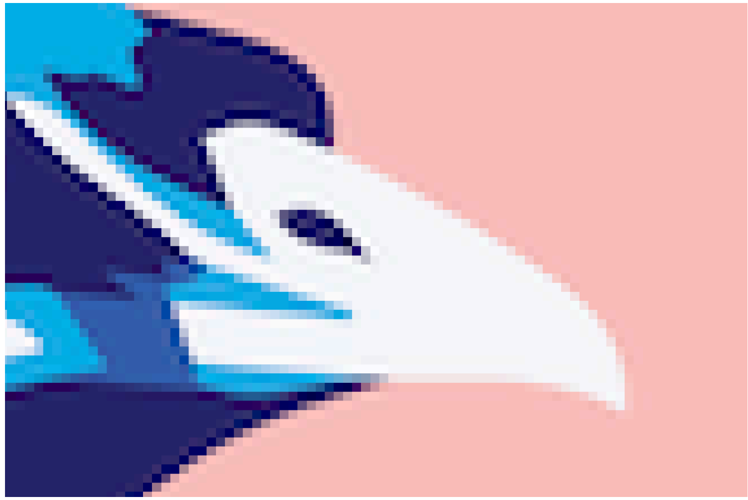
Università di Genova
DIBRIS DIPARTIMENTO
DI INFORMATICA, BIOINGEGNERIA,
ROBOTICA E INGEGNERIA DEI SISTEMI

Vector Graphics on Discrete Surfaces

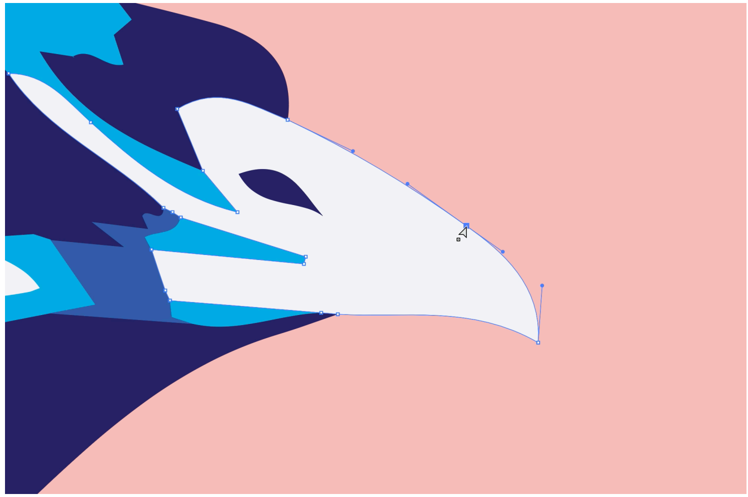
Claudio Mancinelli

Interactive Vector Graphics on Discrete Surfaces

Interactive ***Vector Graphics*** *on Discrete Surfaces*



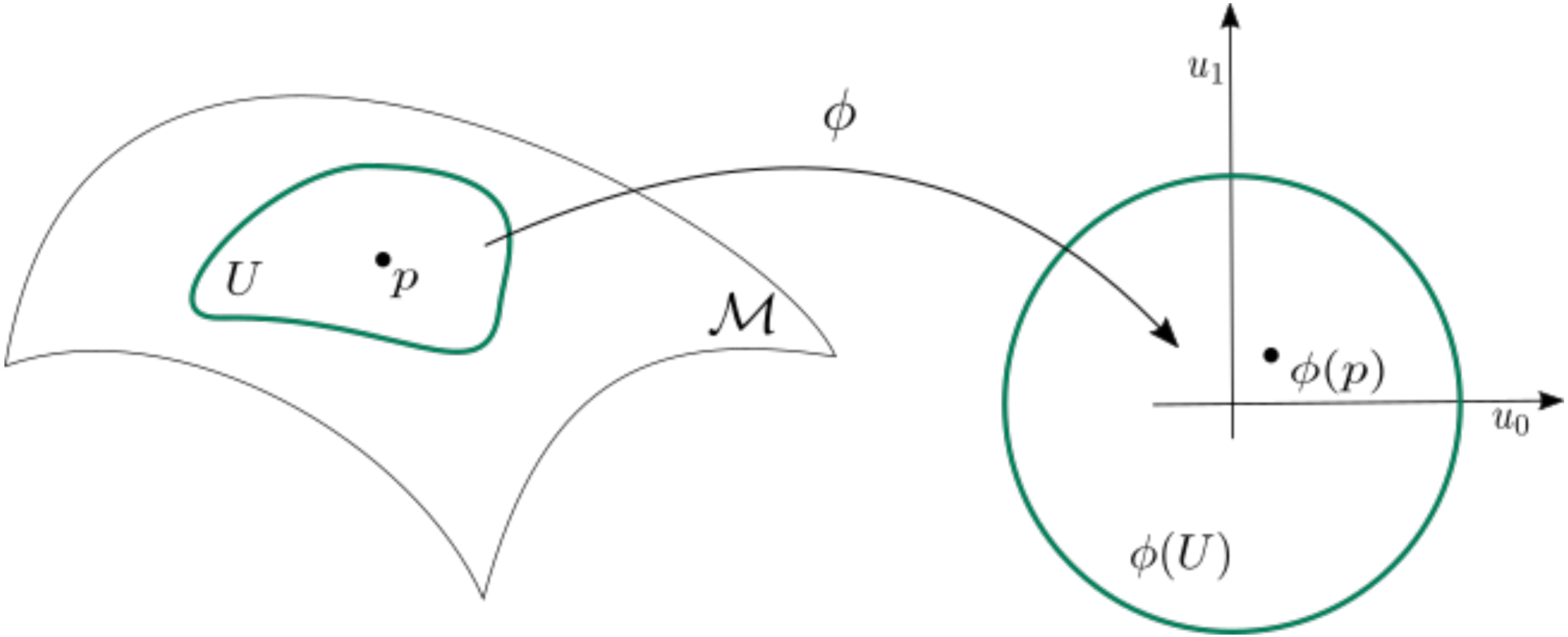
Raster graphics



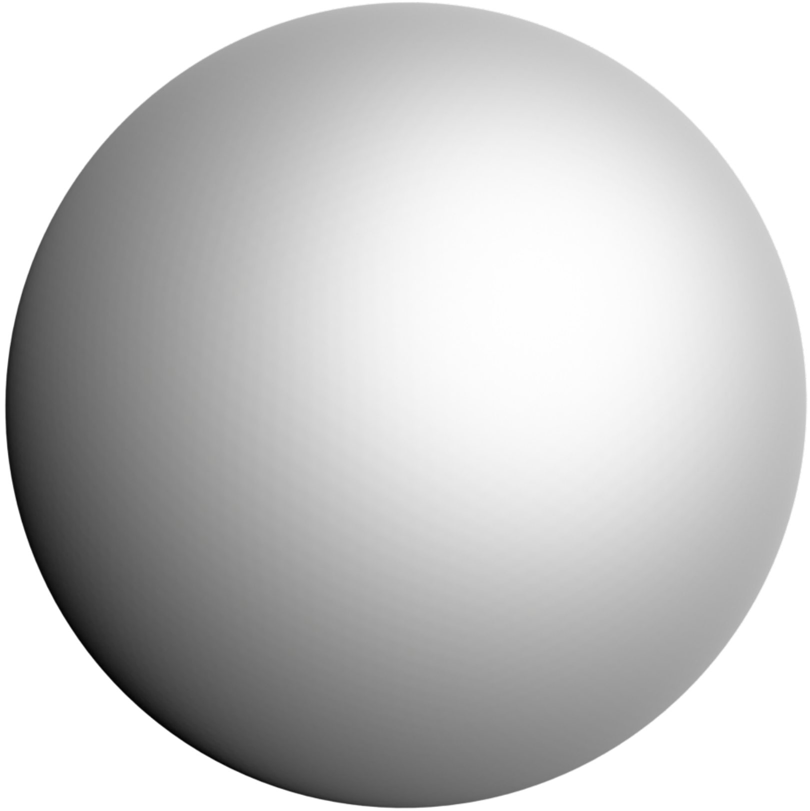
Vector graphics

Interactive Vector Graphics on Discrete **Surfaces**

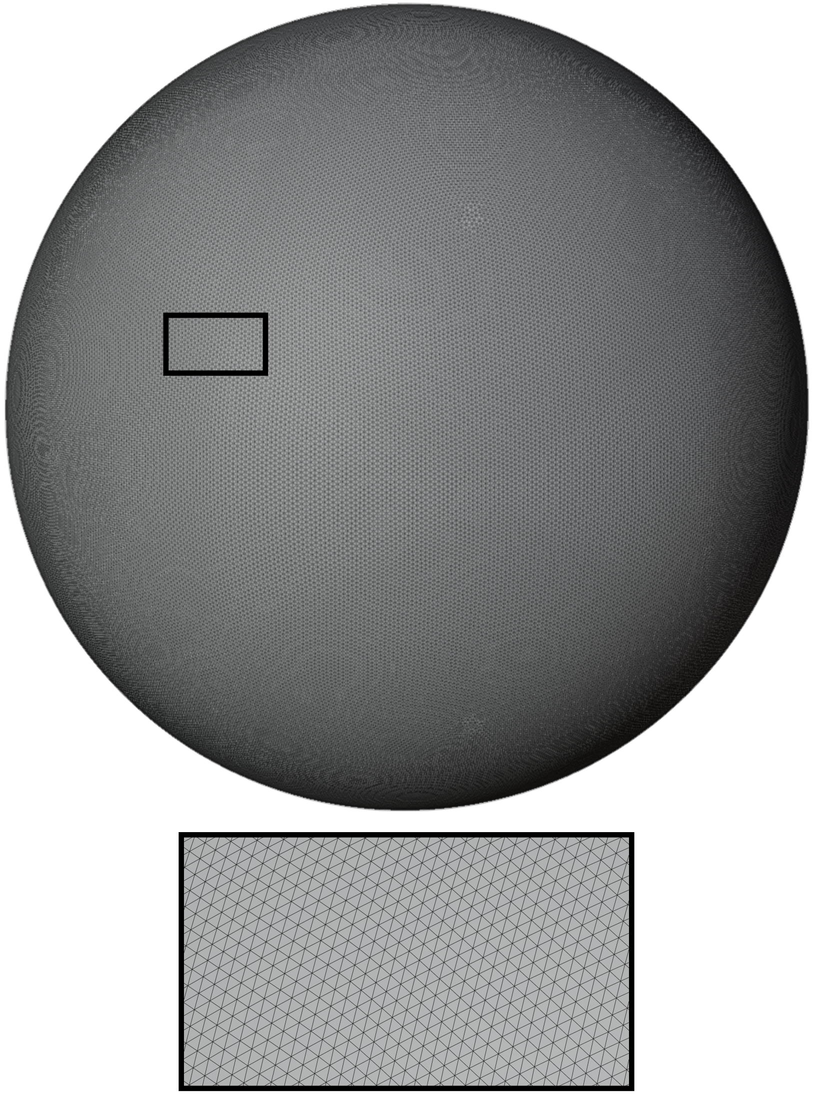
Every point has a neighborhood that "looks like" an open set of the Euclidean plane

