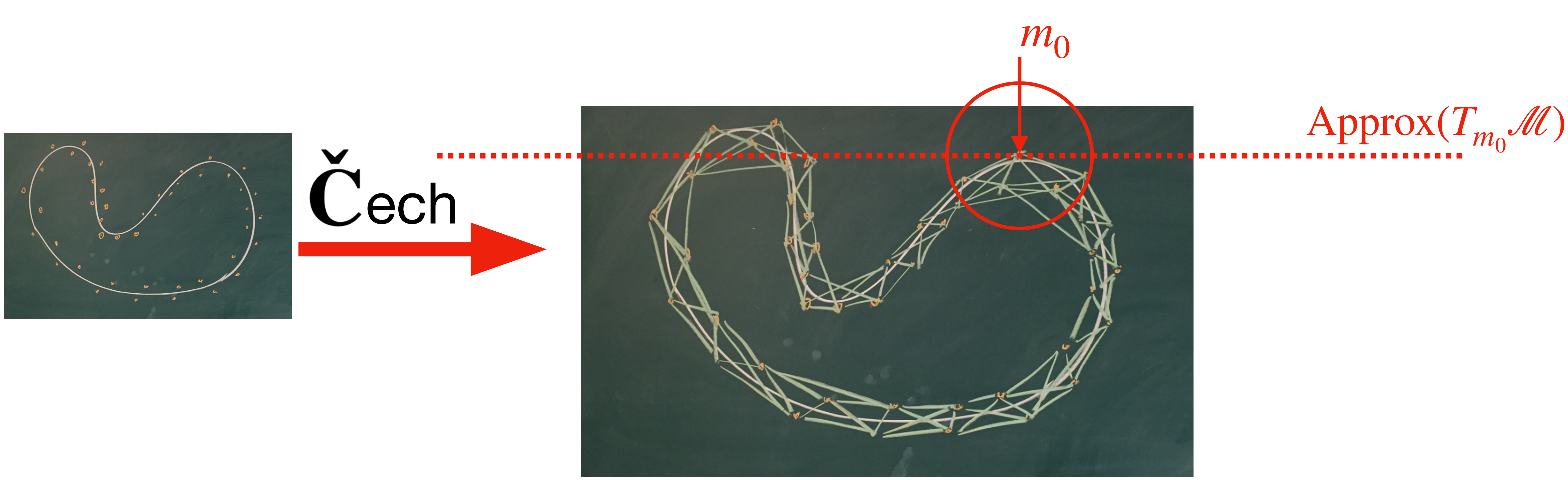
(Attali, Dominique, and A. L. "Delaunay-Like Triangulation of Smooth Orientable Submanifolds by ℓ1-Norm Minimization. » 2022)

The support of the chain that minimizes $\Gamma \mapsto \|\Gamma\|_1$

under constraint $\begin{cases} \partial \Gamma = 0 \\ \text{load}_{m_0,\ \text{Approx}(T_{m_0}\mathcal{M})} = 1, \end{cases}$

triangulates *the manifold.*





Delaunay as linear programming

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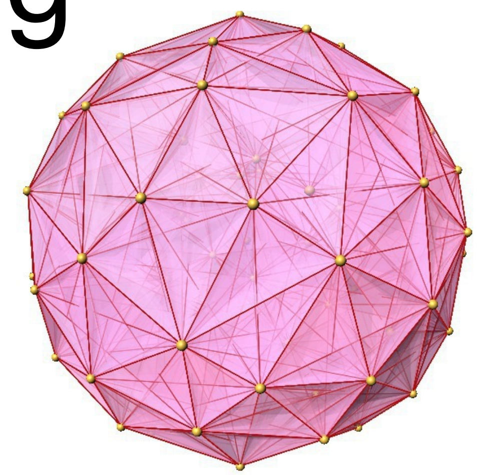

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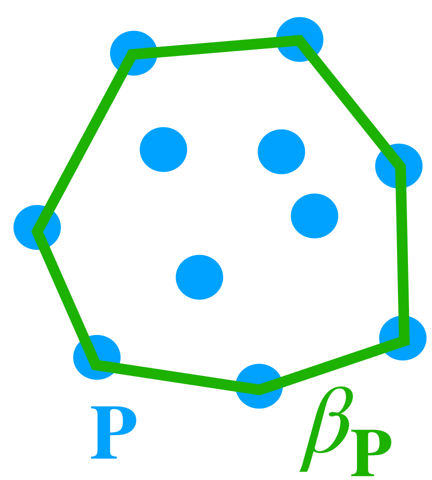

Behavior as $p \to \infty$ **?**

Delaunay order

When lexicographic-minimal chain is Delaunay

Behavior as $p \to \infty$?

$$w_p(\sigma) = \left(\int_{|\sigma|} \delta_\sigma(x)^p dx \right)^{\frac{1}{p}} = \|\delta_\sigma\|_p$$

The weights w_p defines **a preorder $\leq_\infty$ on simplices**:

$$\sigma_1 \leq_\infty \sigma_2 \underset{def.}{\iff} \quad \exists p \in [1, \infty[, \quad \forall p' \in [p, \infty[, \quad w_{p'}(\sigma_1) \leq w_{p'}(\sigma_2)$$

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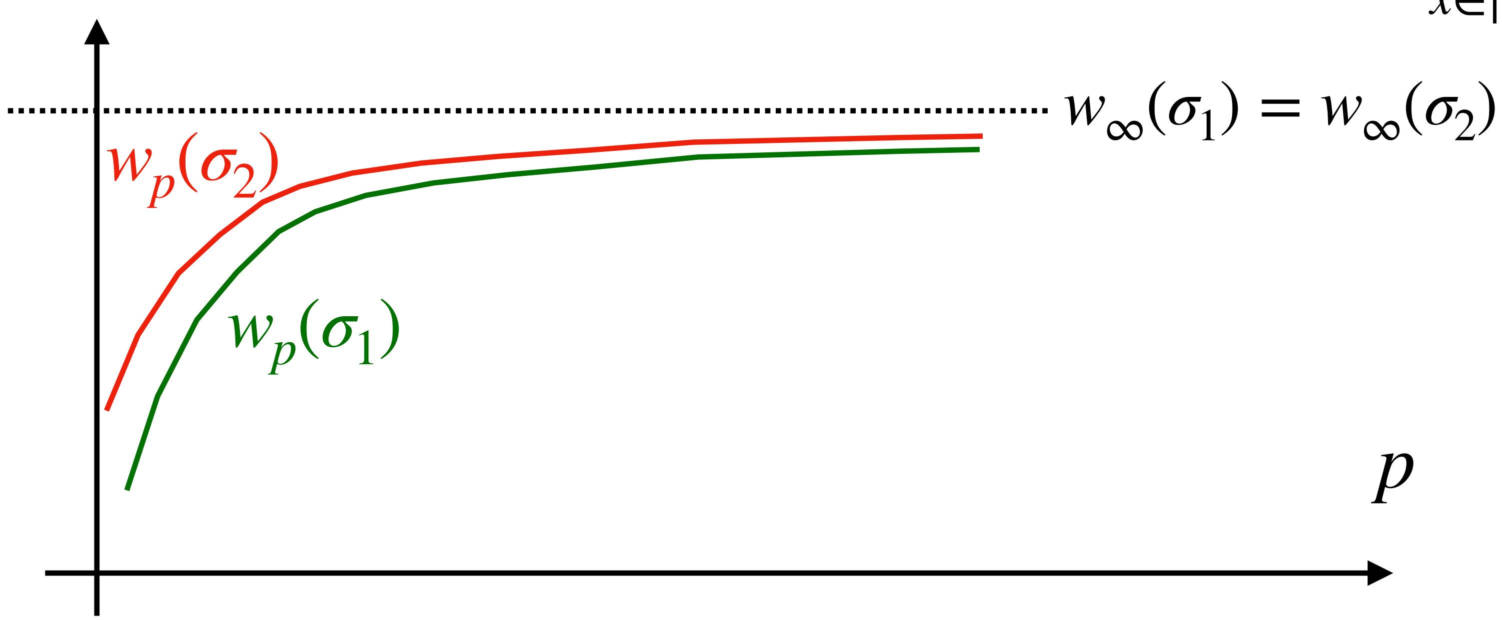
$\leq_\infty$ is a **finer (pre-)order** than comparing $w_\infty = \|\delta_\sigma\|_\infty = \max_{x \in |\sigma|} \delta_\sigma(x) = \lim_{p \to \infty} w_p$



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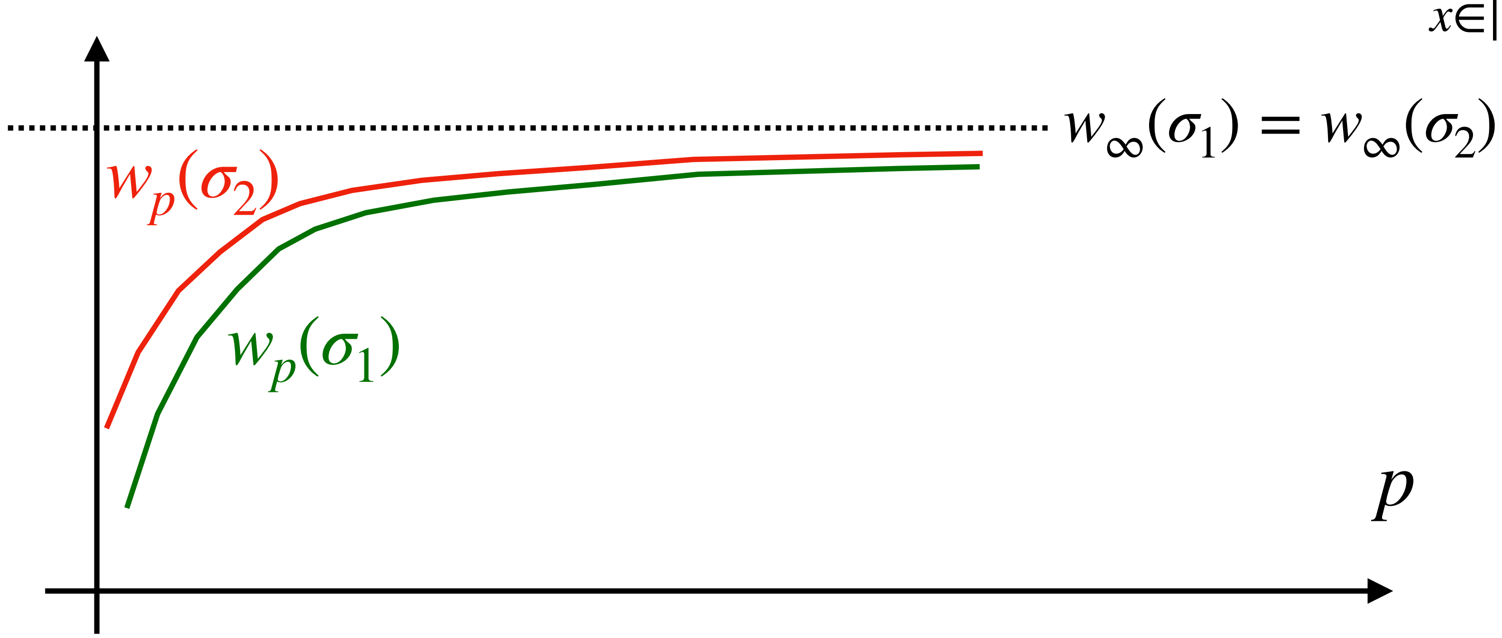

$w_\infty(\sigma_1) = w_\infty(\sigma_2)$
but
$\sigma_1 \leq_\infty \sigma_2$
and
$\sigma_2 \not\leq_\infty \sigma_1$

Delaunay order

When lexicographic-minimal chain is Delaunay

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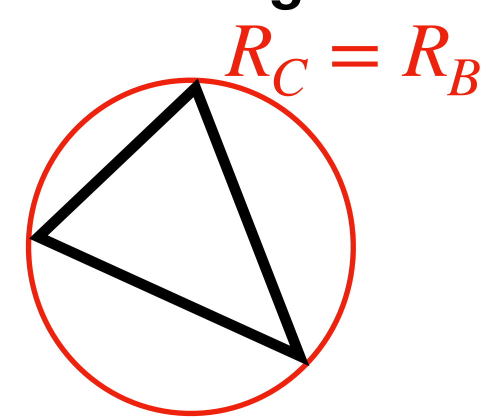
For 2-simplices, under a **generic** condition, one has:

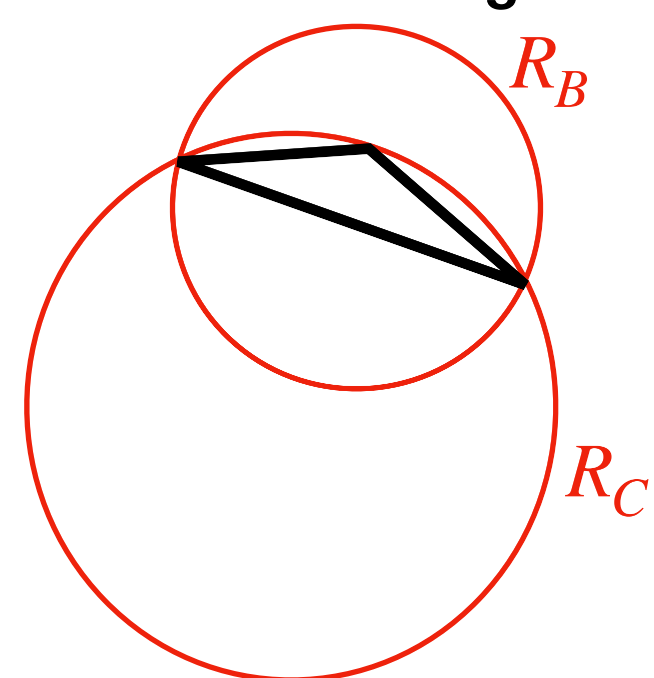
Lemma 7.4. *If Condition 1 holds, $\leq_\infty$ is a total order on the set of 2-simplices of K with:*

$$\sigma_1 \leq_\infty \sigma_2 \iff \begin{cases} \mathrm{R_B}(\sigma_1) < \mathrm{R_B}(\sigma_2) \\ \text{or} \\ \mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2) \quad \text{and} \quad \mathrm{R_C}(\sigma_1) \geq \mathrm{R_C}(\sigma_2) \end{cases}$$

Acute triangle:



Obtuse triangle:



Delaunay order

When lexicographic-minimal chain is Delaunay

Behavior as $p \to \infty$?

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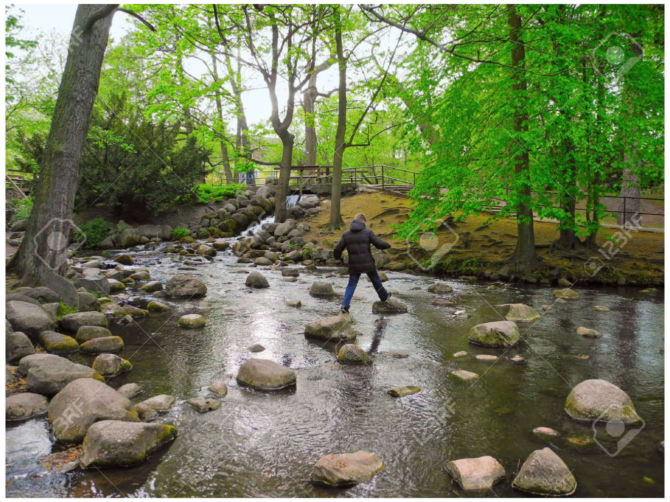
When $\leq_\infty$ is a total order, it defines a **lexicographic order** $\sqsubseteq_{lex}$ on chains:

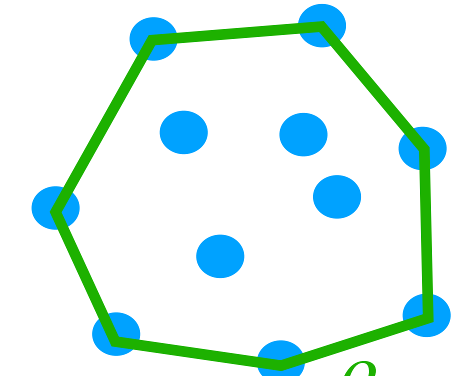
$$\Gamma_1 \sqsubseteq_{lex} \Gamma_2 \underset{def.}{\iff} \begin{cases} \Gamma_1 = \Gamma_2 \\ \text{or} \\ \sigma_{\max} = \max_{\leq_\infty} \{\sigma \in \Gamma_1 - \Gamma_2\} \in \Gamma_2 \end{cases}$$



(With coefficients in $\mathbb{Z}_2$, $\Gamma_1 - \Gamma_2$ is the symmetric difference between Γ_1 and Γ_2)

Delaunay triangulation

When lexicographic-minimal chain is Delaunay



Theorem 1 *Let* $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, *with* $N \geq n+1$, *be weighted points in general position and* $K_{\mathbf{P}}$ *the* n*-dimensional full simplicial complex over* $\mathbf{P}$. *Denote by* $\beta_{\mathbf{P}} \in \boldsymbol{C}_{\boldsymbol{n-1}}(K_{\mathbf{P}})$ *the* $(n-1)$*-chain, set of simplices belonging to the boundary of the convex hull* $\mathcal{CH}(\mathbf{P})$.

Then the simplicial complex $|\Gamma_{\min}|$ *support of*

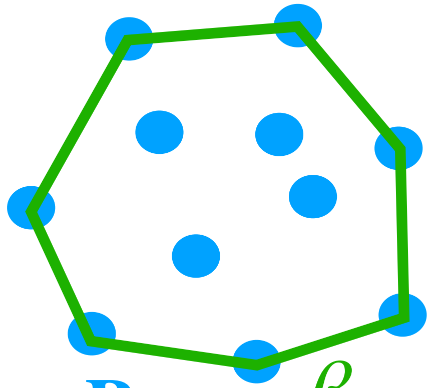
$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \left\{ \Gamma \in \boldsymbol{C}_{\boldsymbol{n}}(K_{\mathbf{P}}), \partial\Gamma = \beta_{\mathbf{P}} \right\}$$

is the regular triangulation of $\mathbf{P}$.

(Cohen-Steiner, L.,Vuillamy 2020)

Delaunay triangulation

When lexicographic-minimal chain is Delaunay



Theorem 1 *Let* $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$*, with* $N \geq n+1$*, be weighted points in general position and* $K_{\mathbf{P}}$ *the* n*-dimensional full simplicial complex over* $\mathbf{P}$*. Denote by* $\beta_{\mathbf{P}} \in \boldsymbol{C_{n-1}}(K_{\mathbf{P}})$ *the* $(n-1)$*-chain, set of simplices belonging to the boundary of the convex hull* $\mathcal{CH}(\mathbf{P})$*.*

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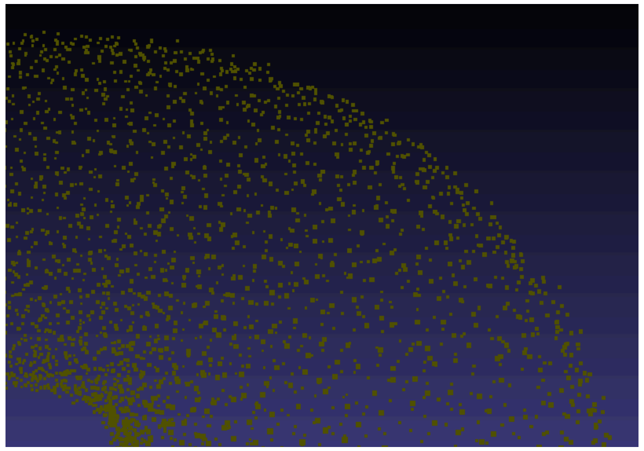
is the regular triangulation of $\mathbf{P}$*.*

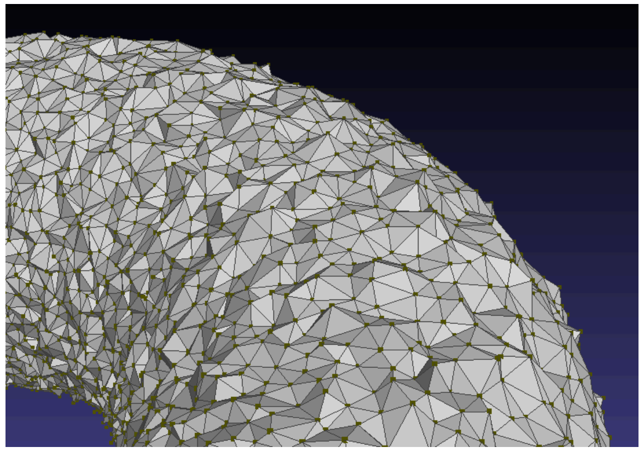
This extends to smooth (positive reach) 2-manifolds

Triangulation of positive reach 2-manifolds

$\mathbf{P} \subset \mathcal{M}$ is an (ϵ, η)-sampling of $\mathcal{M}$ iff:

- $d_H(\mathbf{P}, \mathcal{M}) < \epsilon$
- $\forall p, q \in \mathbf{P}, p \neq q \Rightarrow d(p,q) > \eta$





Theorem 1. *There are constants C_1, C_2, C_3 such that:*

If $\mathcal{M}$ is a smooth 2-manifold embedded in $\mathbb{R}^n$ with reach $\mathcal{R}$, $\mathbf{P}$ an (ϵ, η)-sampling of $\mathcal{M}$ and K a Čech or Vietoris-Rips complex on K with parameter λ, such that:

$$C_1 \epsilon < \lambda < C_2 \mathcal{R}$$

→ **K captures the homotopy type $\Rightarrow \beta_2 = 1$**

and:

$$\frac{\epsilon}{\mathcal{R}} < C_3 \left(\frac{\eta}{\epsilon}\right)^{10}$$

→ **Lexicographic minimal chain in $H_2(K, \mathbb{Z}_2)$ is a triangulation**

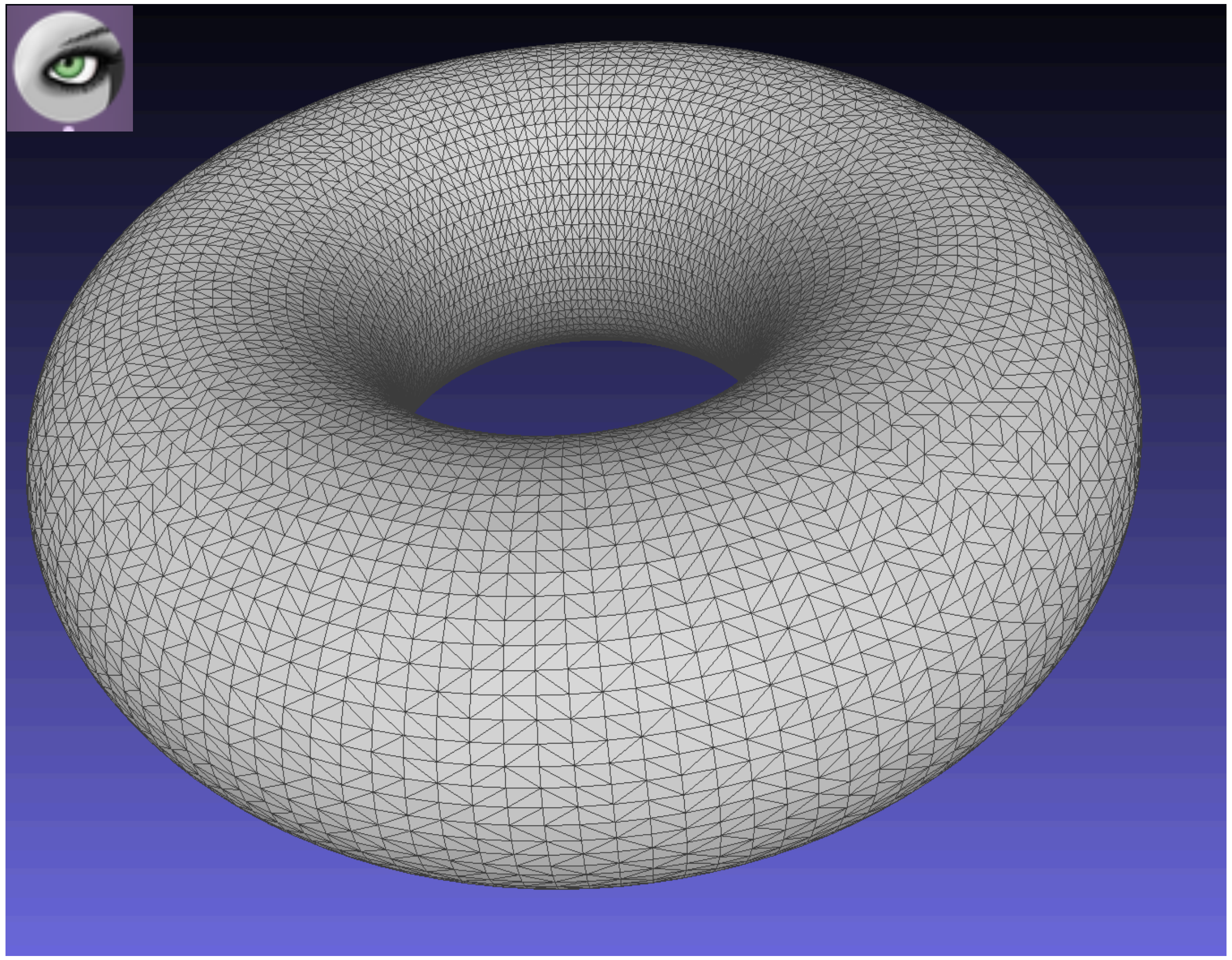
Then if:

$$\mathcal{T} = \min_{\sqsubseteq_{lex}} \mathrm{Ker}(\partial_2) \setminus \mathrm{Im}(\partial_3)$$

The restriction of $\pi_{\mathcal{M}}$ to $|\mathcal{T}|$ is an homeomorphism on $\mathcal{M}$. It follows that $(|\mathcal{T}|, \pi_{\mathcal{M}})$ is a triangulation of $\mathcal{M}$.

Minimal homology representative cycle

Examples of lexicographic-minimal cycle



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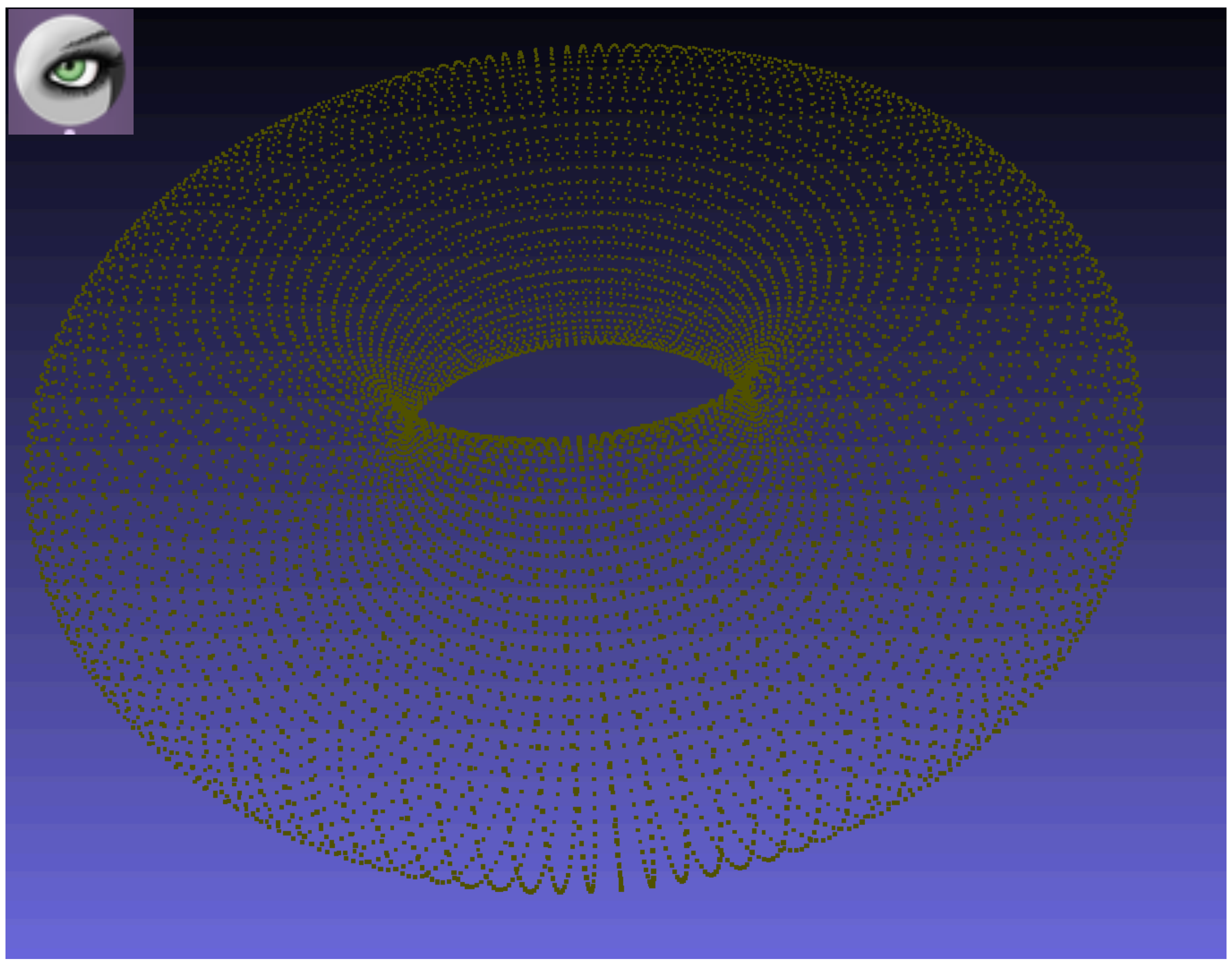