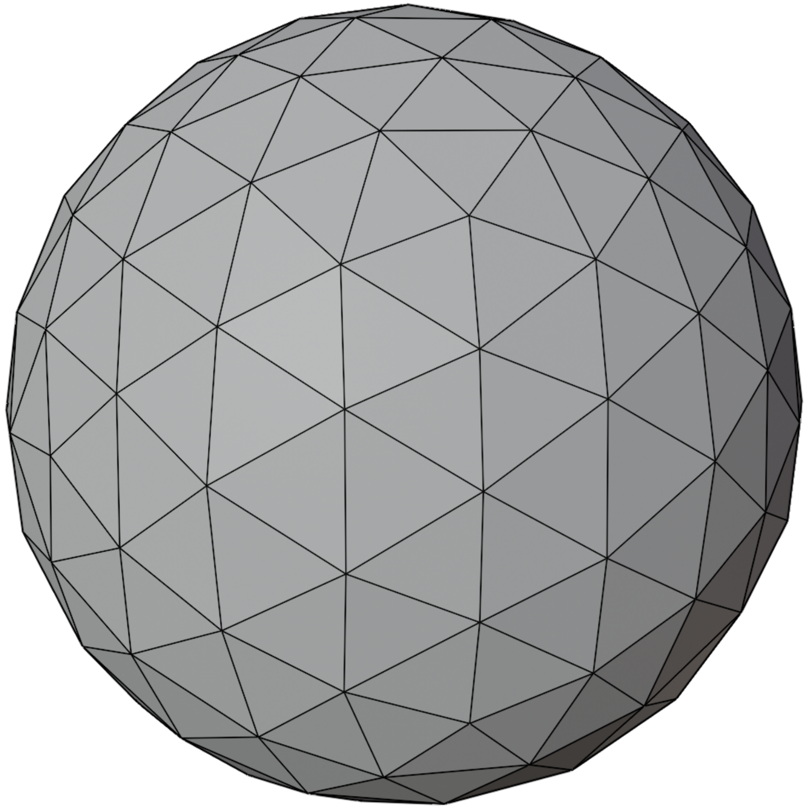


Interactive Vector Graphics on ***Discrete Surfaces***



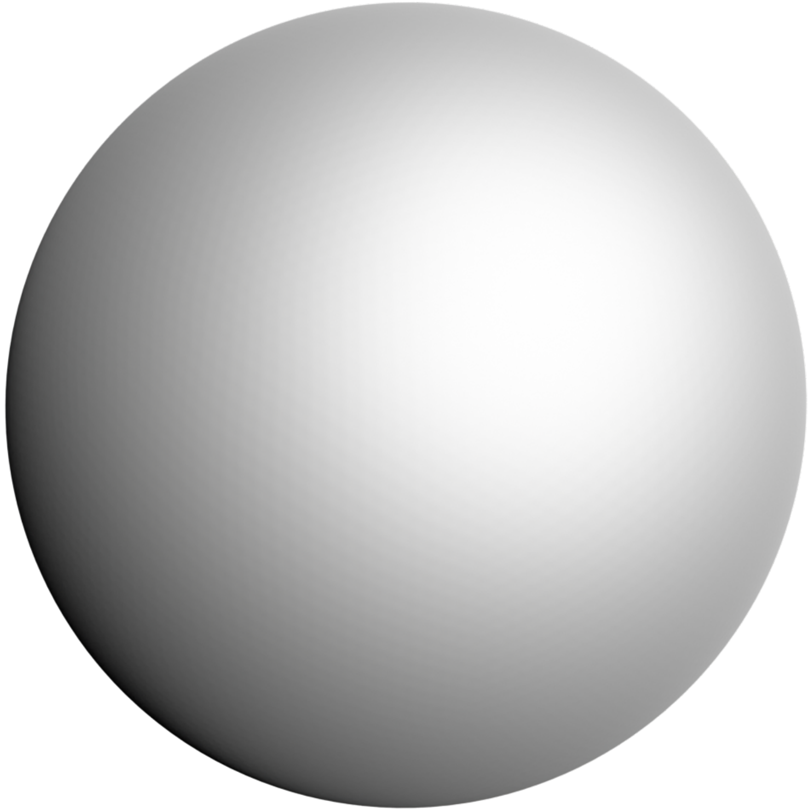
$$\{x \in \mathbb{R}^3 : \|x\| = 1\}$$



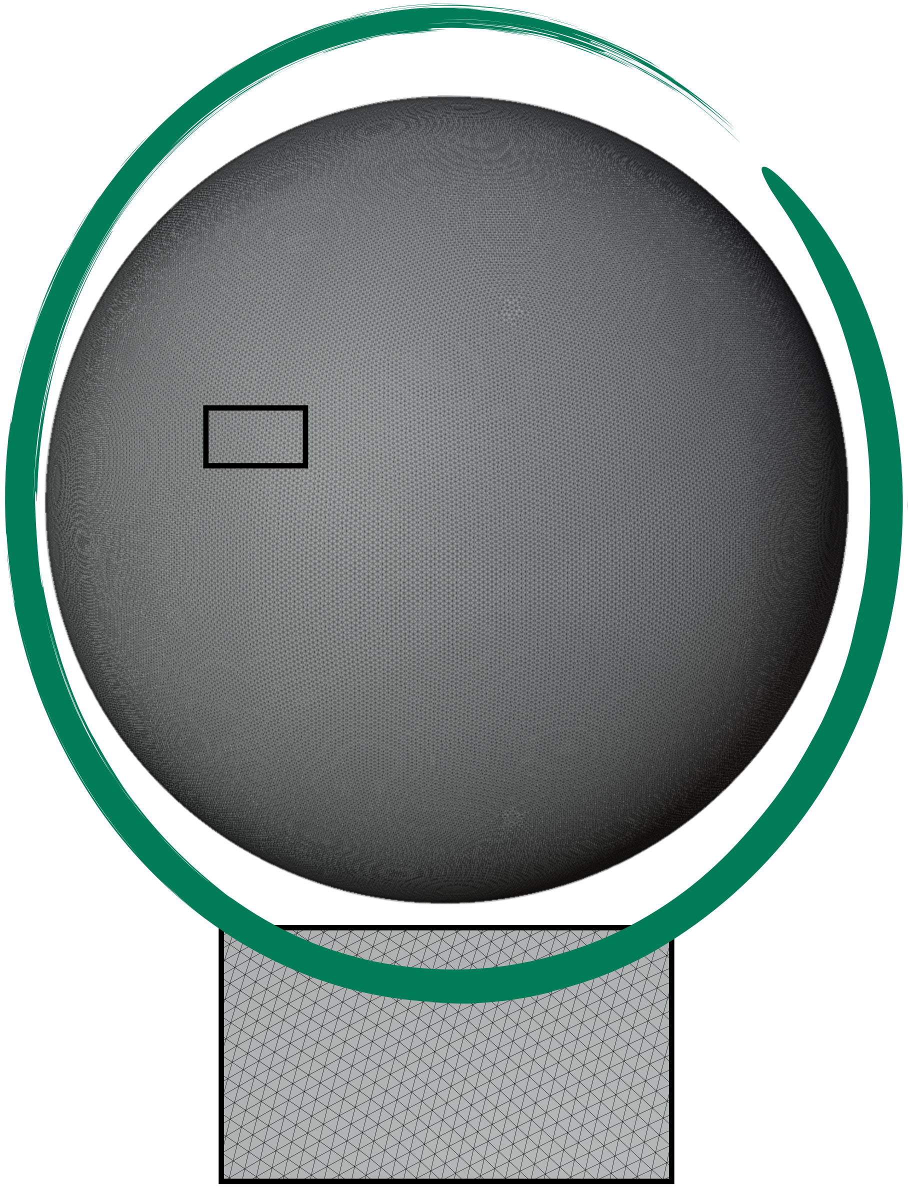


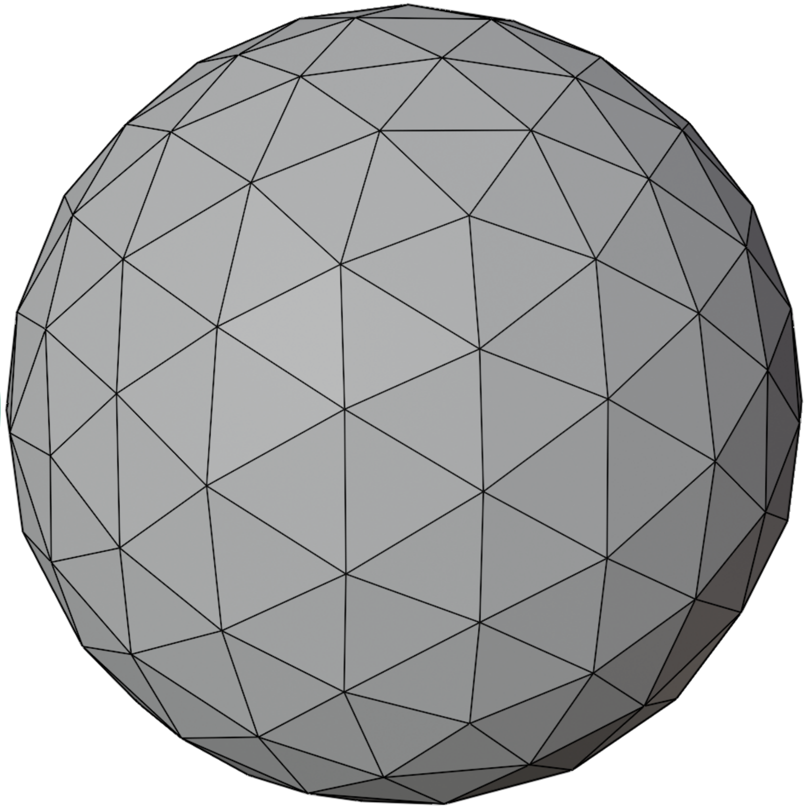
162 vertices
320 triangles

Interactive Vector Graphics on Discrete Surfaces



$$\{x \in \mathbb{R}^3 : \|x\| = 1\}$$

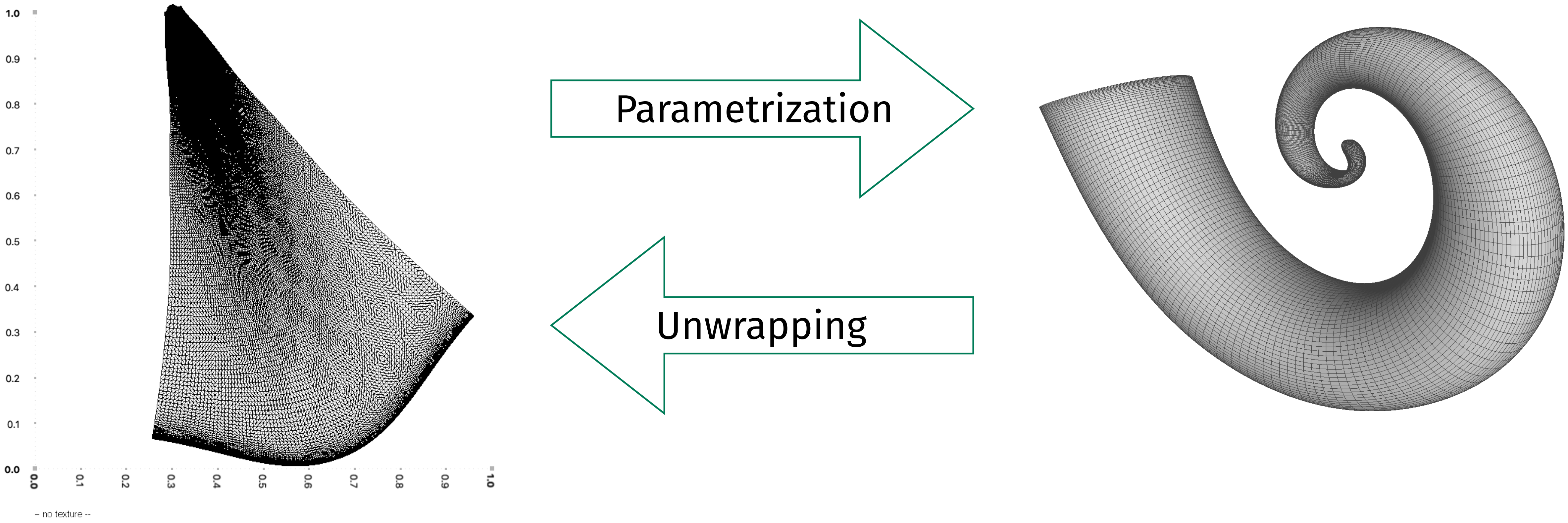




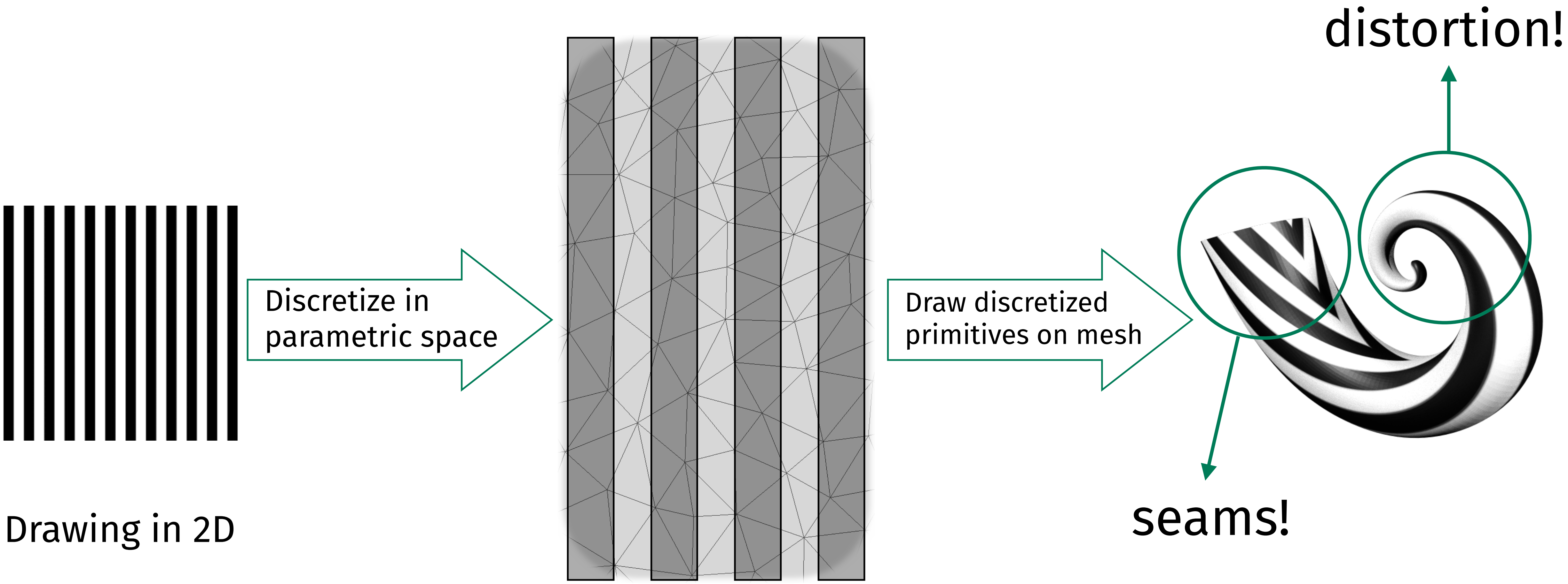
162 vertices
320 triangles

The Traditional Way

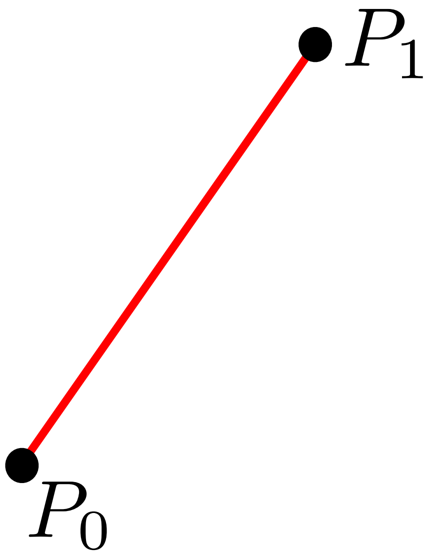
Parametrization maps from 2D plane to the surface

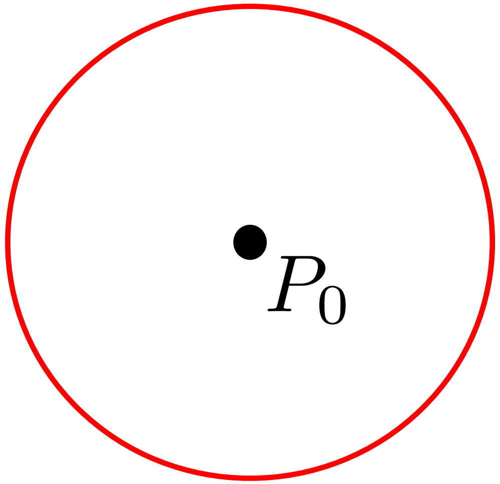


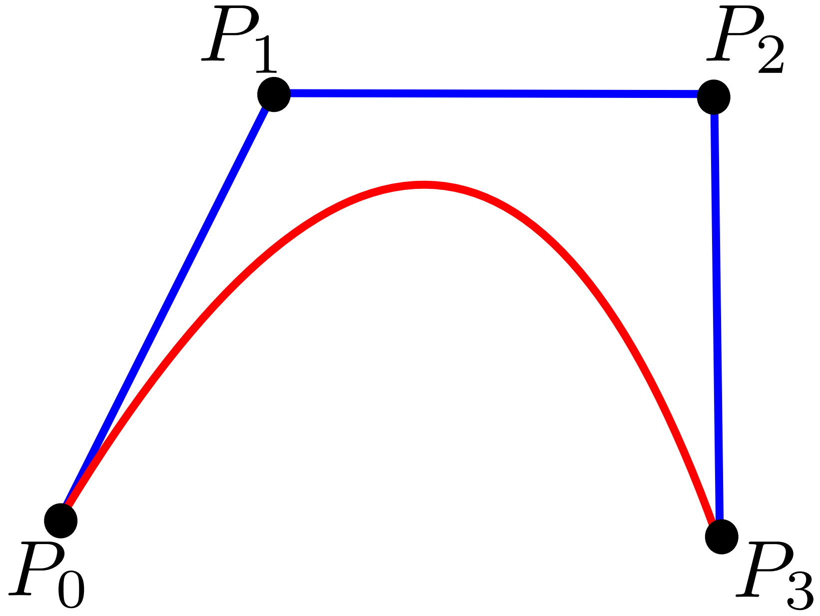
The Traditional Way



Geometric Primitives in the Euclidean setting





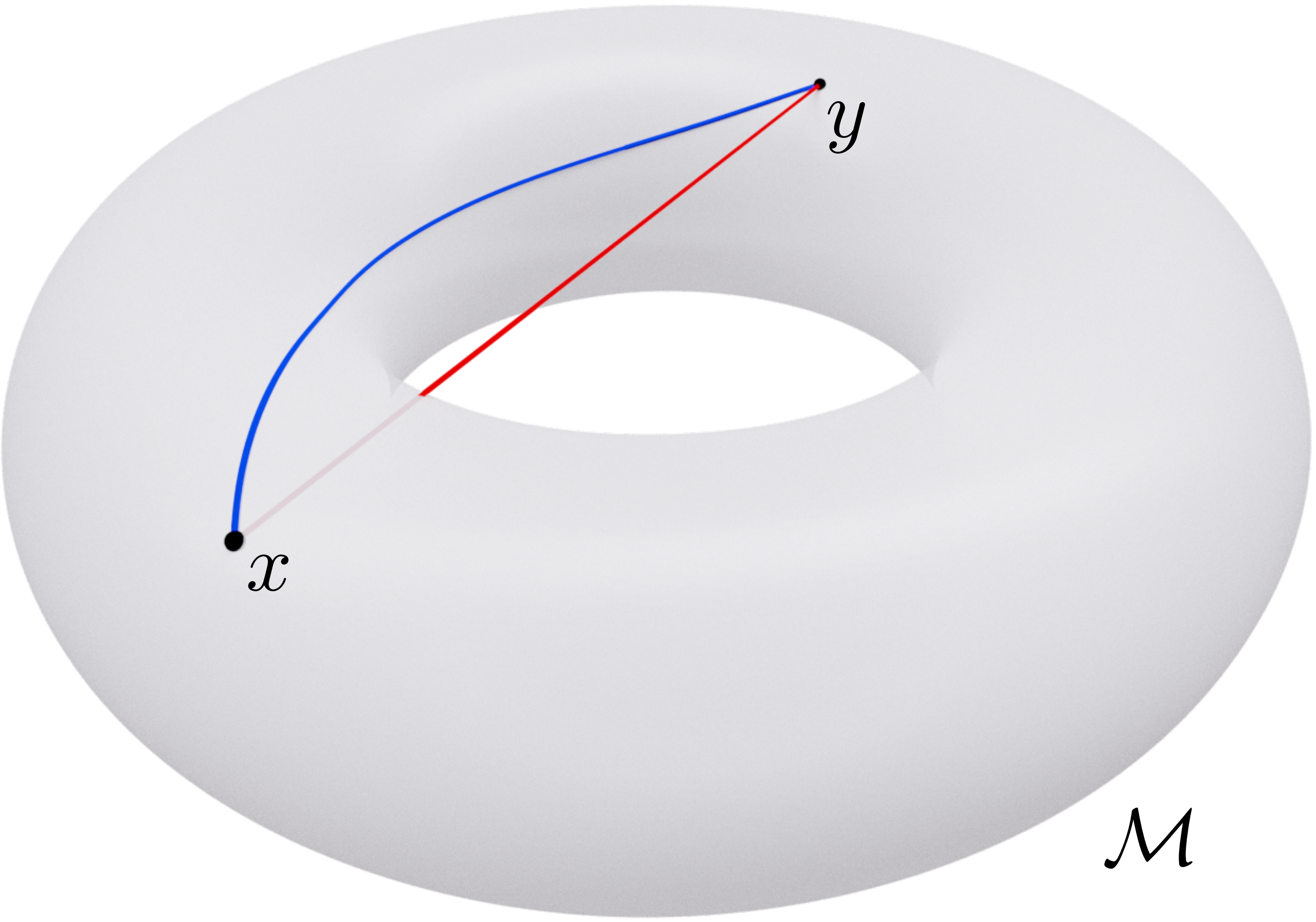


$$(1-t)P_0 + tP_1$$

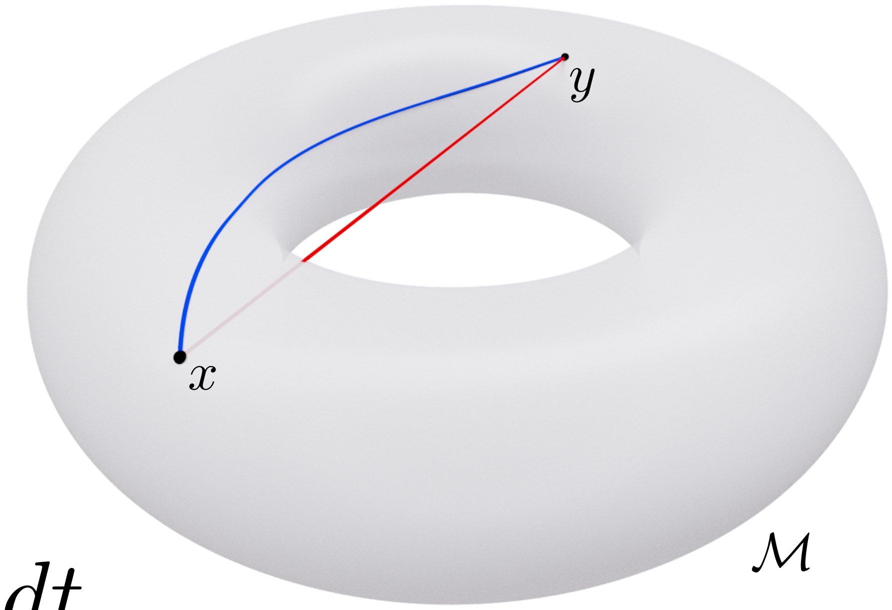
$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

$$\sum_{i=0}^{3} B_i^3(t)P_i$$

Geodesic Distance



Geodesic Distance



Let $\gamma : [a, b] \to \mathcal{M}$ be a curve on $\mathcal{M}$.