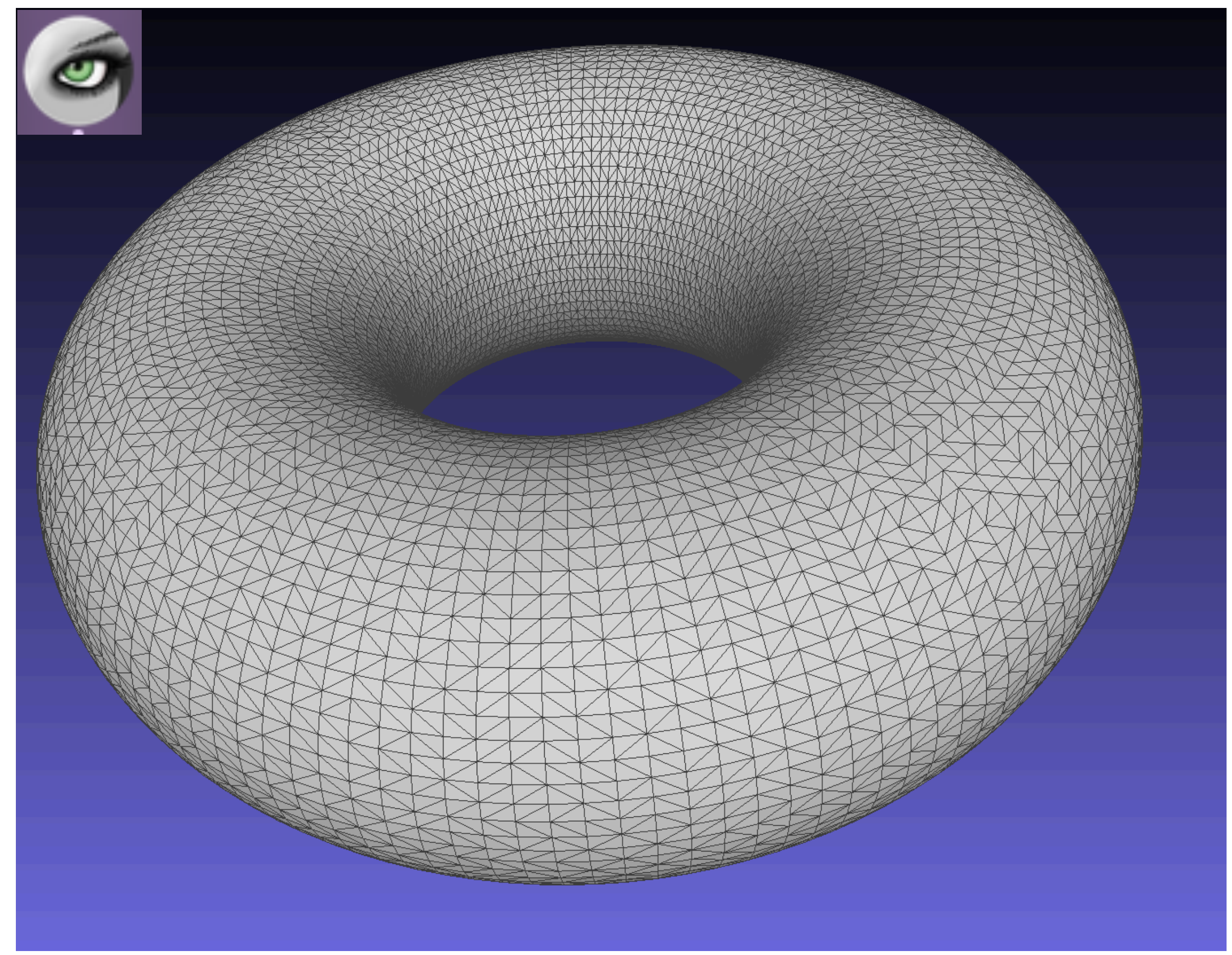

Minimal homology representative cycle

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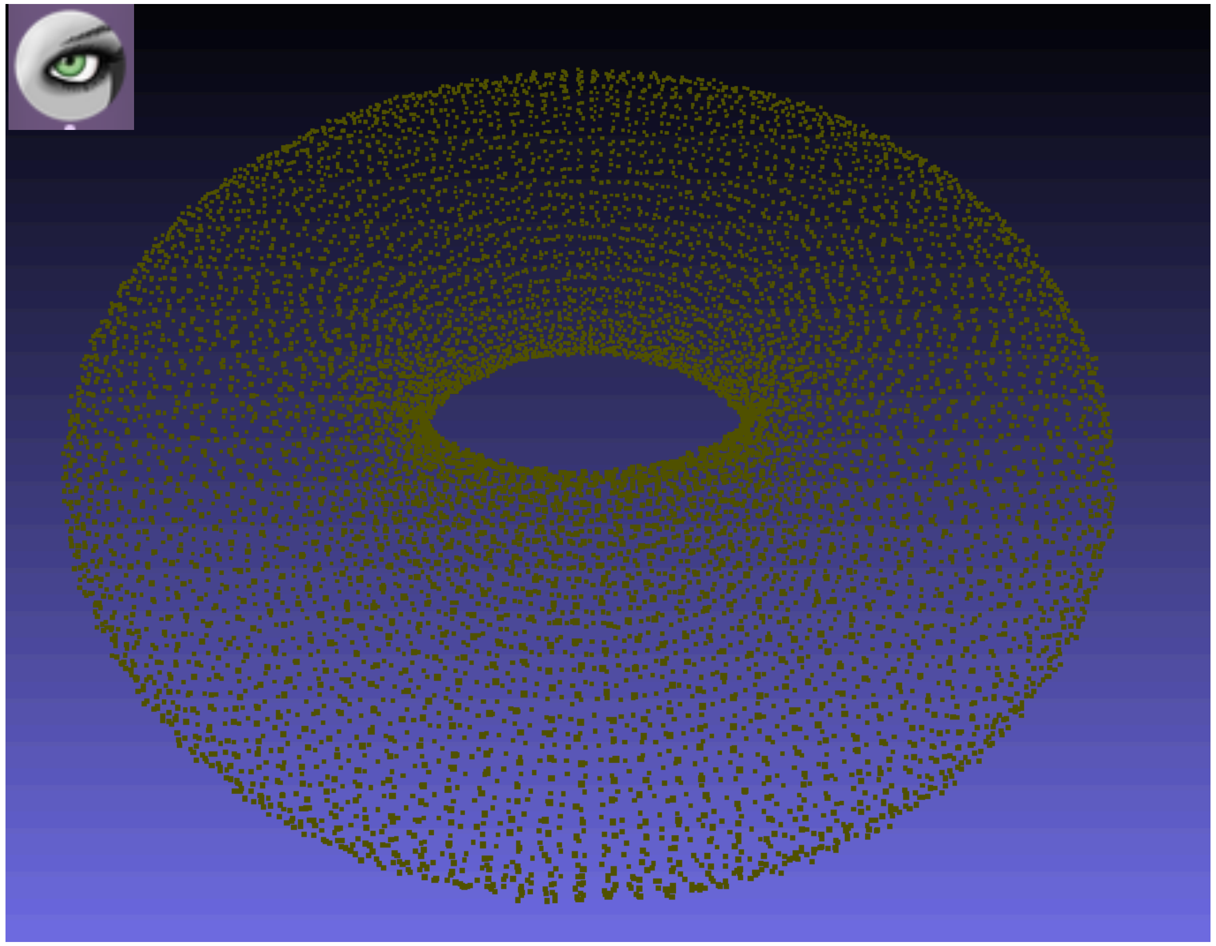

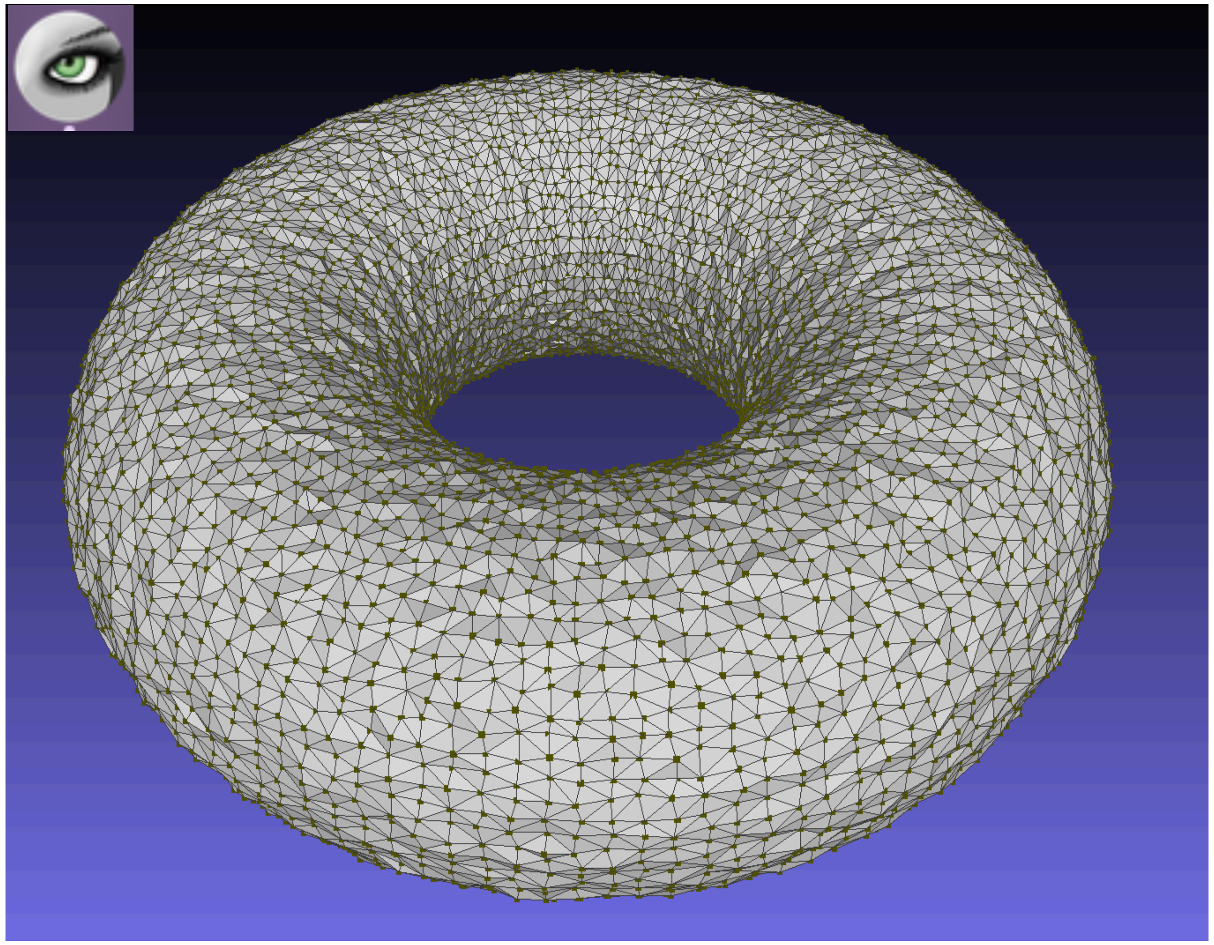

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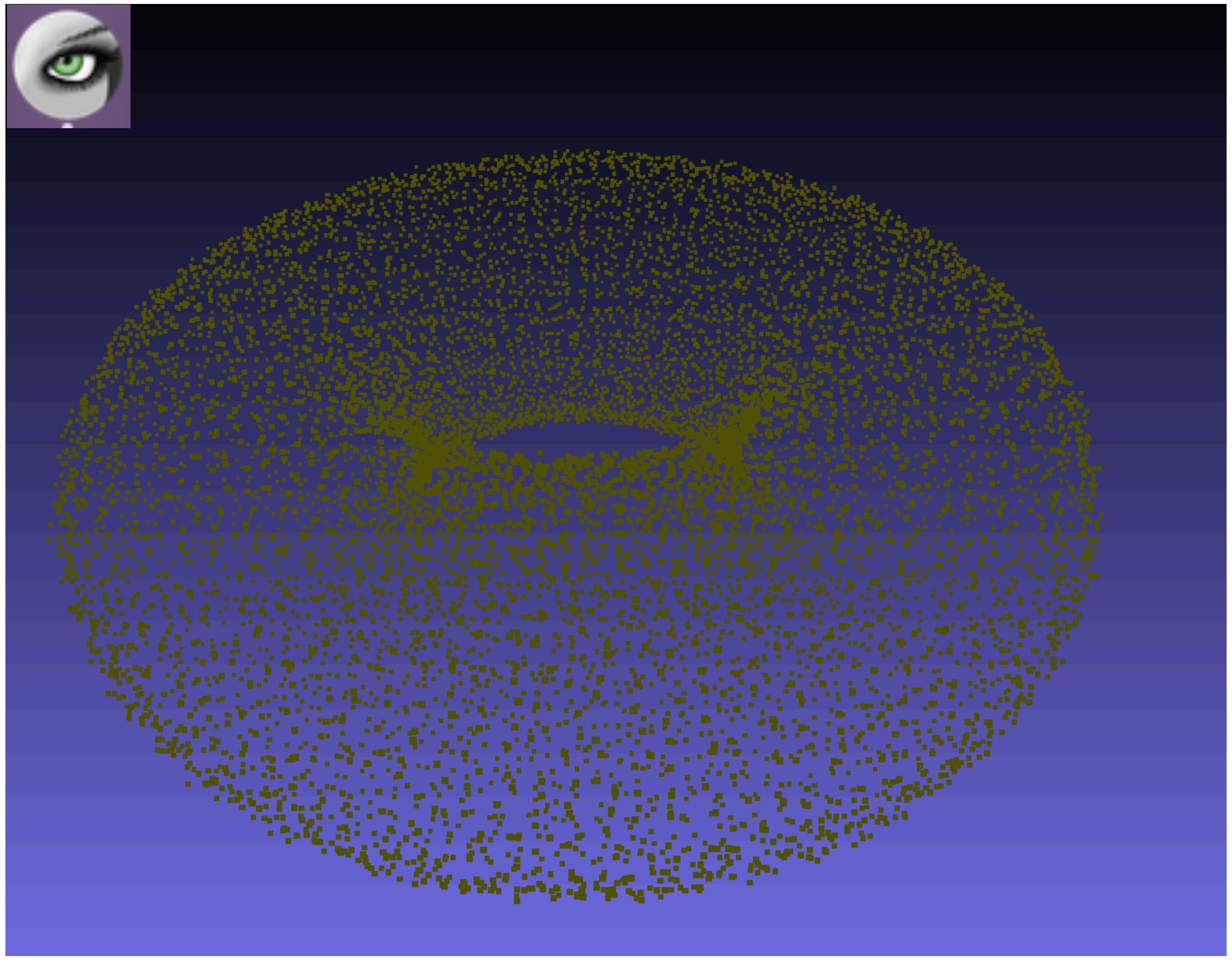