Then its *length* is defined as

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)} dt$$

The *geodesic distance* between two points on $\mathcal{M}$ can be defined as

$$d(x, y) = \inf\{L(\gamma) : \gamma \in \mathcal{C}_{xy}\}$$

Geodesic Distance

The *geodesic distance* between two points on $\mathcal{M}$ can be defined as

$$d(x, y) = \inf\{L(\gamma) : \gamma \in \mathcal{C}_{xy}\}$$

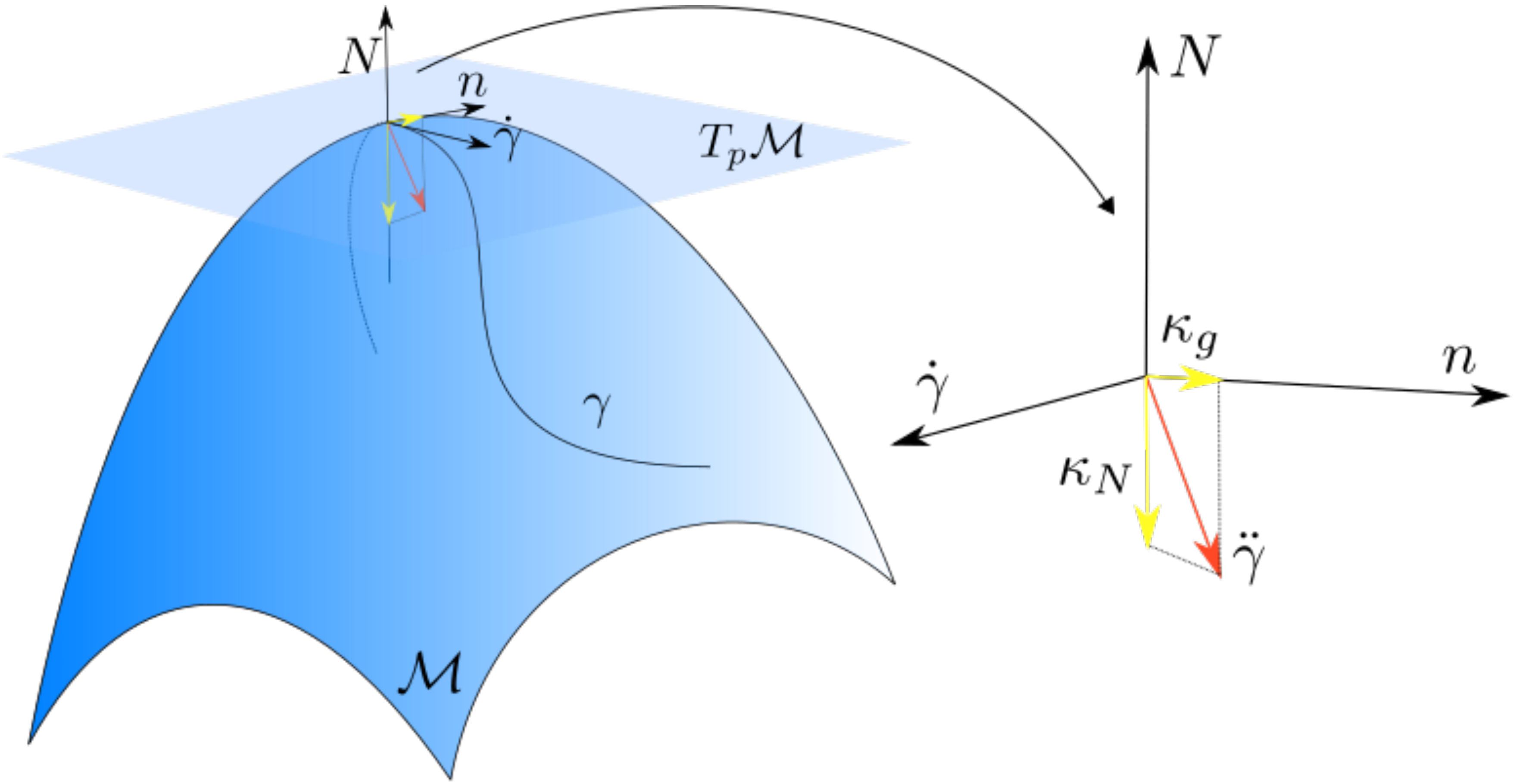
We will often consider the *geodesic distance field*

$$\begin{aligned} d_x : \mathcal{M} &\to \mathbb{R} \\ y &\mapsto d(x, y) \end{aligned}$$

Geodesics

If a curve $\gamma : [a,b] \to \mathcal{M}$ realizes the distance between two points, then γ is called a *geodesic*.

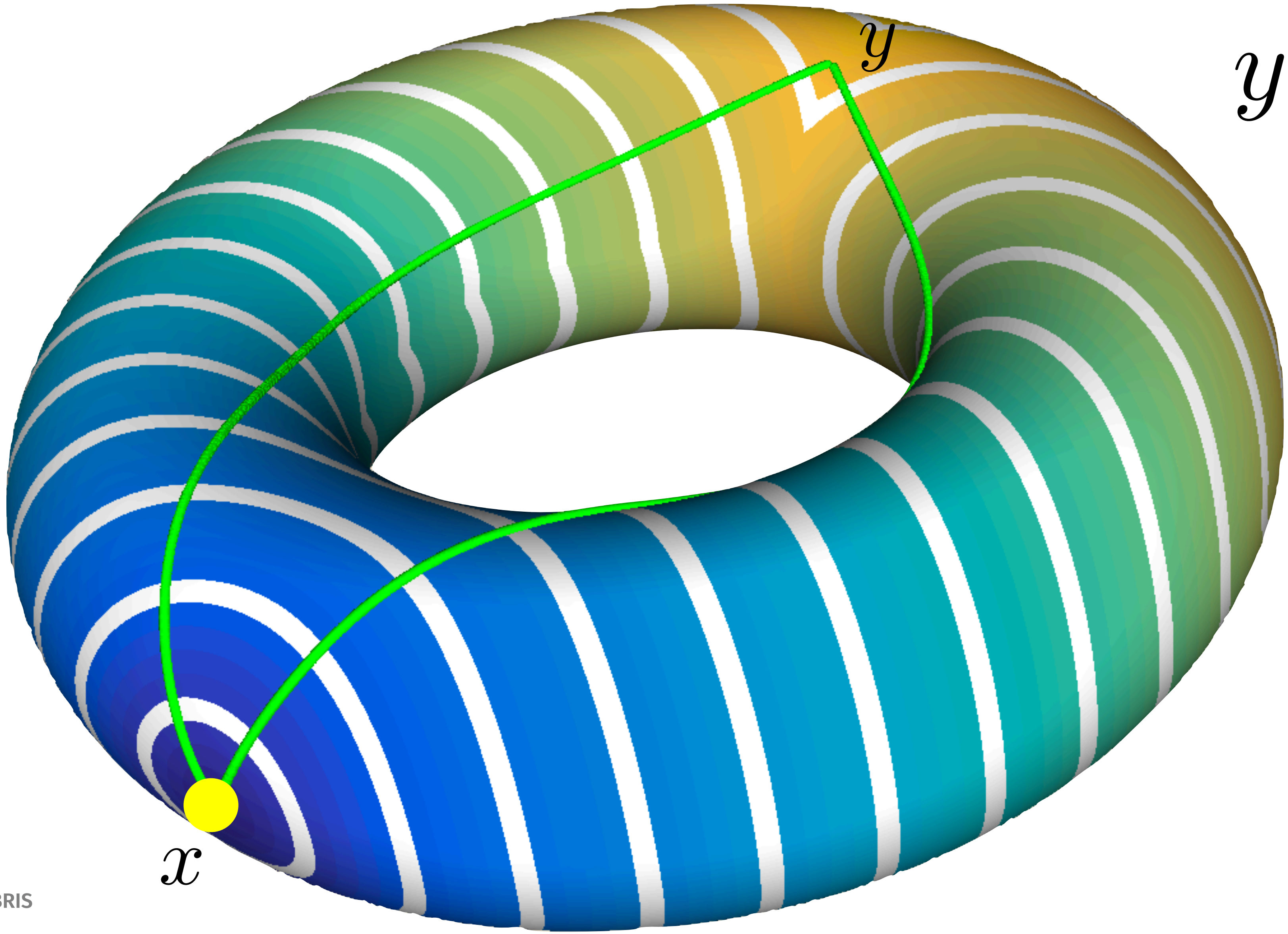
Geodesics have null (geodesic) curvature.



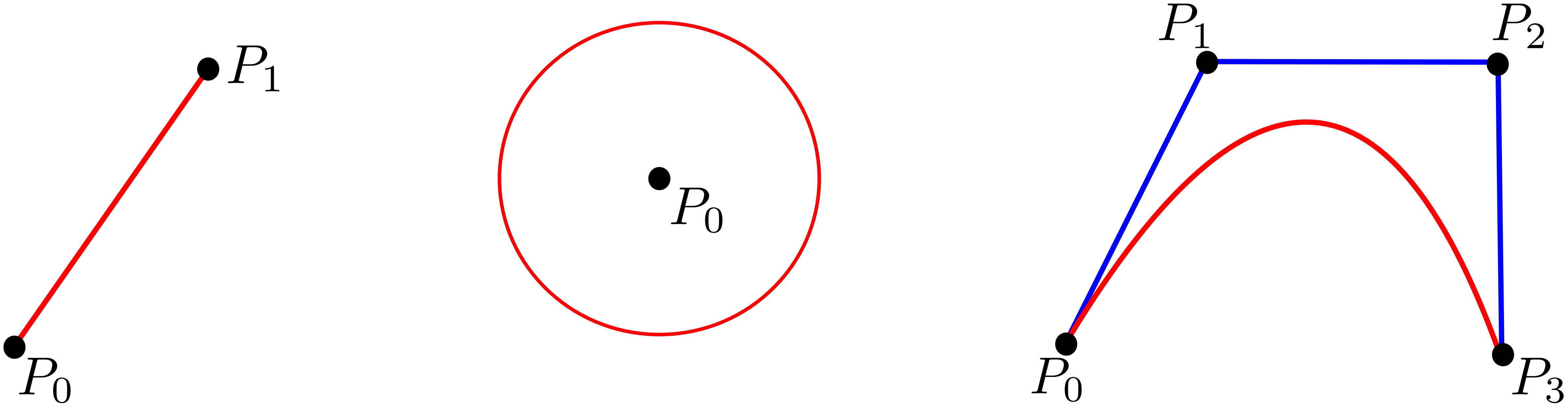
Geodesic distance field

$$d_x : \mathcal{M} \to \mathbb{R}$$

$$y \mapsto d(x, y)$$

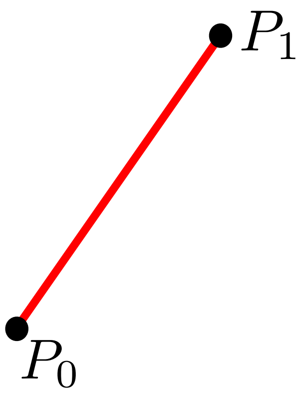


Moving to the manifold setting

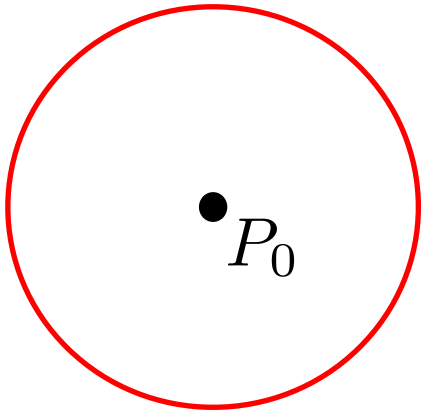


Moving to the manifold setting

$\mathbb{R}^3$ $\mathcal{M}$



Straight line → *Geodesic*

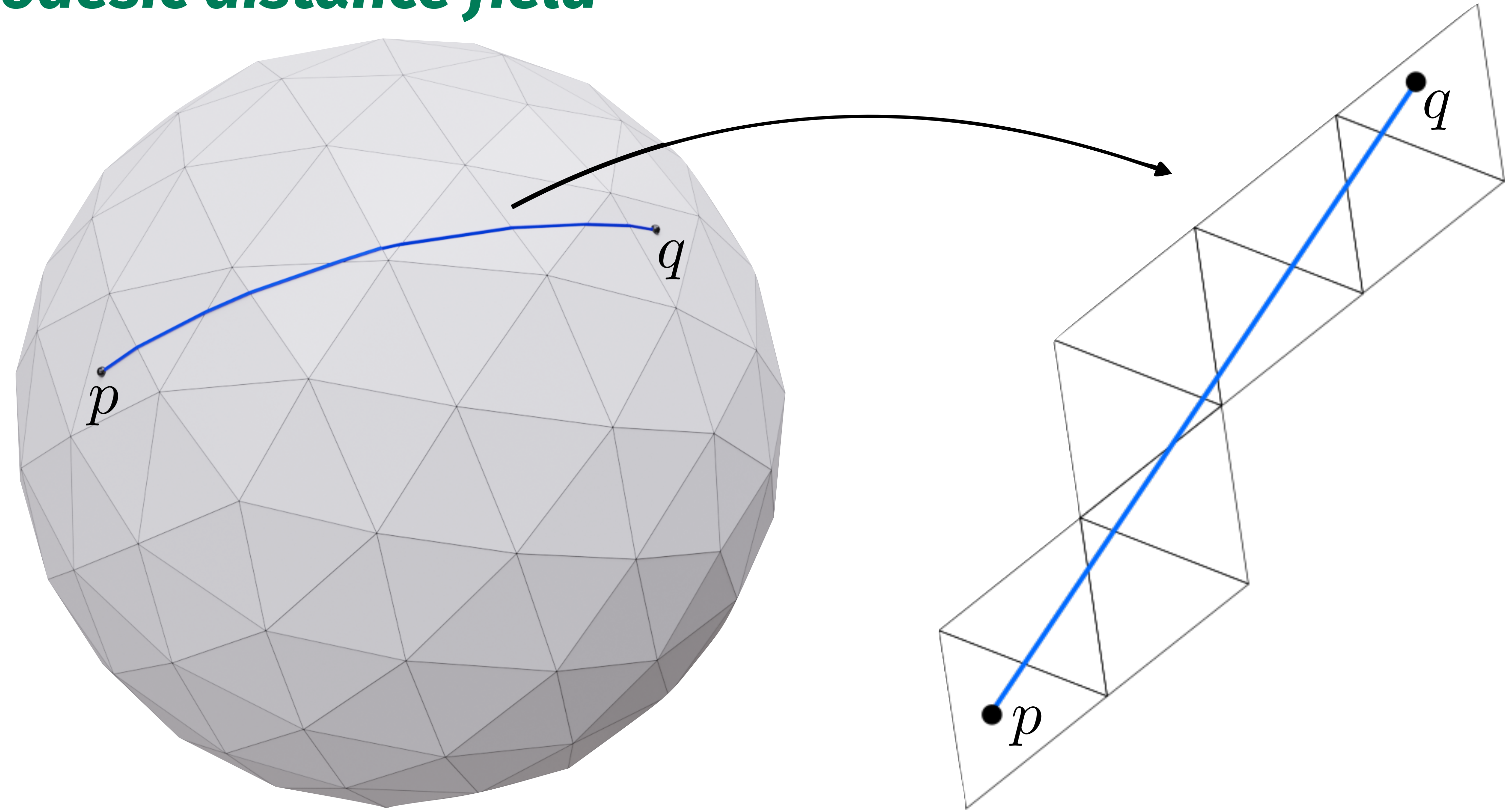


Circle → *Geodesic circle*

Moving to the manifold setting

$\mathbb{R}^3$		$\mathcal{M}$		M (discretization of $\mathcal{M}$)
Straight line	→	Geodesic	→	Shortest path
Circle	→	Geodesic circle	→	$d_x(\cdot)$

Geodesic distance field

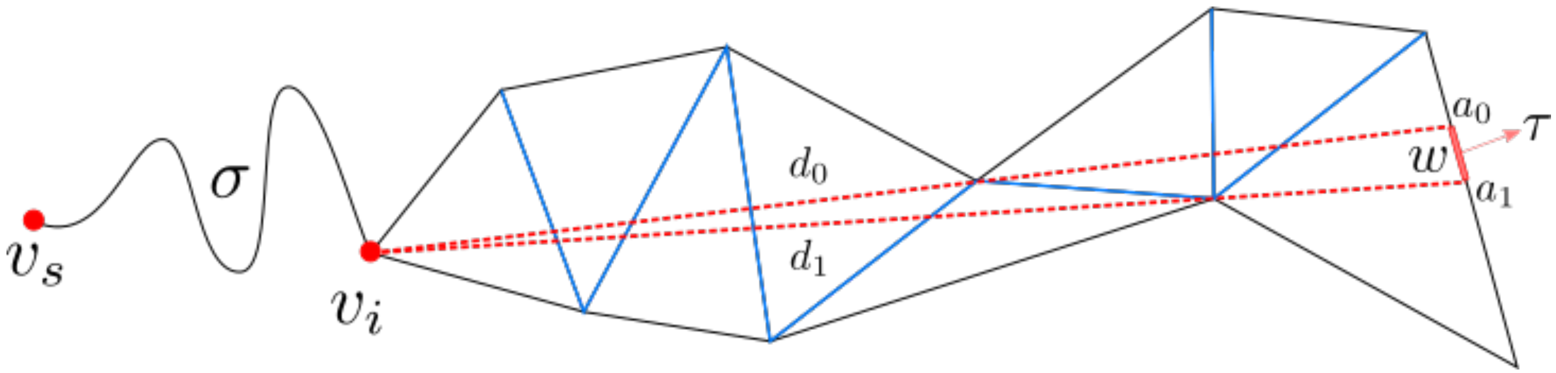


[Mitchell et al., 1987]

Geodesic distance field

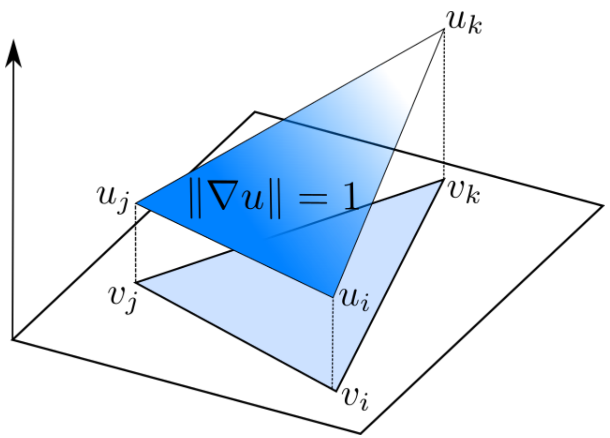
Exact polyhedral methods

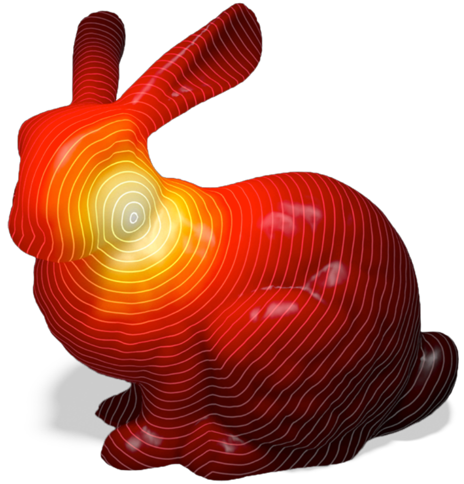
[Mitchell et al., 1987; Chen and Han, 1990; Surazhsky et al., 2005; Xin and Wang, 2007; Xu et al., 2015; Qin et al., 2016]



PDE-based methods

[Kimmel and Sethian, 1998; Crane et al. 2013]

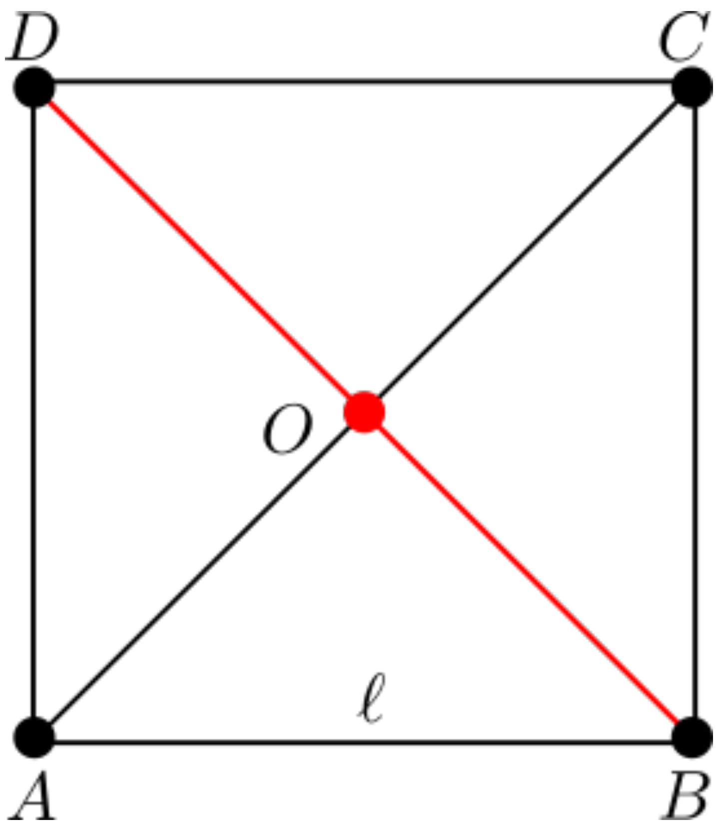




Geodesic distance field

Graph-based methods

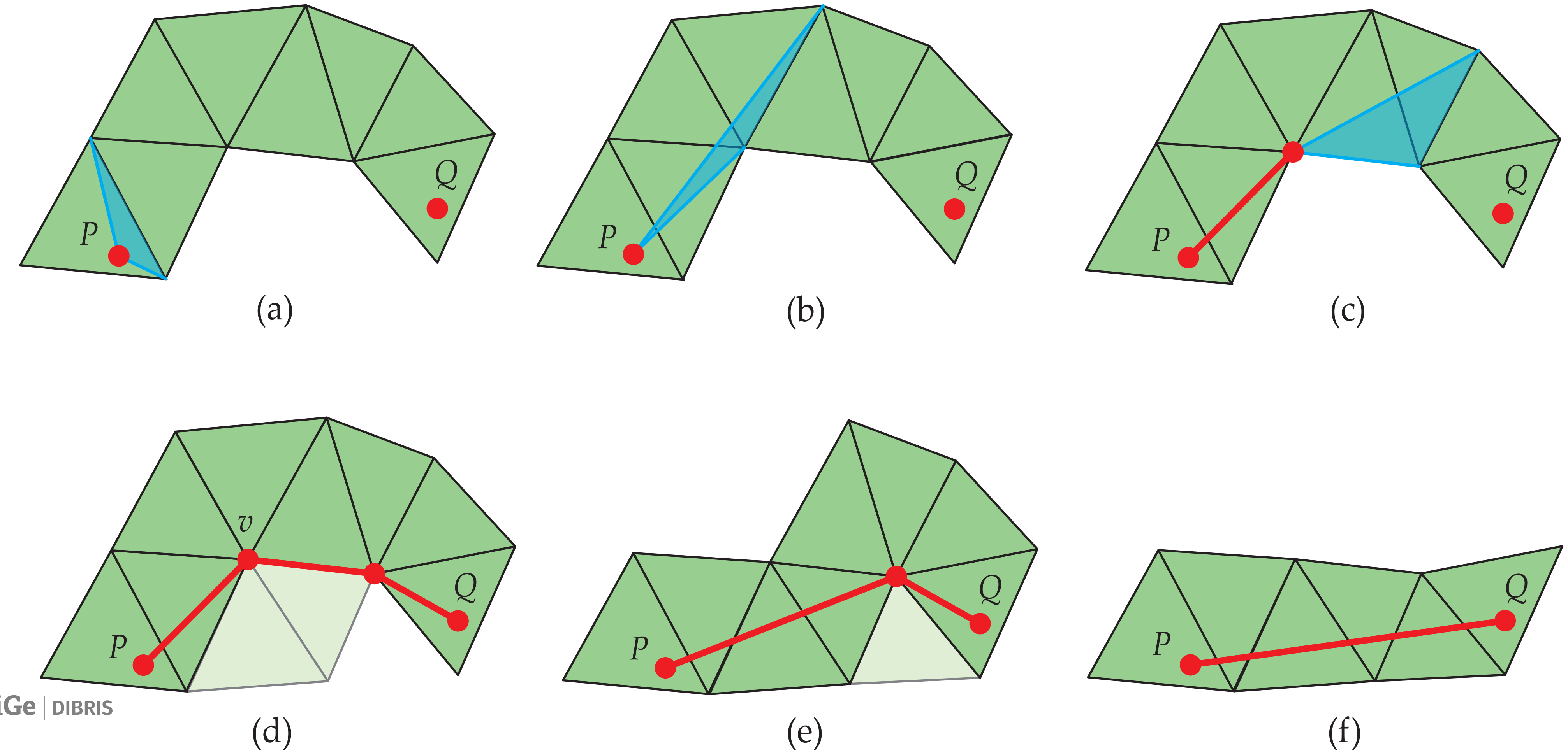
[Lanthier, 1997; Mata and Mitchell, 1997; Aleksandrov et al., 2000; Aleksandrov et al., 2005; Ying et al., 2013; Wang et al., 2017]