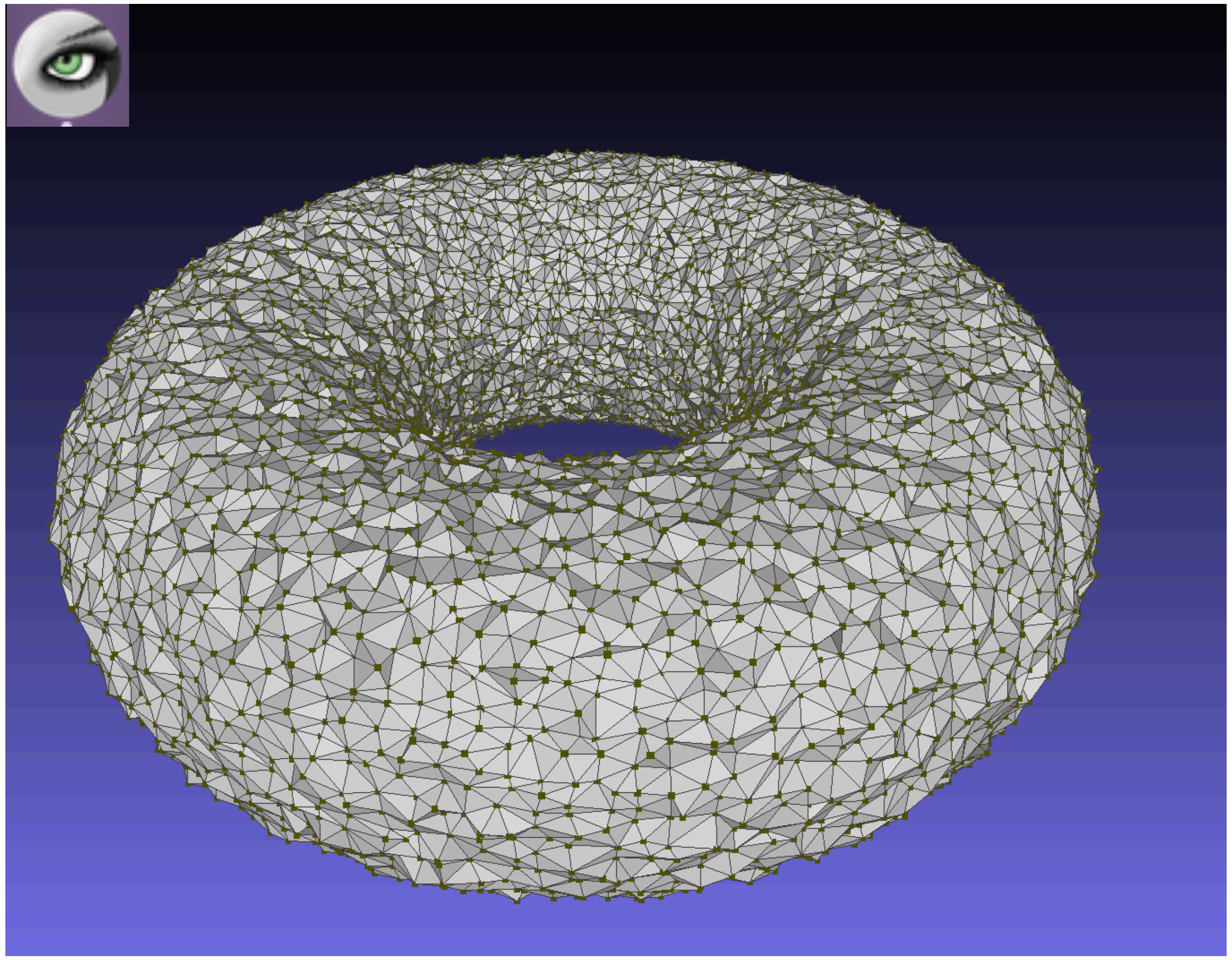

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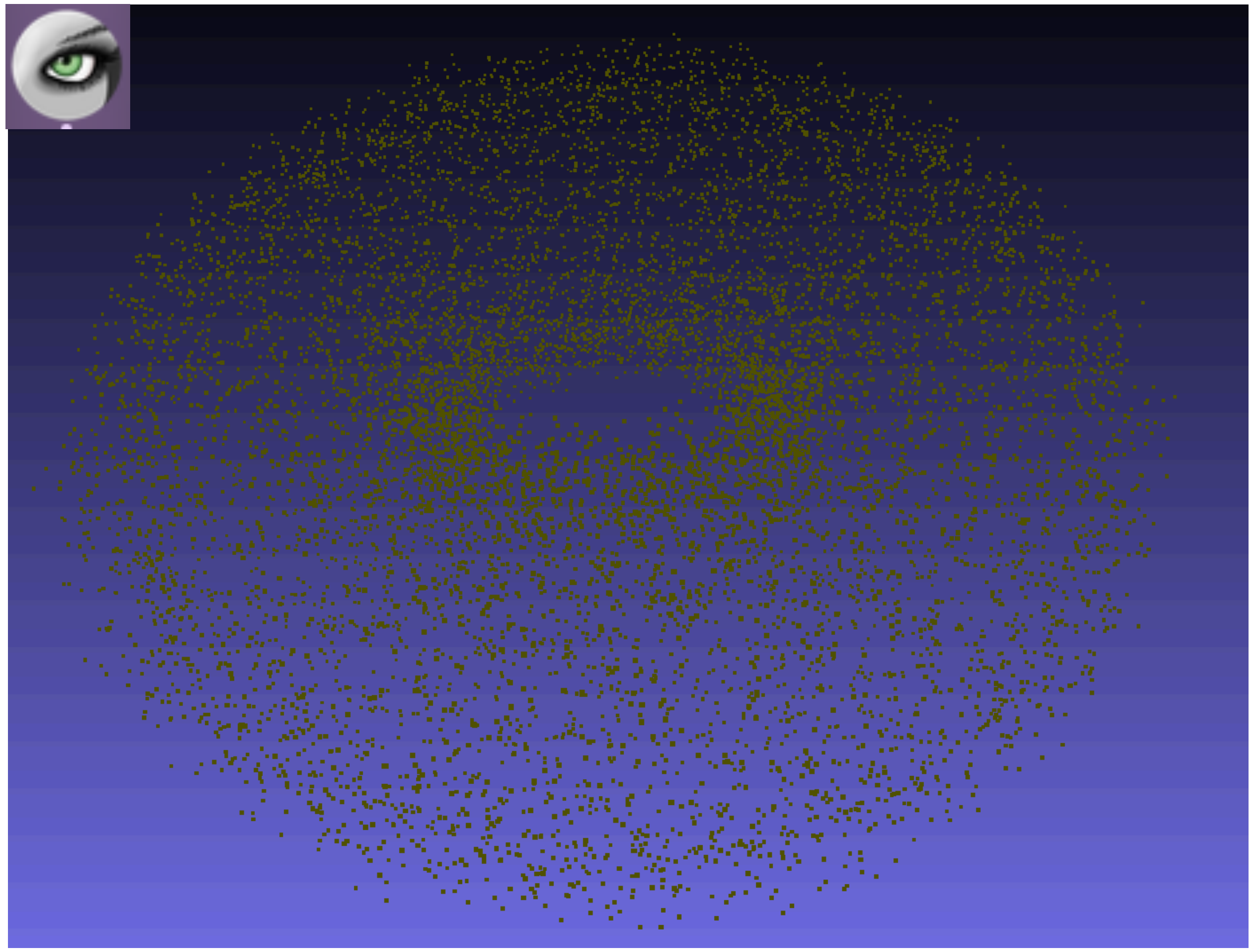

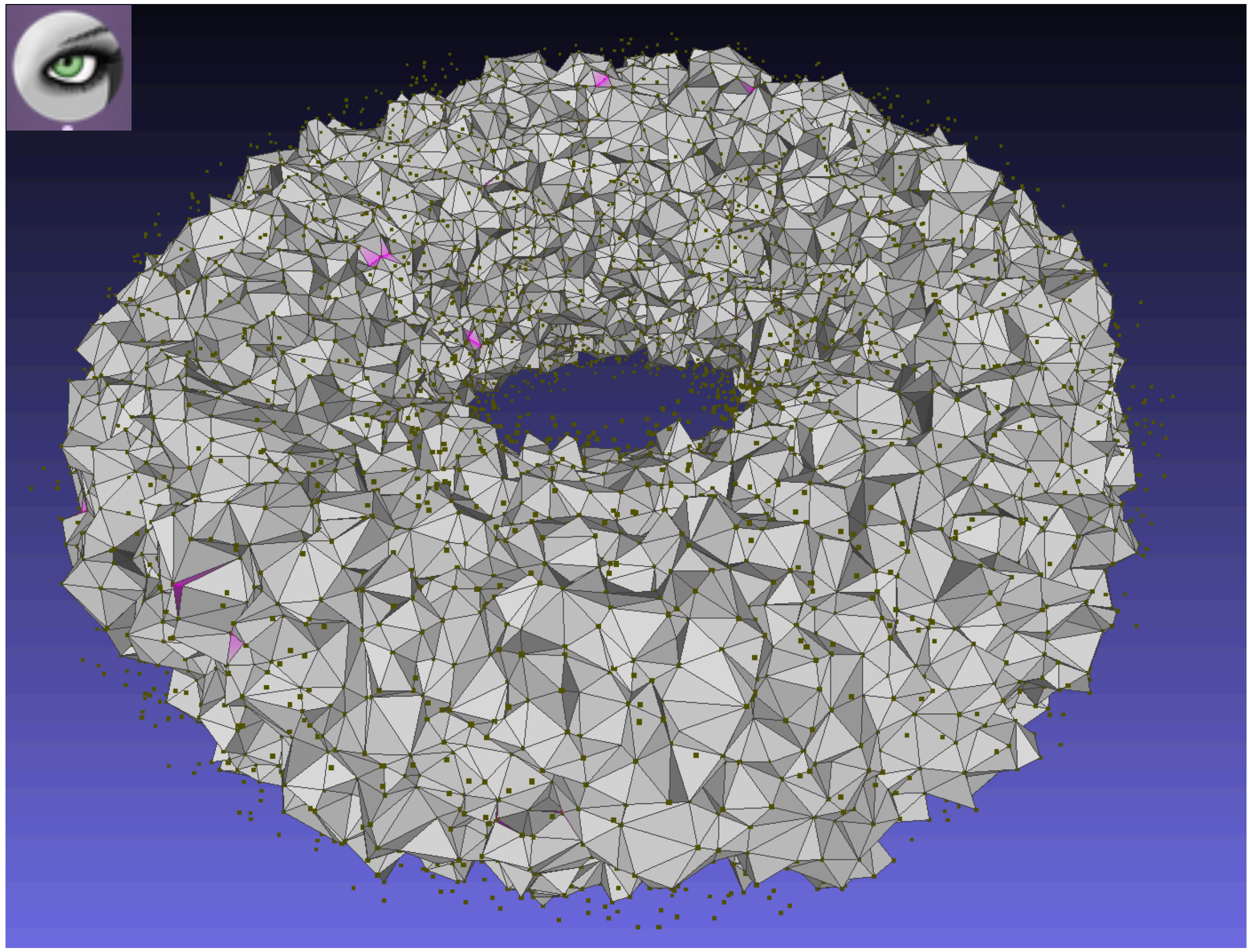

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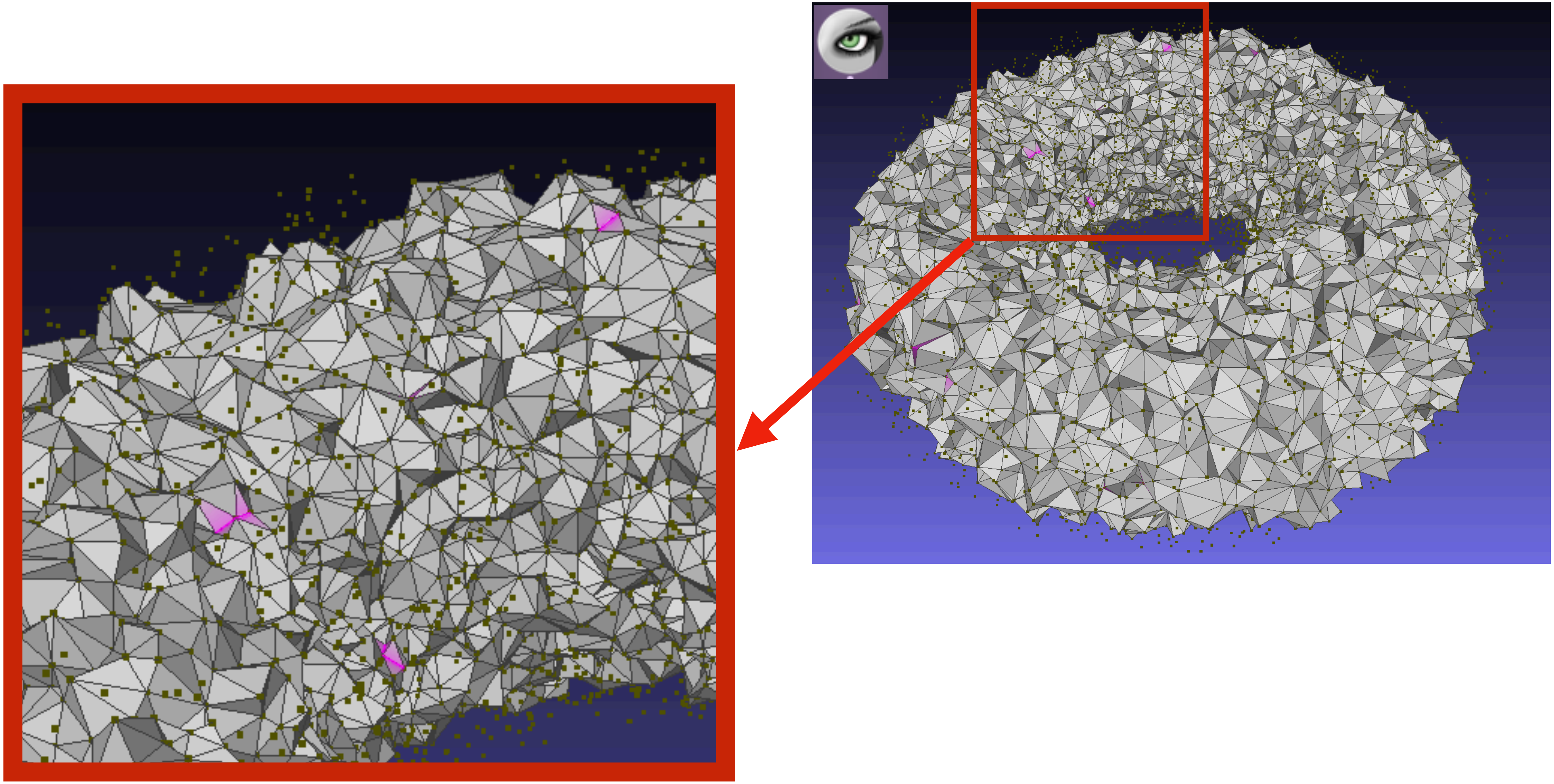

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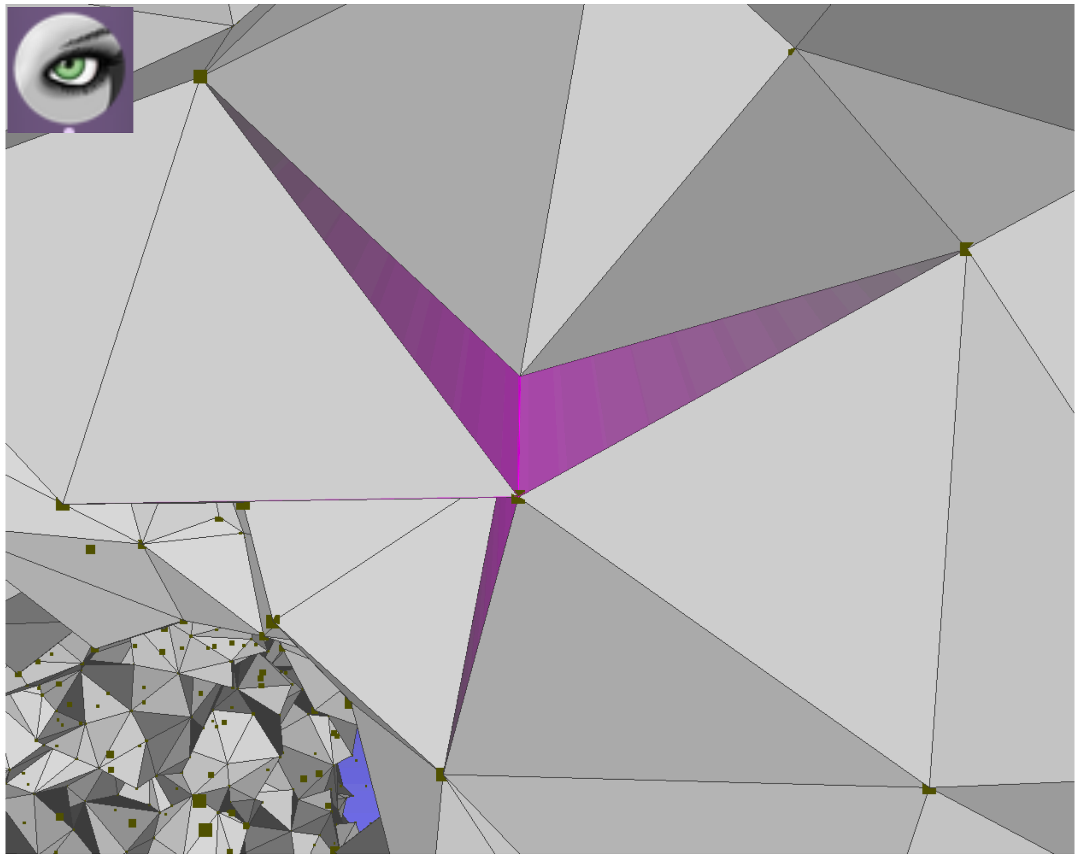

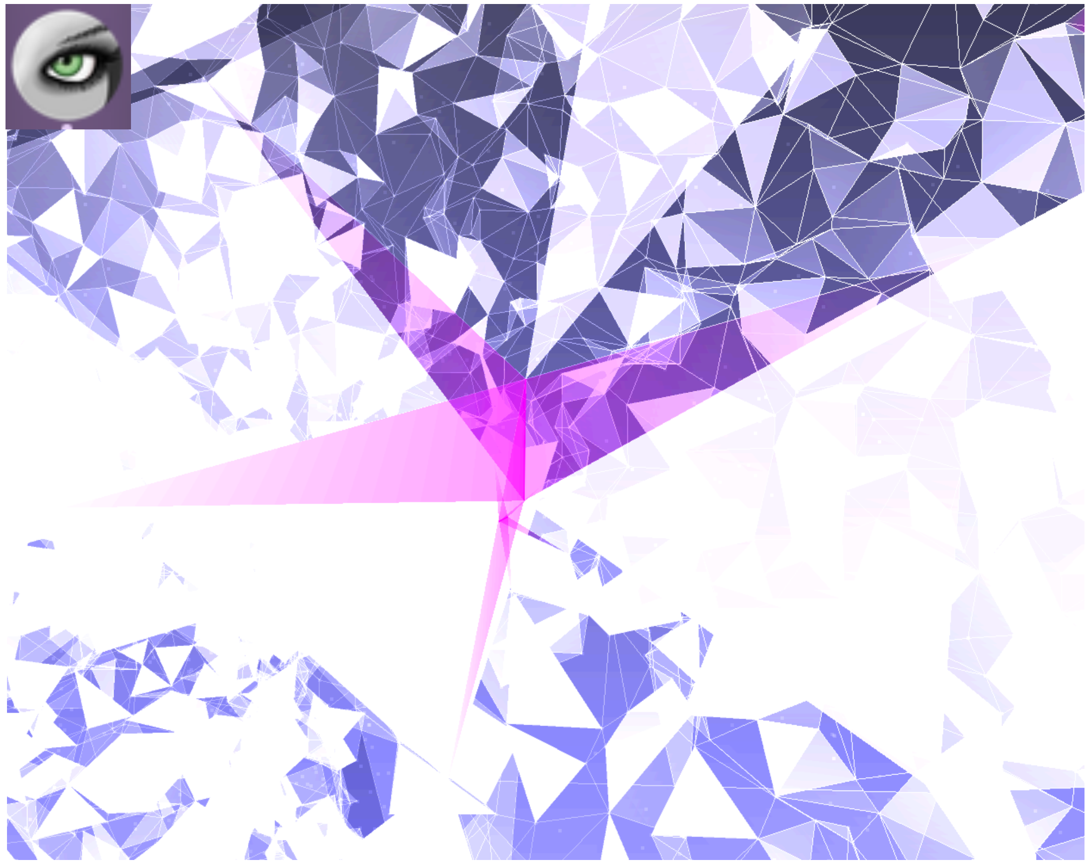

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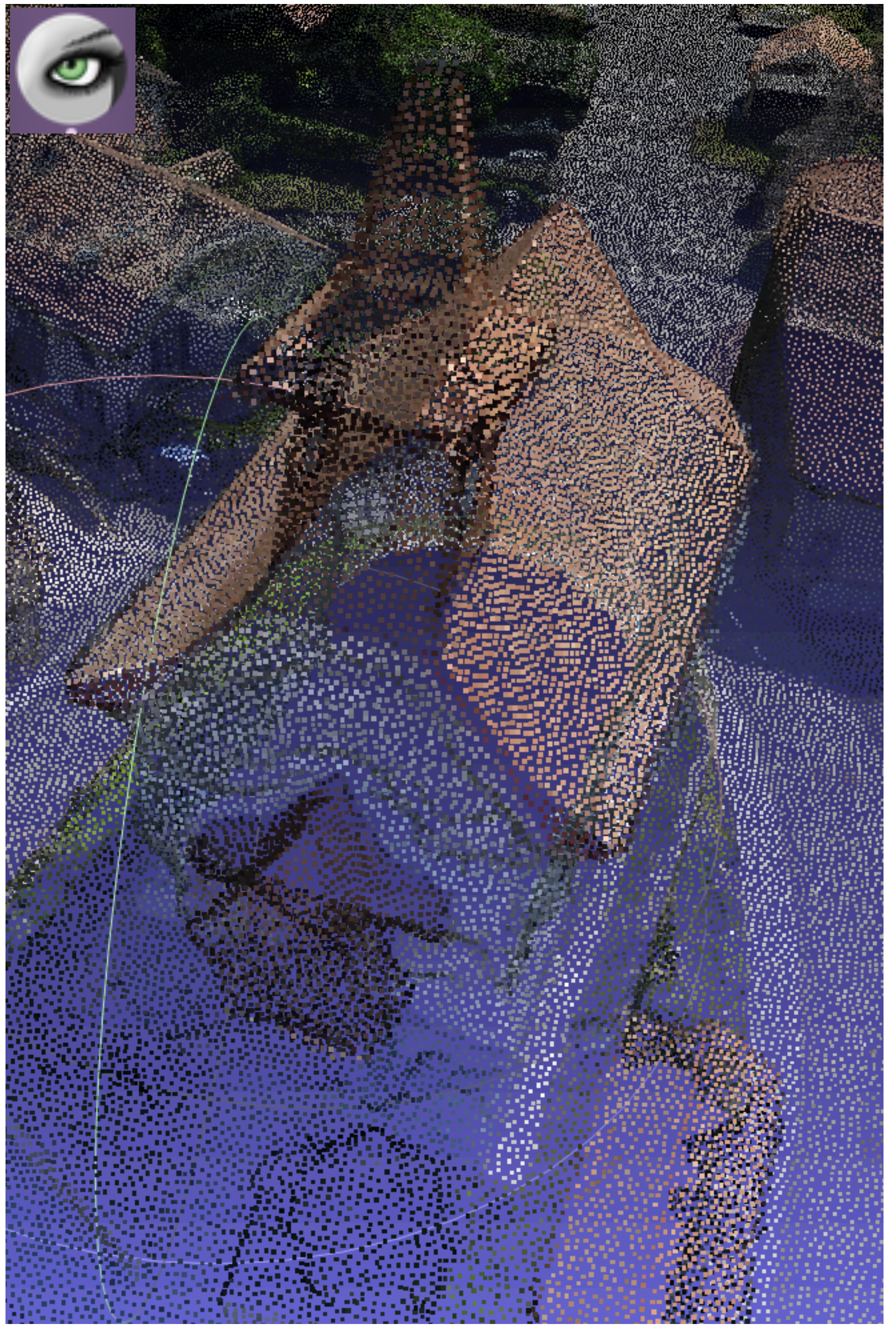

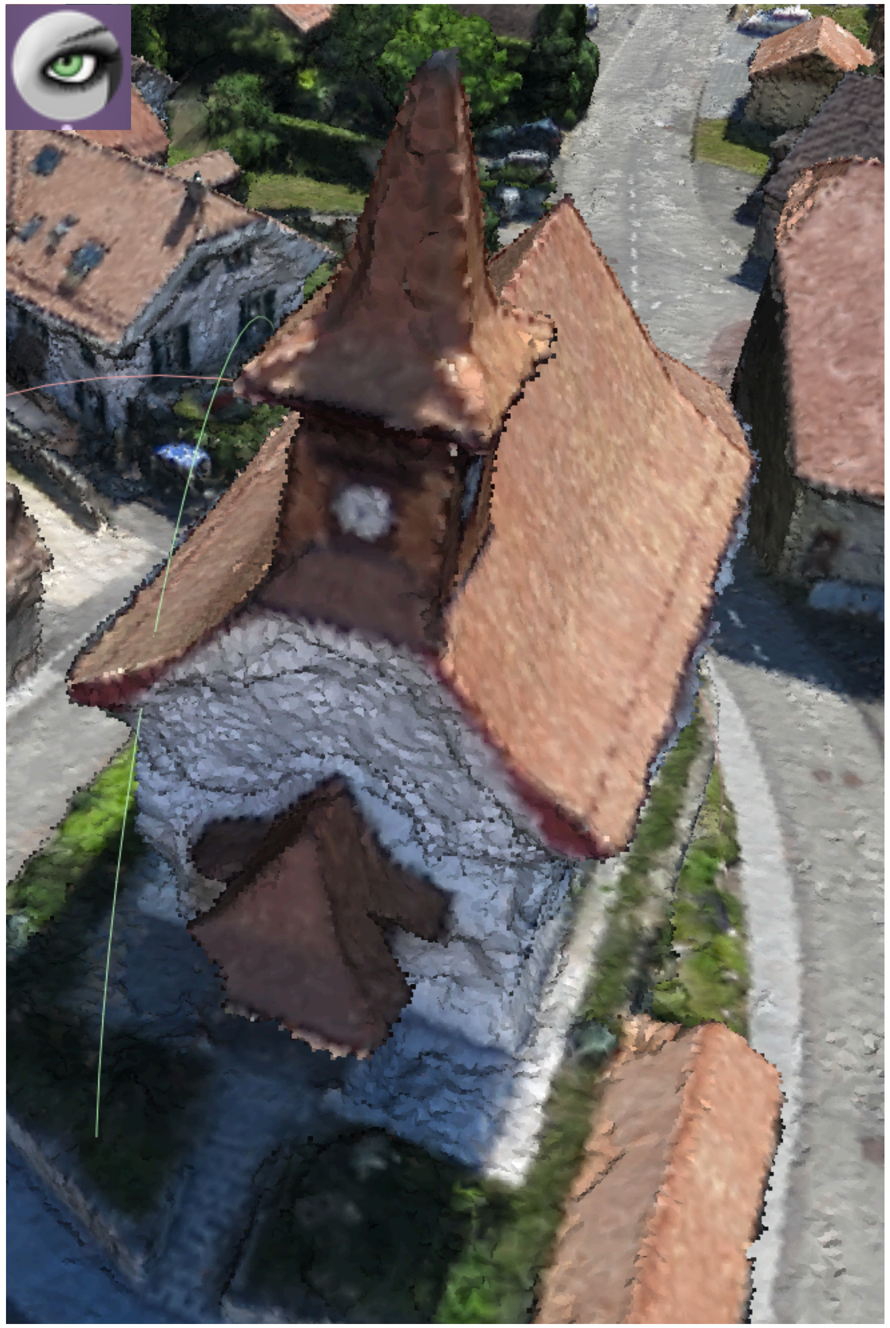

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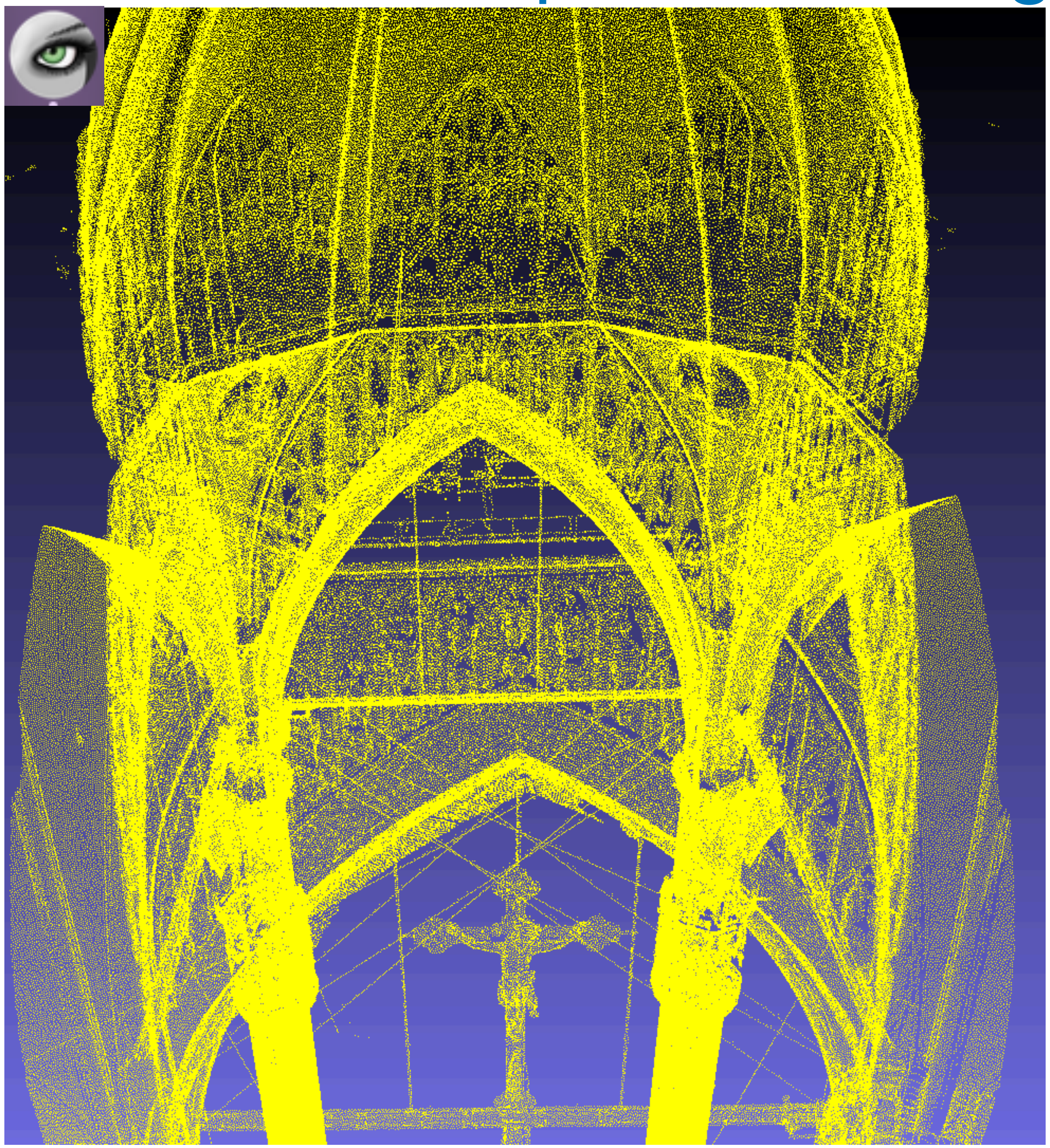