Shortest paths tracing

Local methods

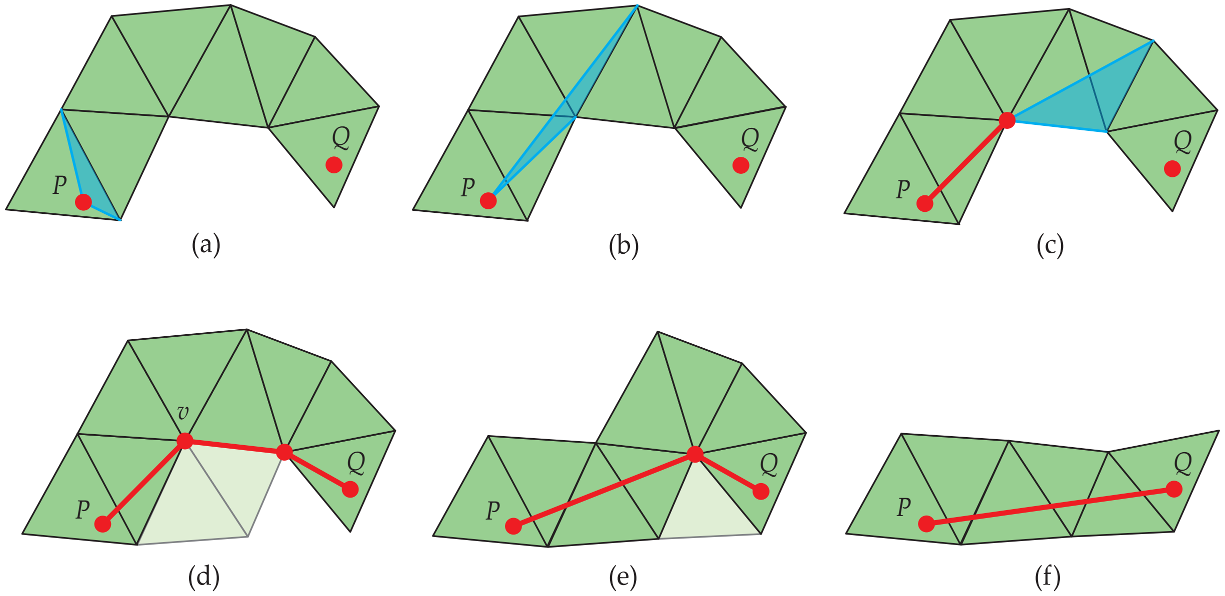
[Wan, 2004; Martínez et al., 2005; Xin et al., 2007; Sharp and Crane, 2020]



Geodesic queries on triangle mesh

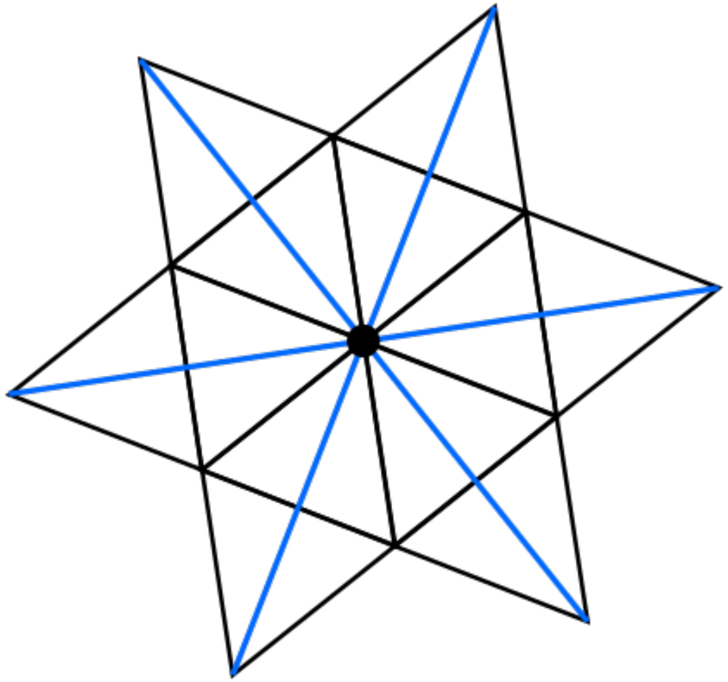
Shortest paths

[Mancinelli et al., 2022]



Geodesic distance fields

[Nazzaro et al., 2022]

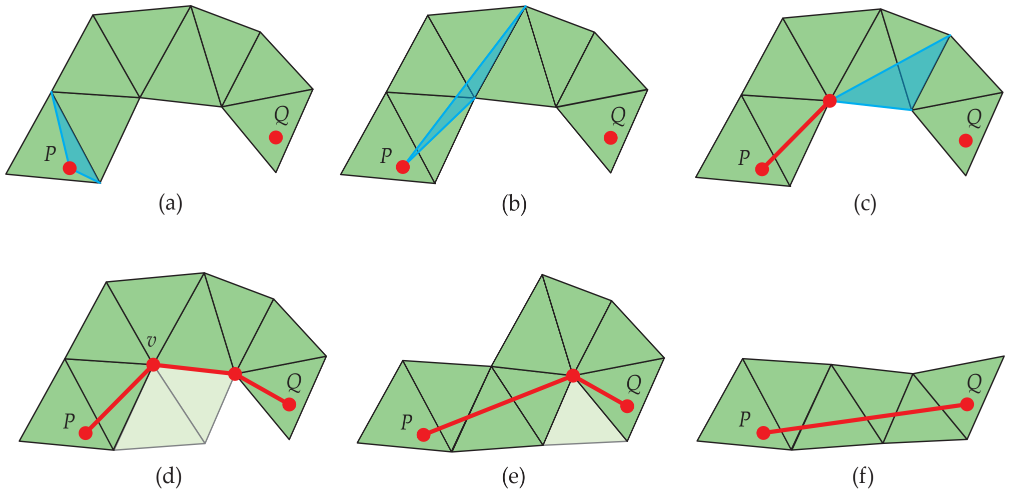


For more details see
[Crane et al., 2020]

Geodesic queries on triangle mesh

Shortest paths

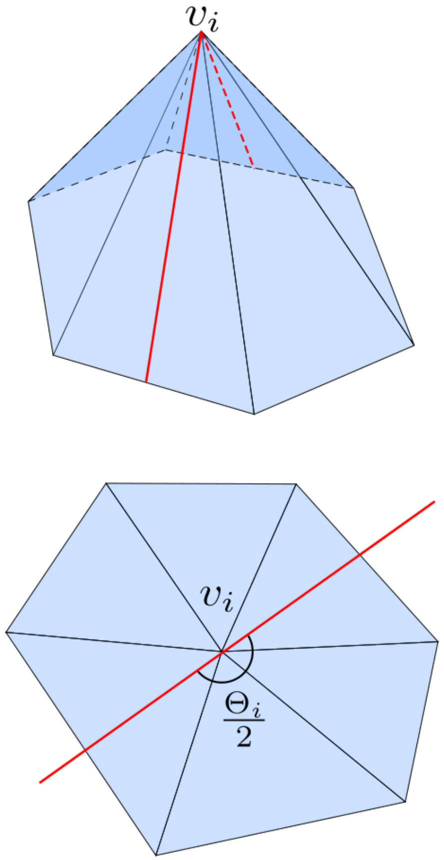
[Mancinelli et al., 2022]



For more details see [Crane et al., 2020]

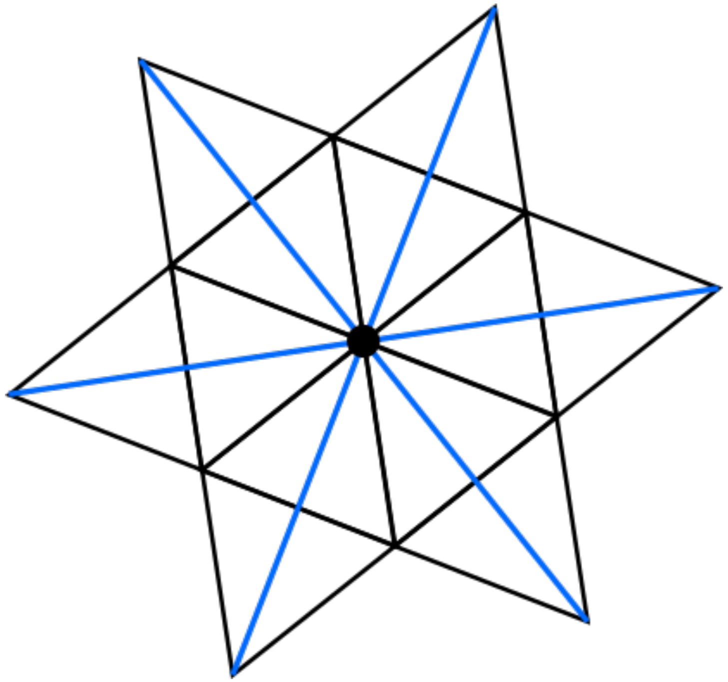
Straightest paths

[Polthier and Schmies, 1998]



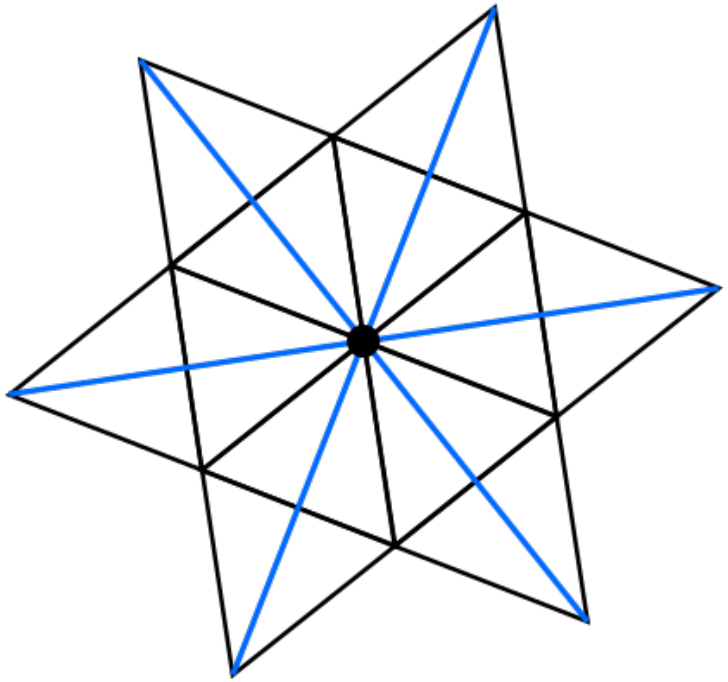
Geodesic distance fields

[Nazzaro et al., 2022]