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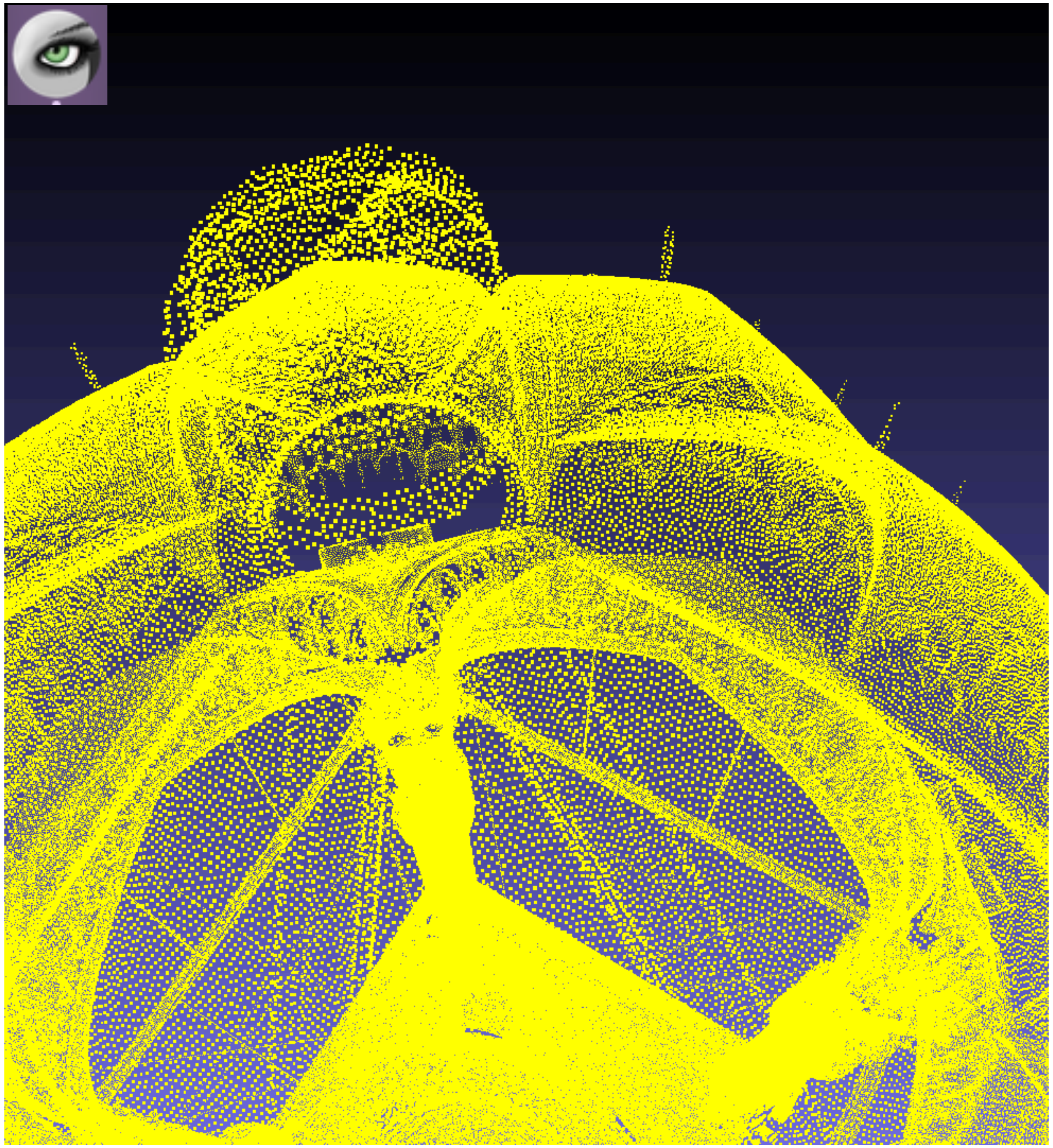

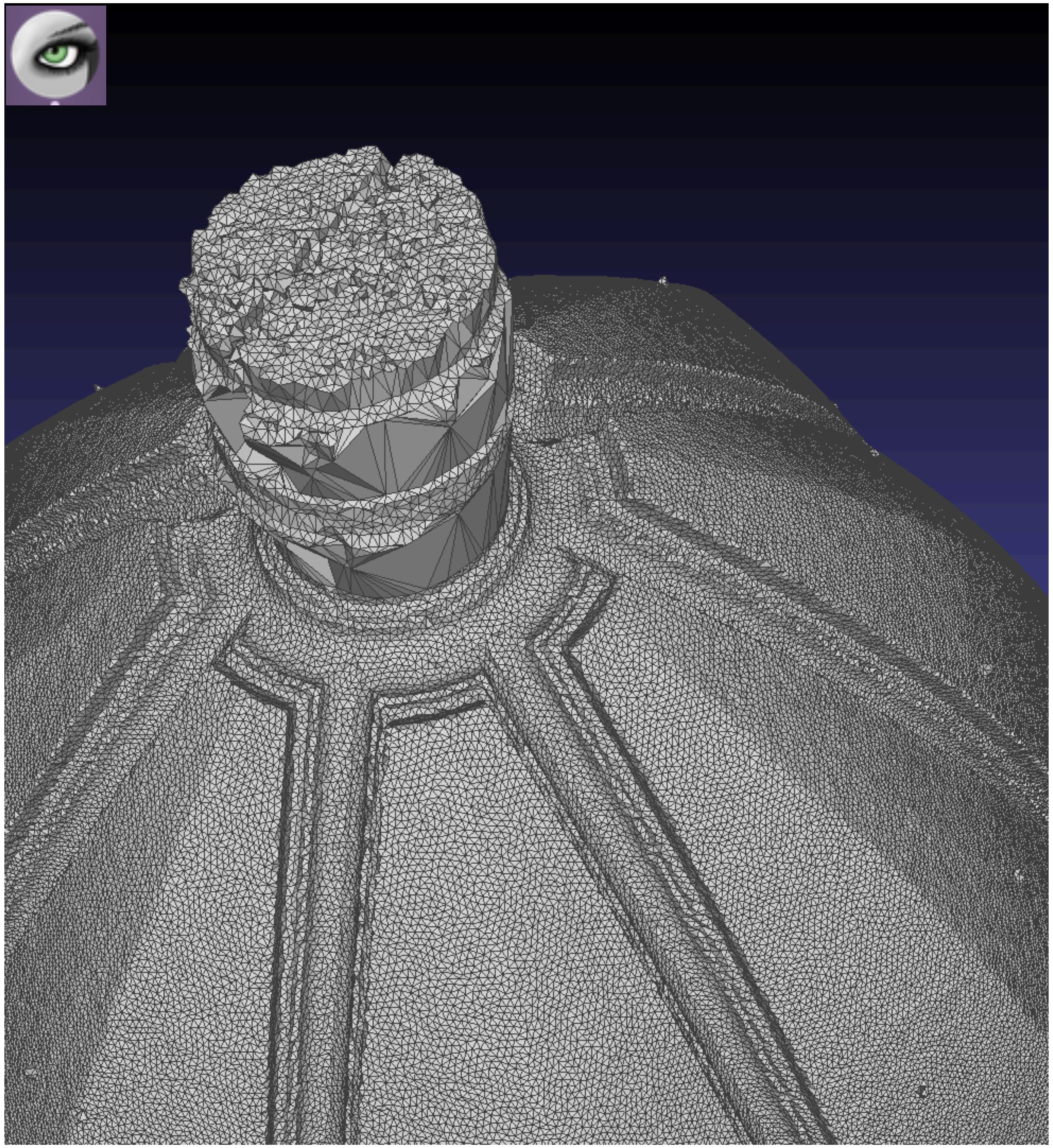

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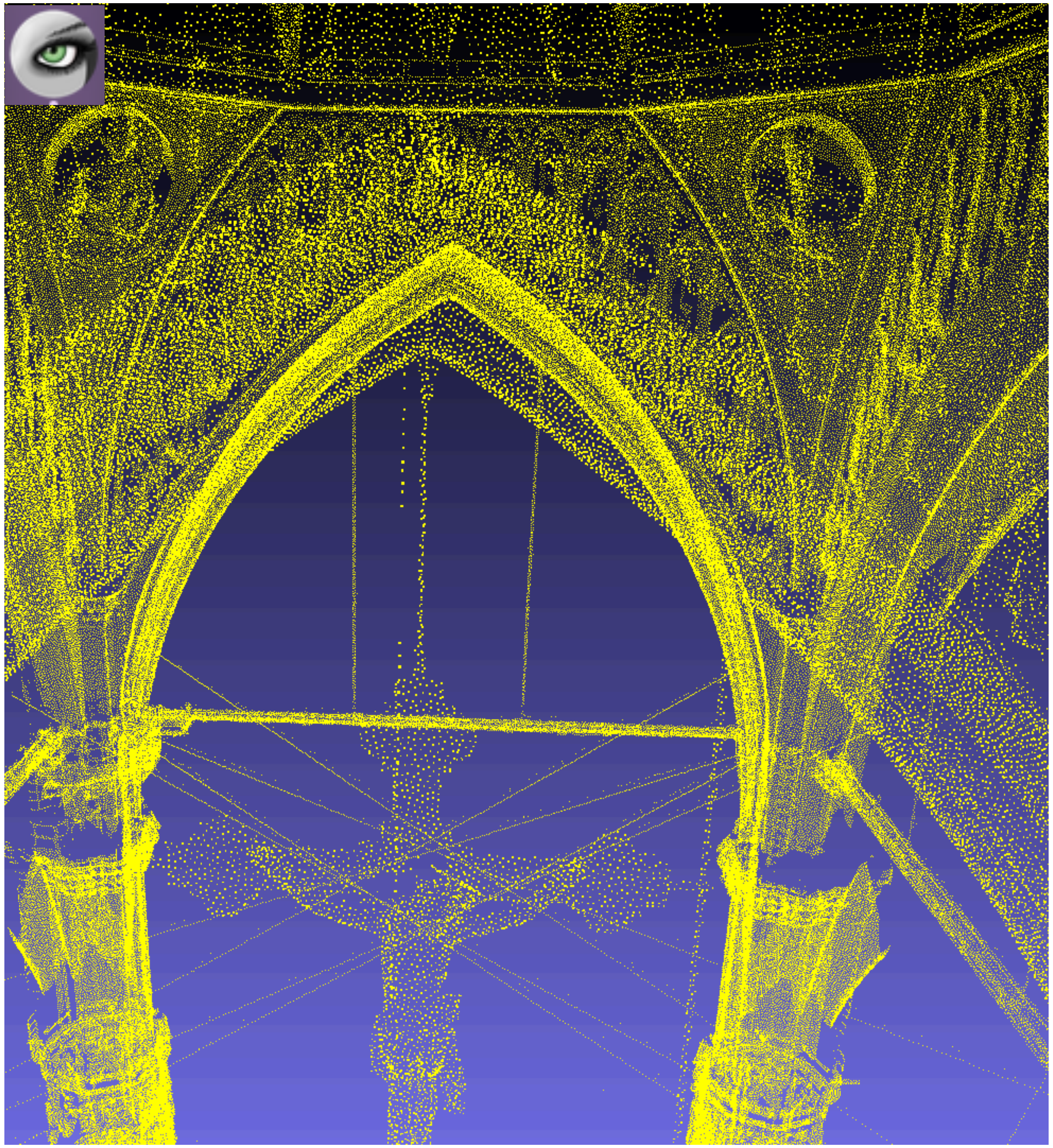

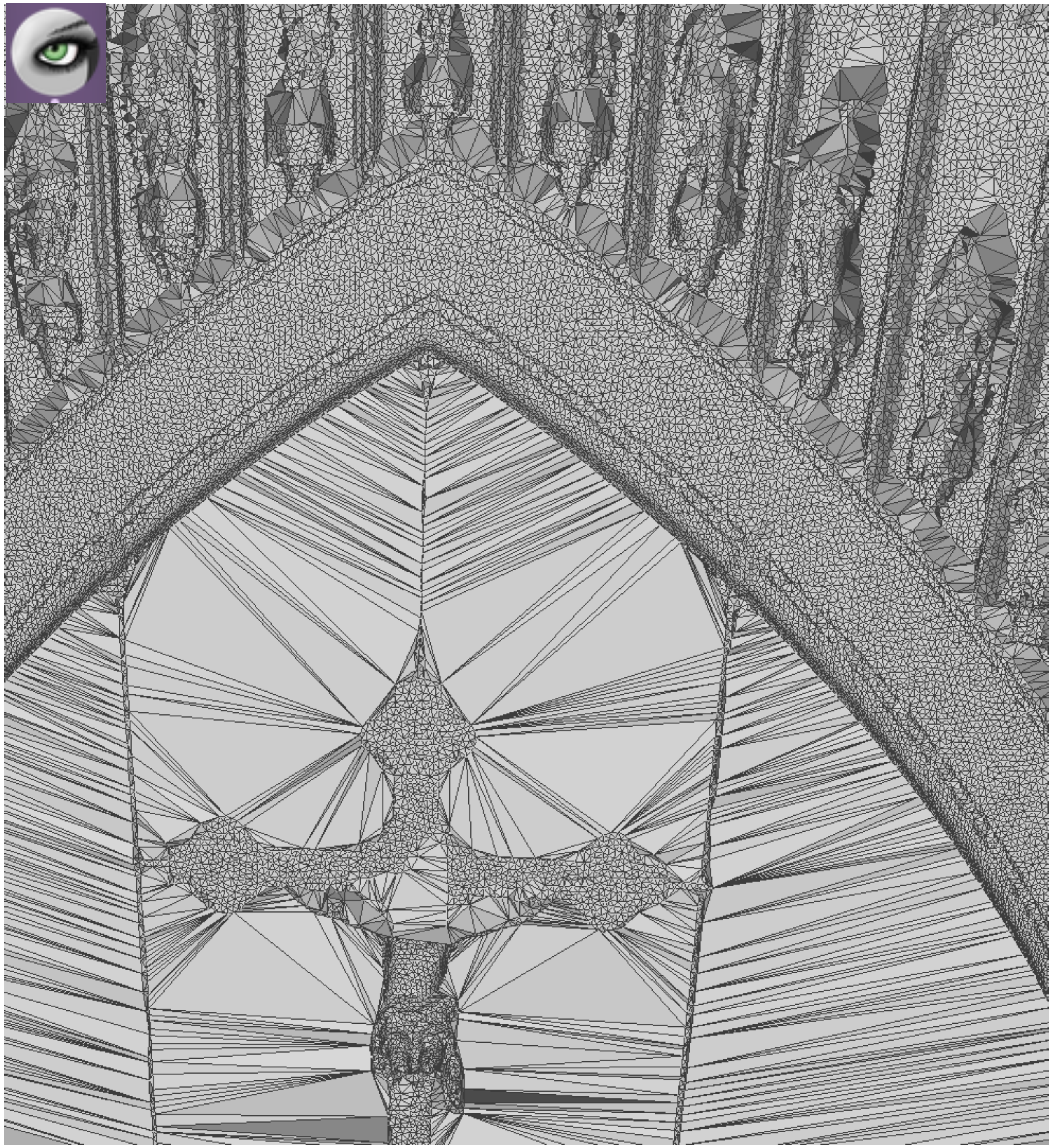

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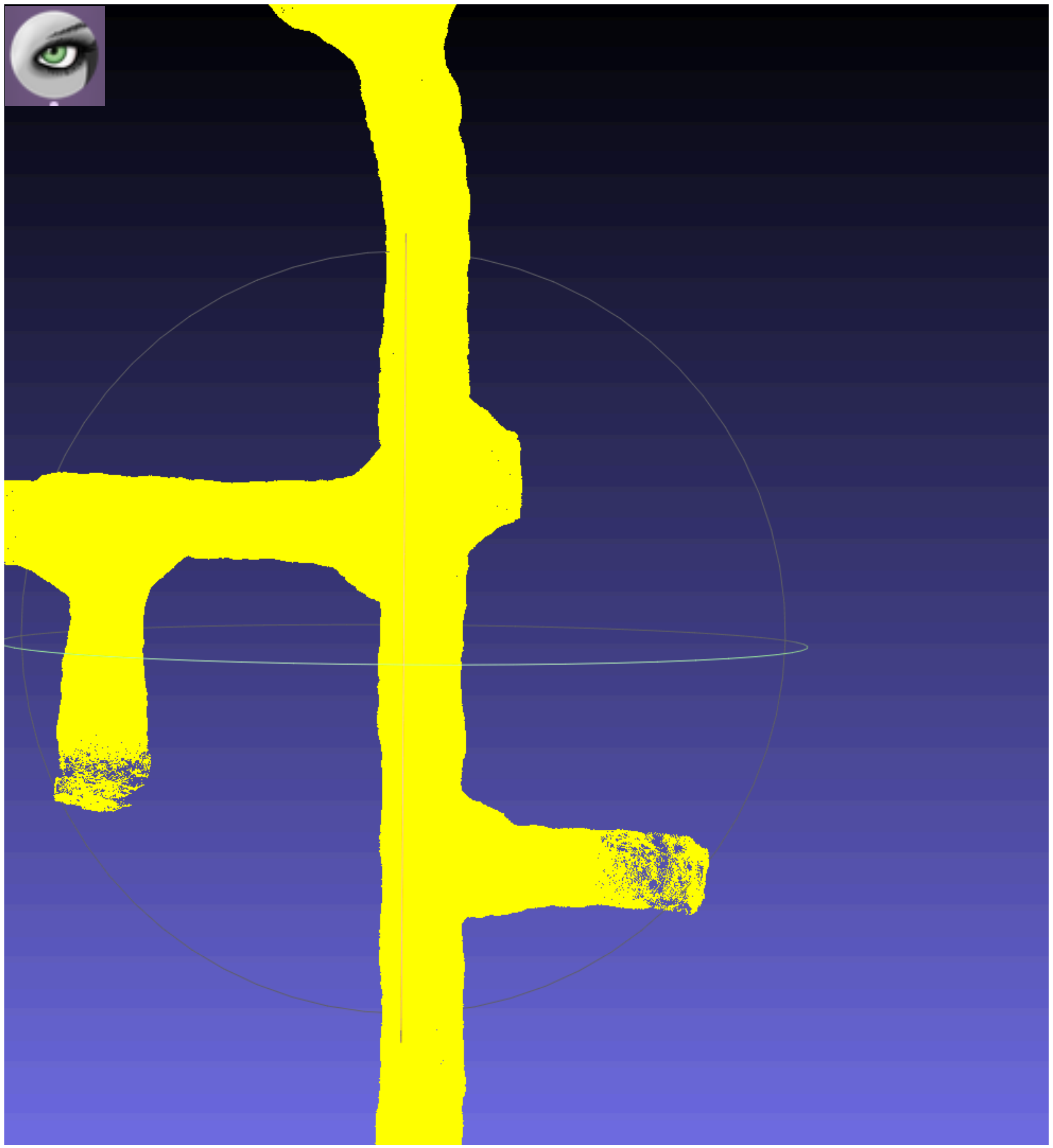

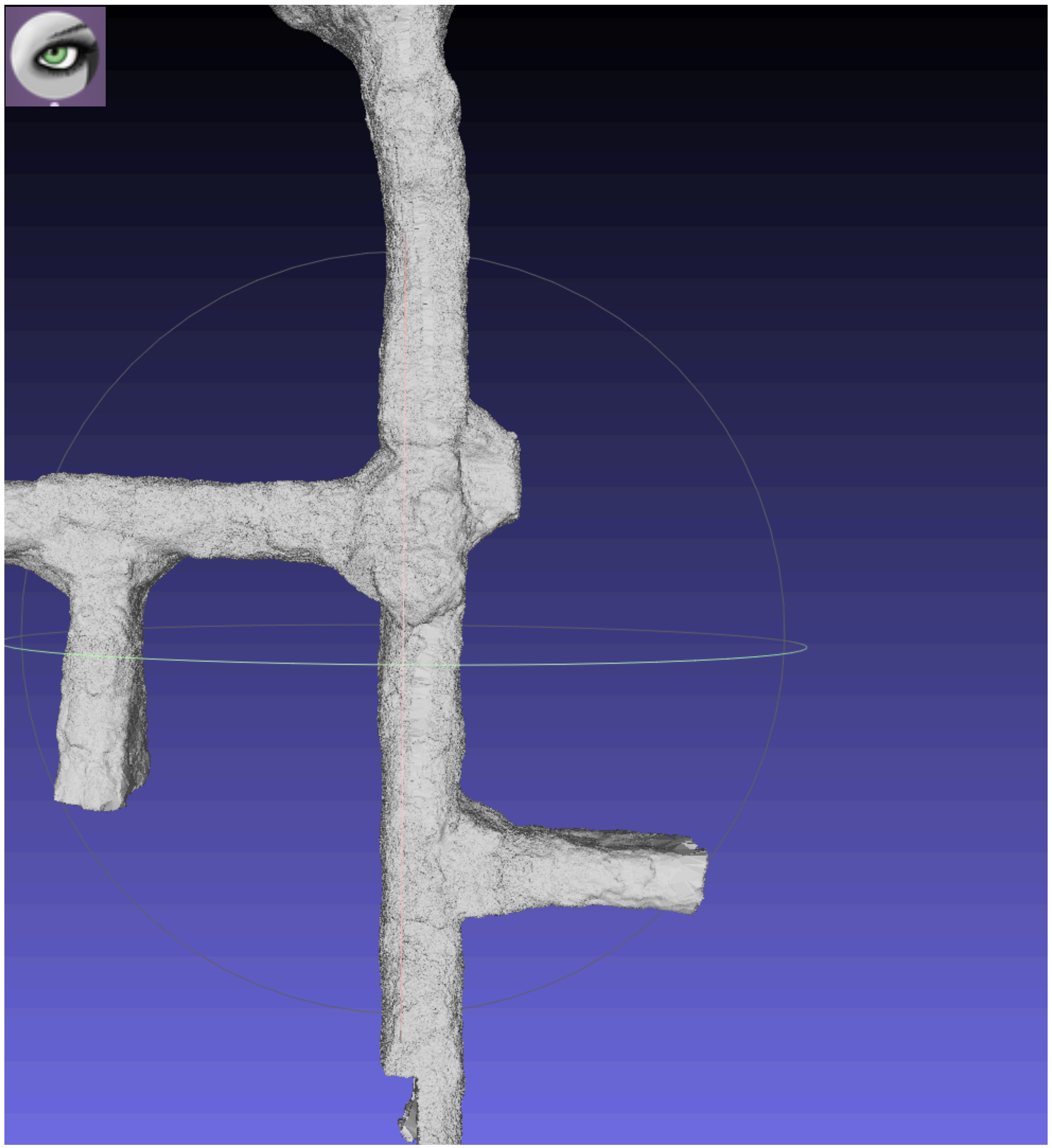

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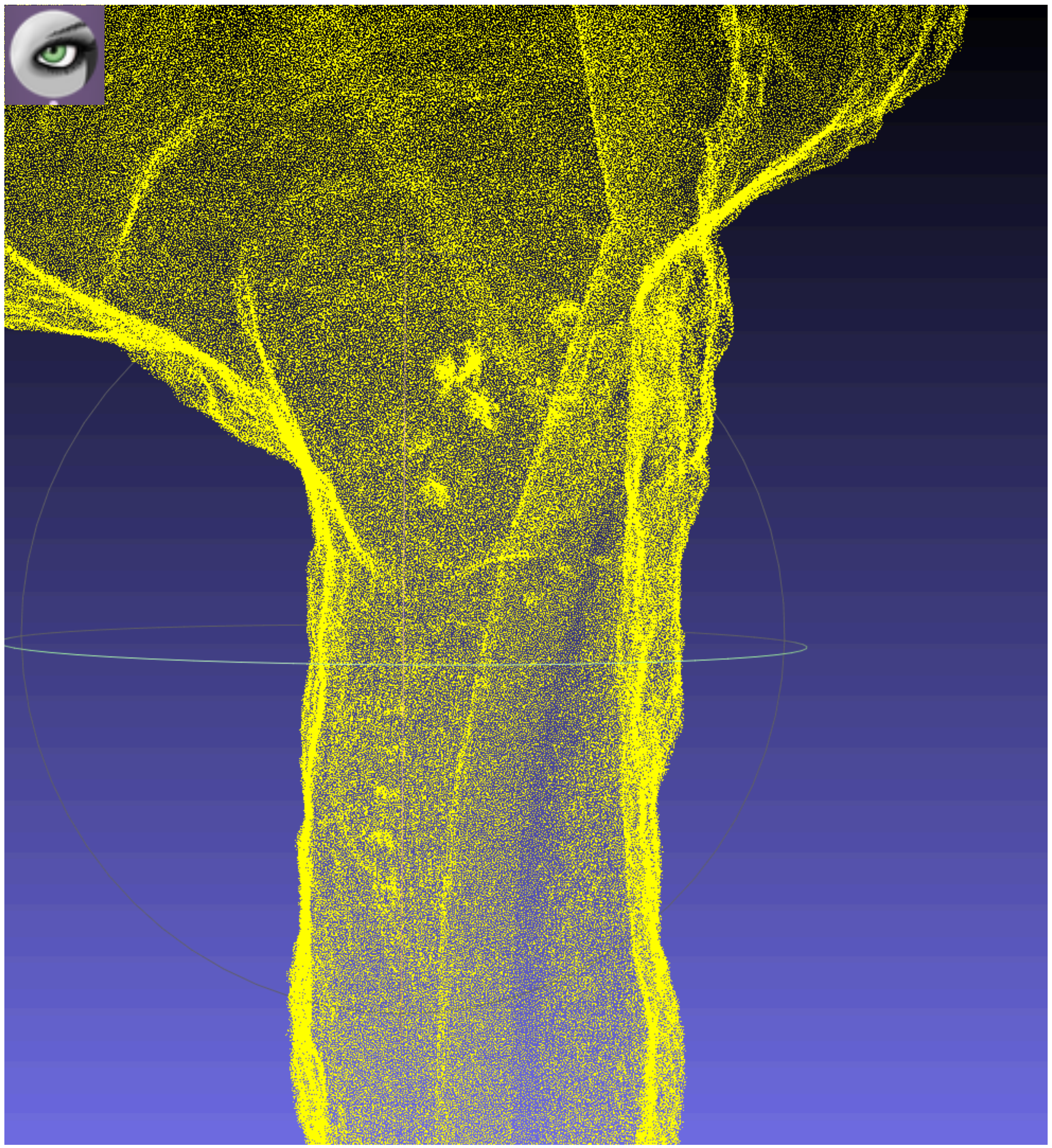
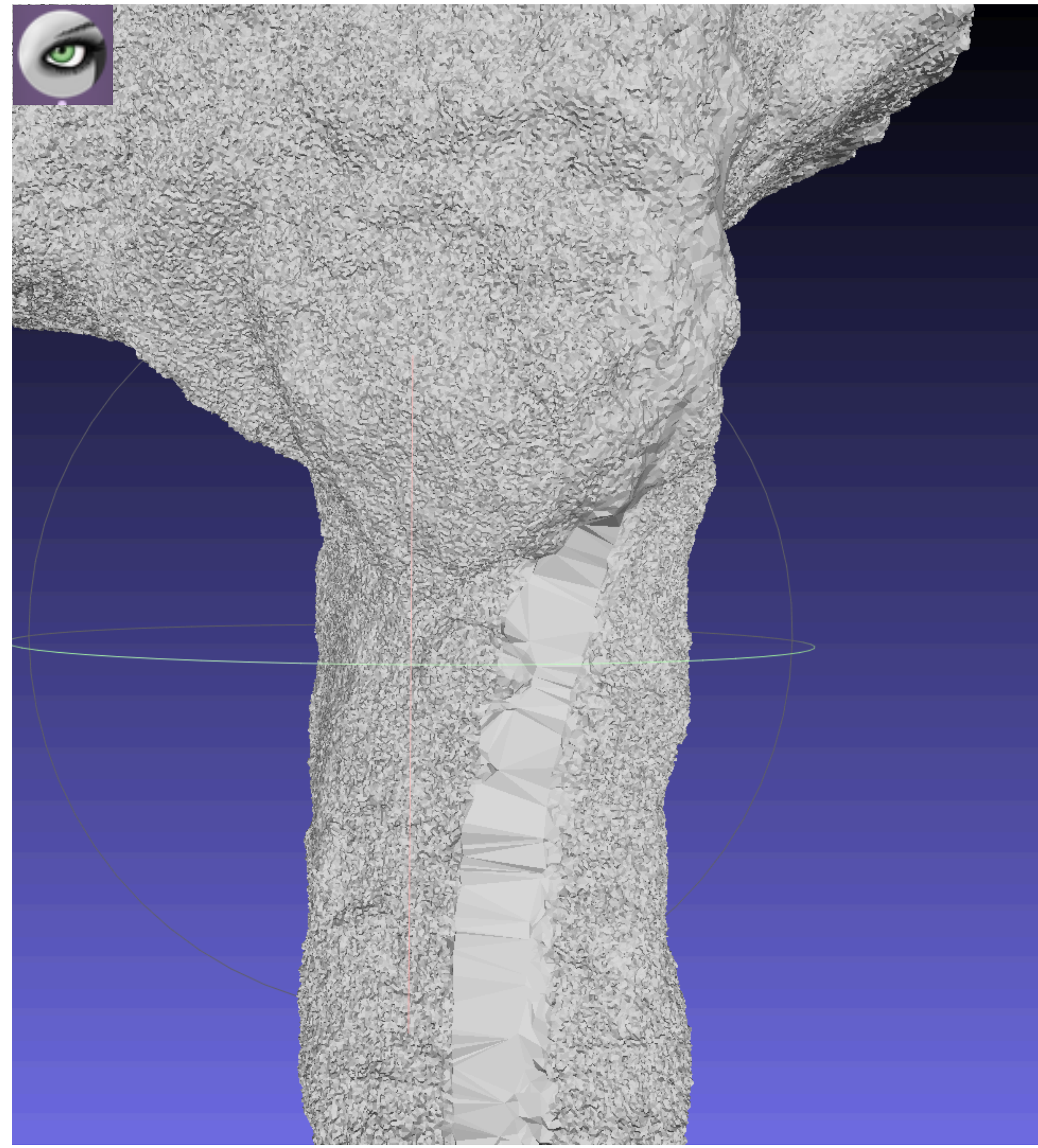