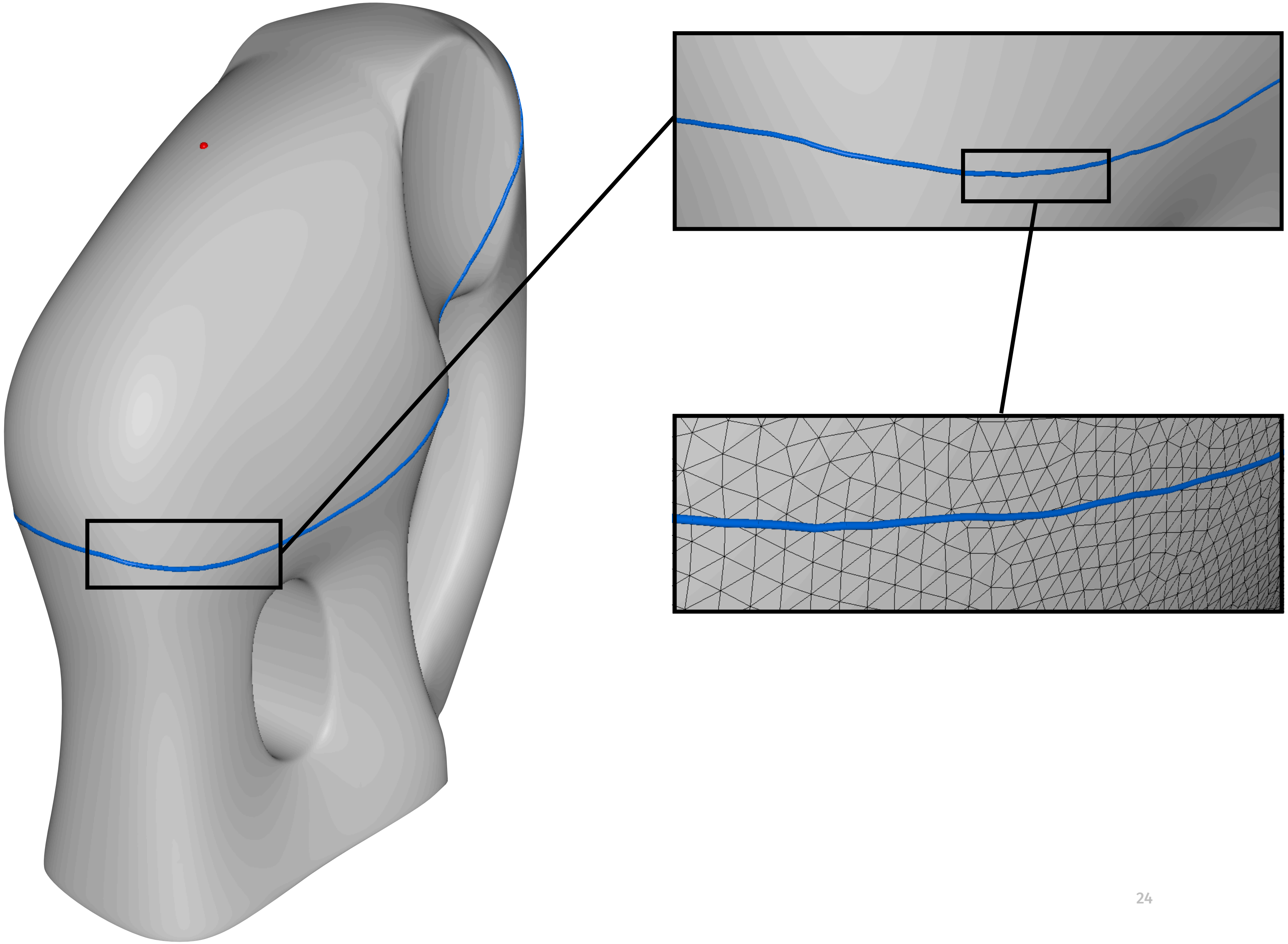


Geodesic queries on triangle mesh

1M triangles

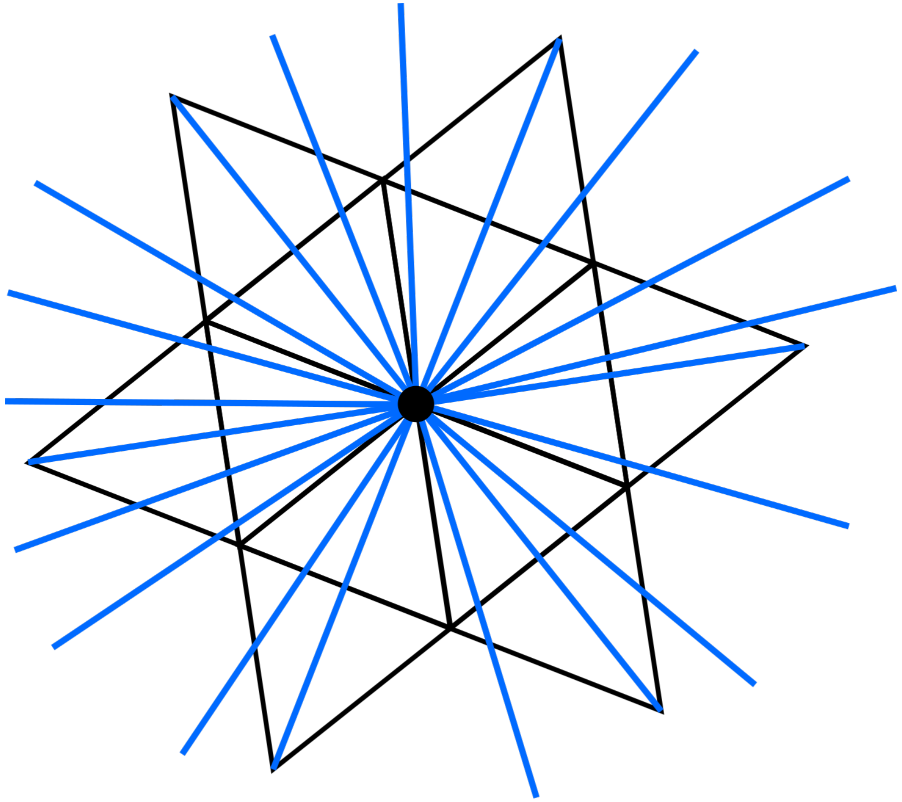


k=1.5

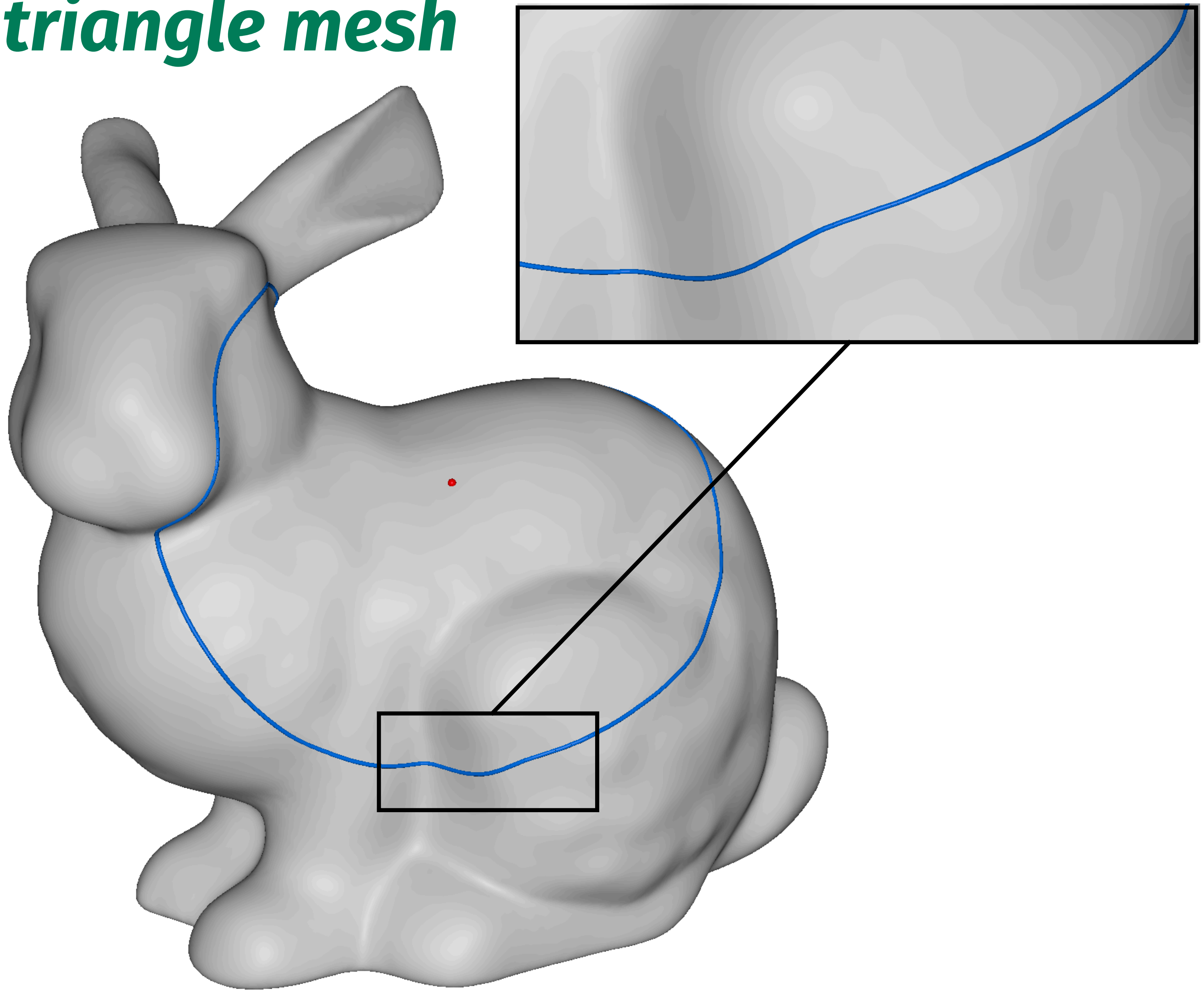


Geodesic queries on triangle mesh

200k triangles



k=4



Straightedge and compass constructions

- Straightedge: can be positioned between any two points and extended indefinitely in both directions

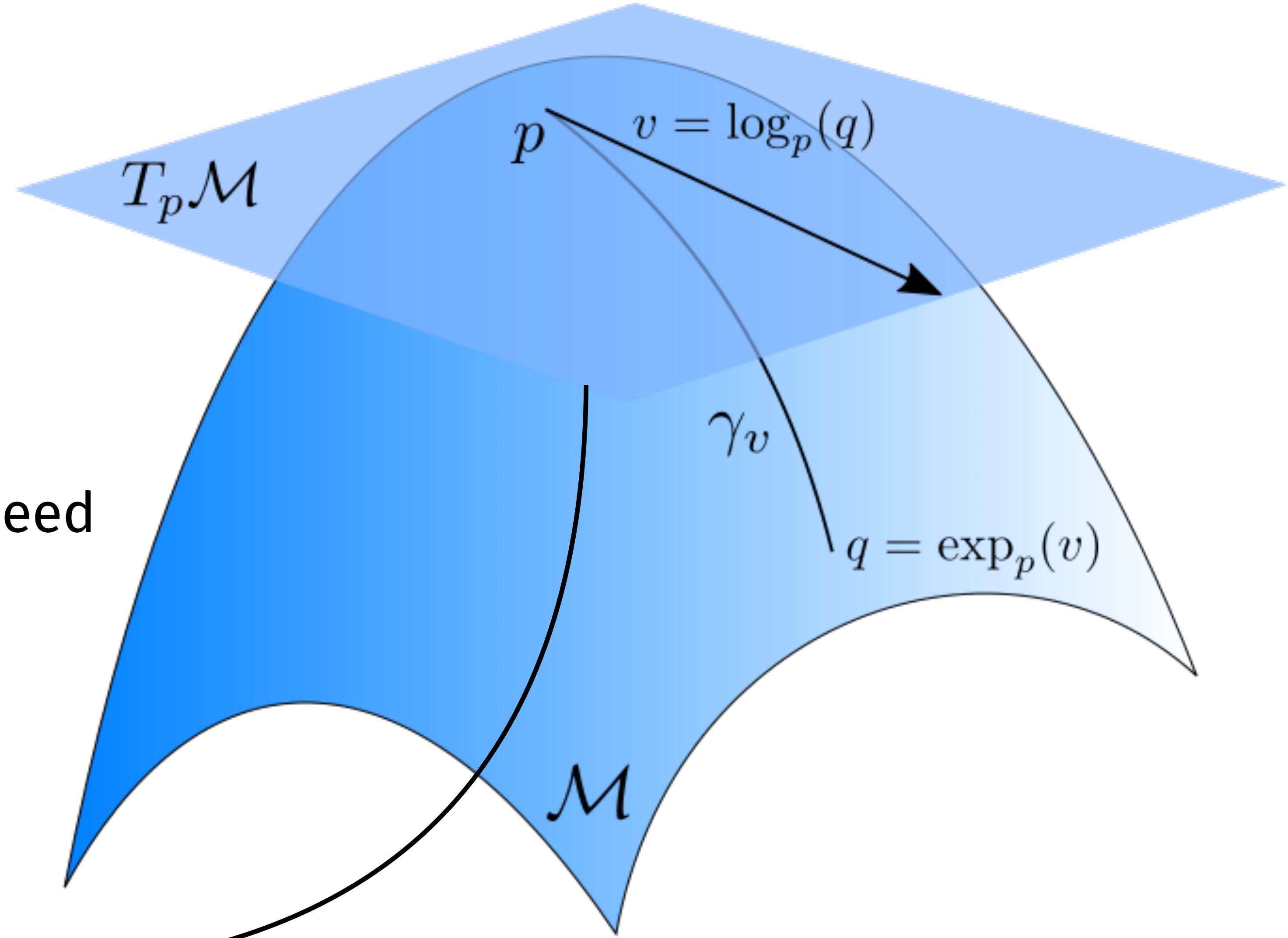
- Compass: can trace circles of any radius by starting with its needle and pencil points at two given points in the plane

Vector Graphics on Surfaces

Preliminaries

The exponential map “shoots” geodesics from a given point

The logarithmic map returns the direction of the geodesic “you need to shoot”



Vector space of dimension 2

Straightedge and compass constructions

How do we port straightedge and compass constructions on surface?