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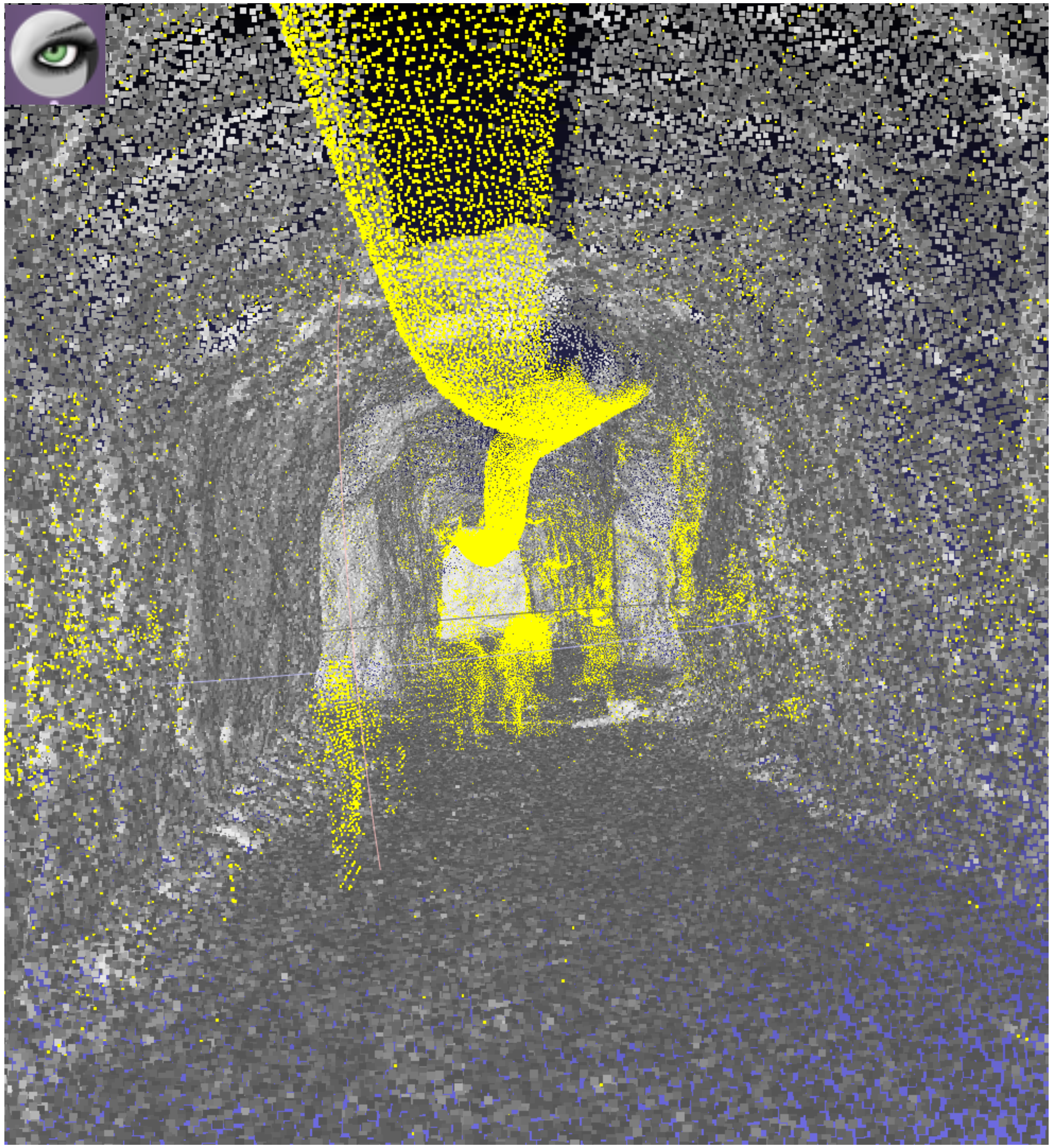

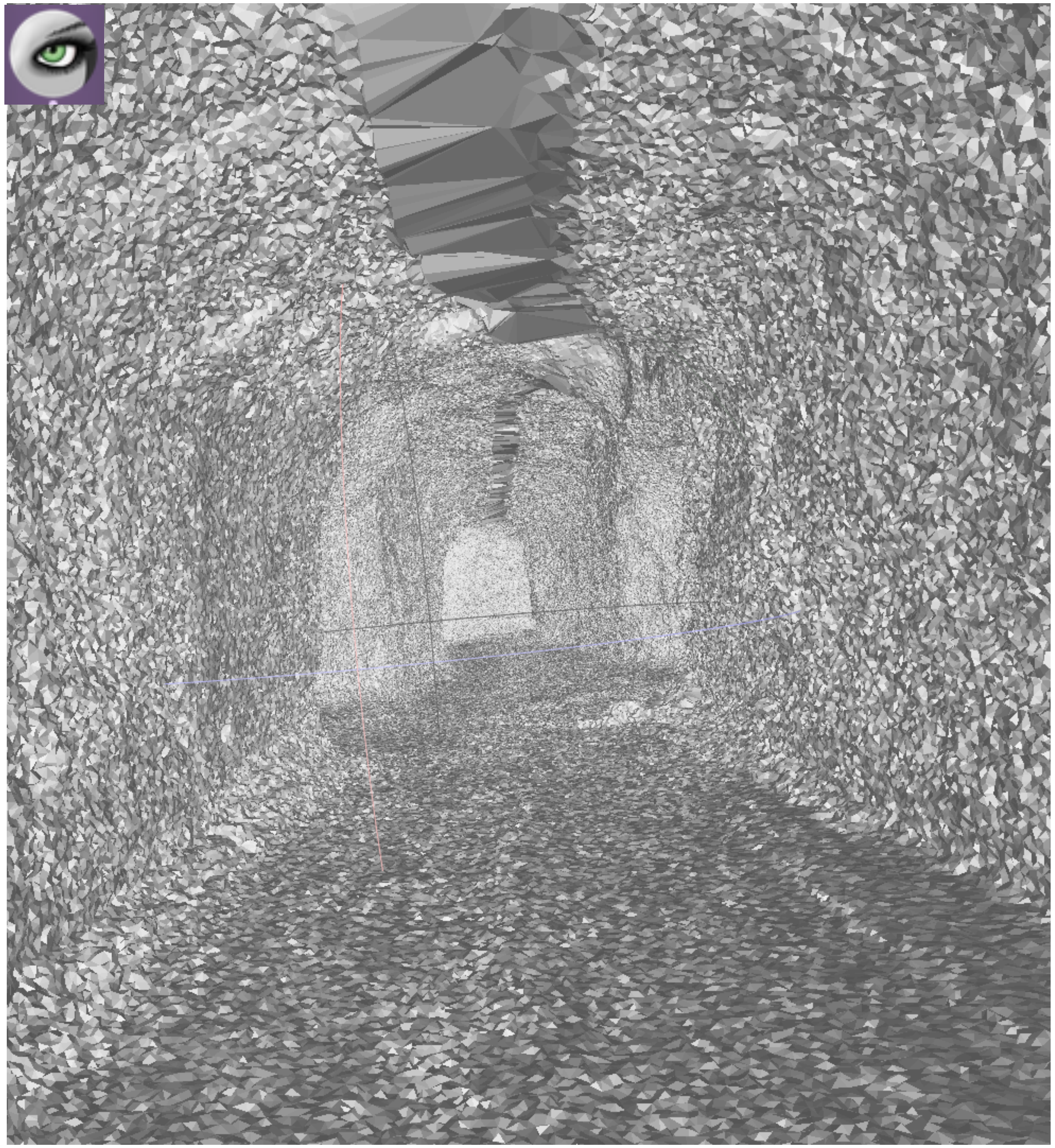

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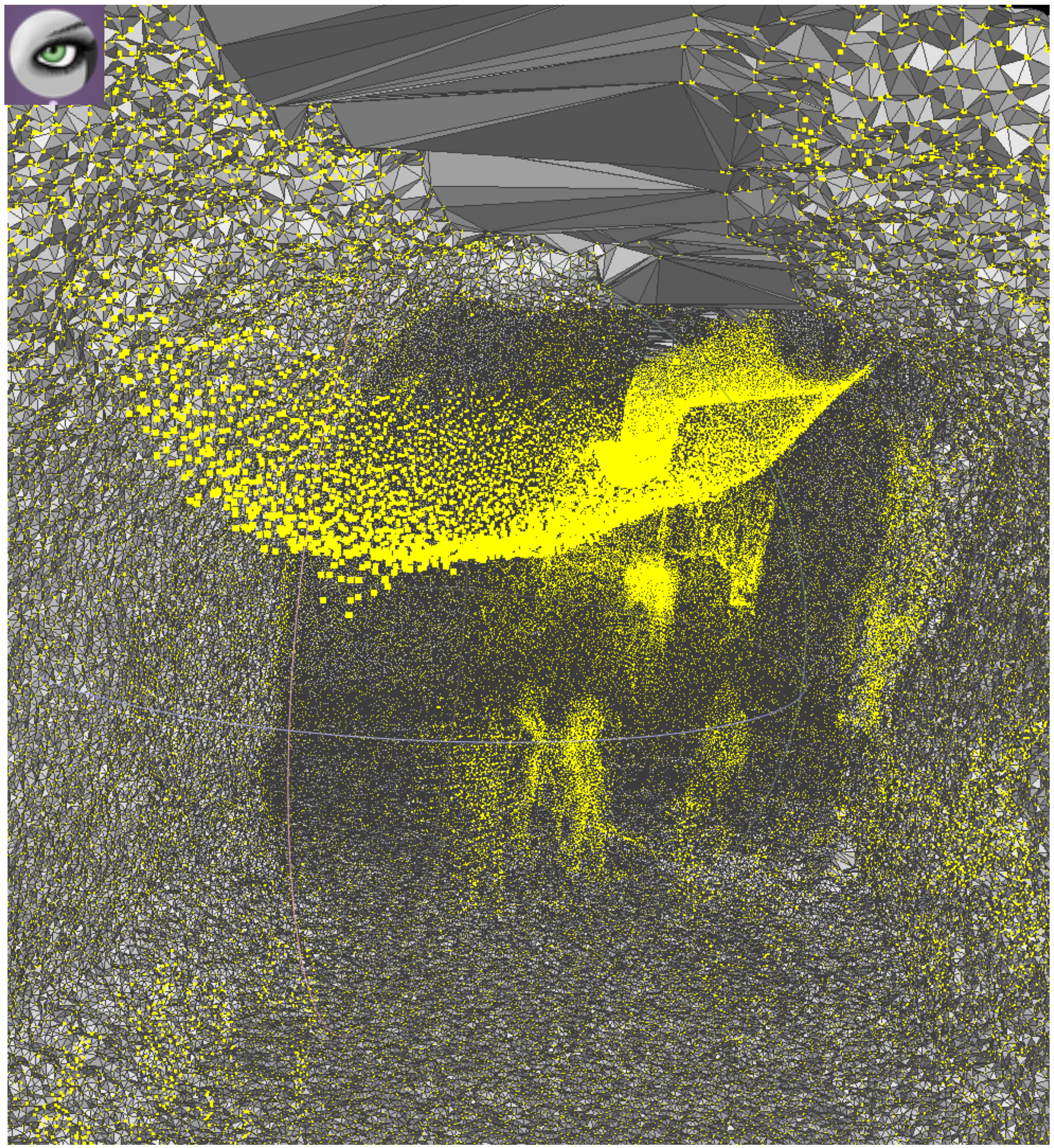

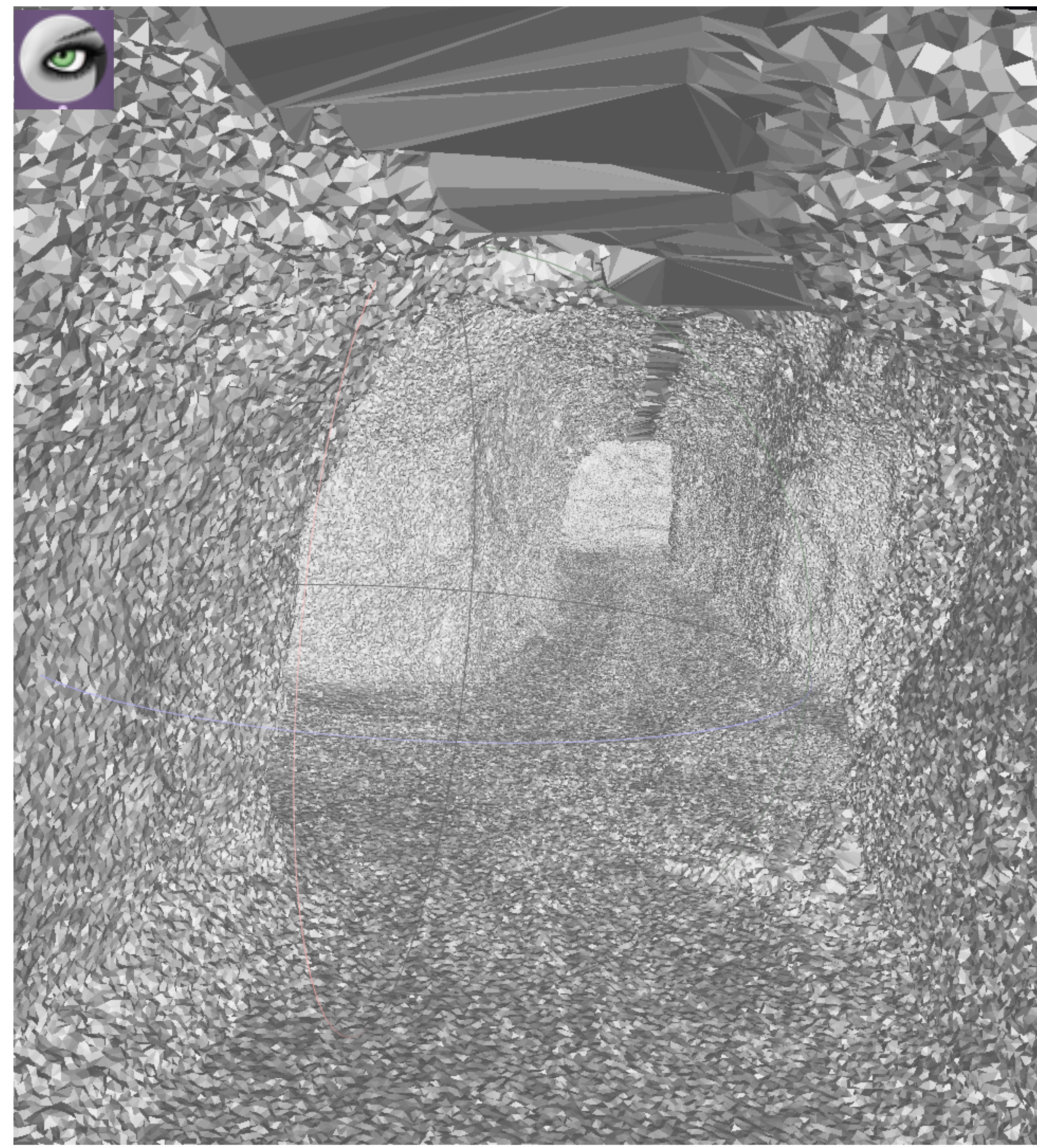

Thank you !