Draw on the tangent space and map the result to the surface

Draw directly on the surface

Vector Graphics on Surfaces

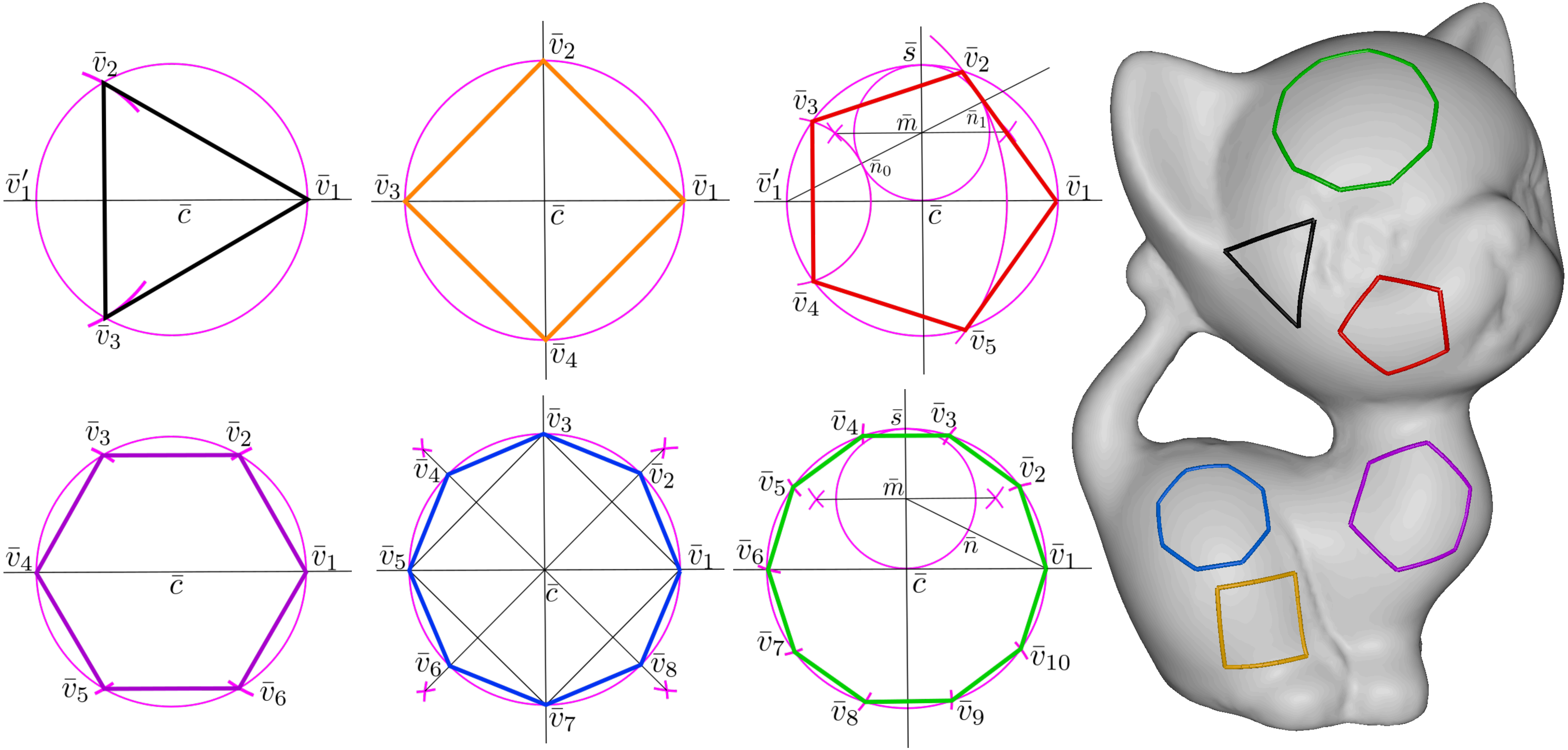
How do we port straightedge and compass constructions on surface?

Draw on the tangent space and map the result to the surface

Draw directly on the surface

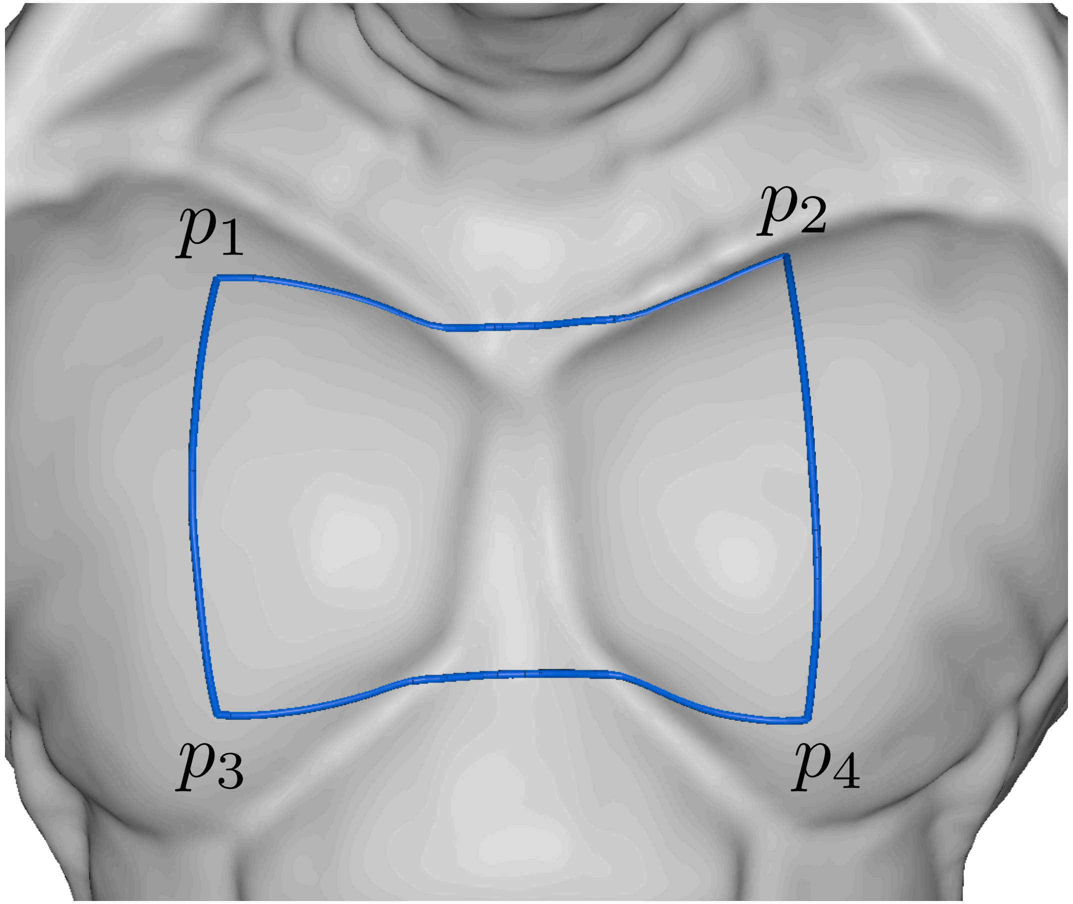
Vector Graphics on Surfaces

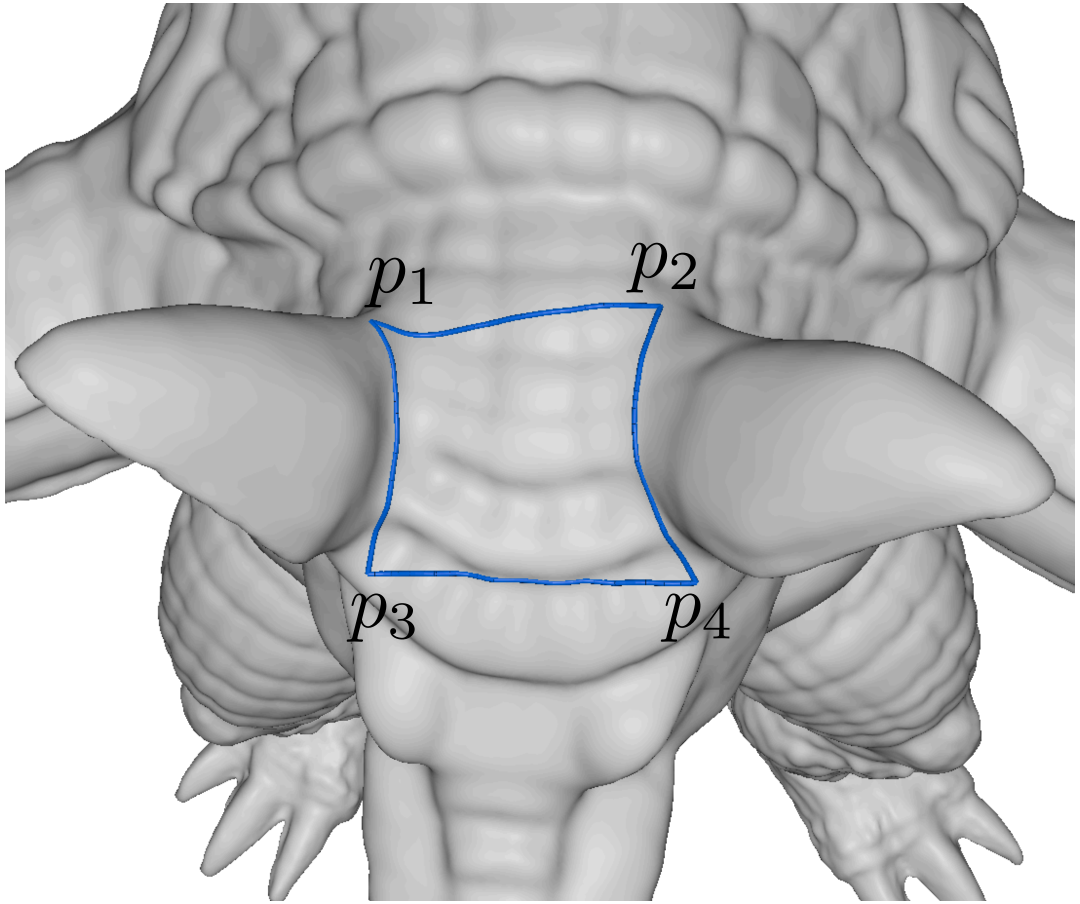
Tangent Space Constructions



Vector Graphics on Surfaces

Tangent Space Constructions





Vector Graphics on Surfaces

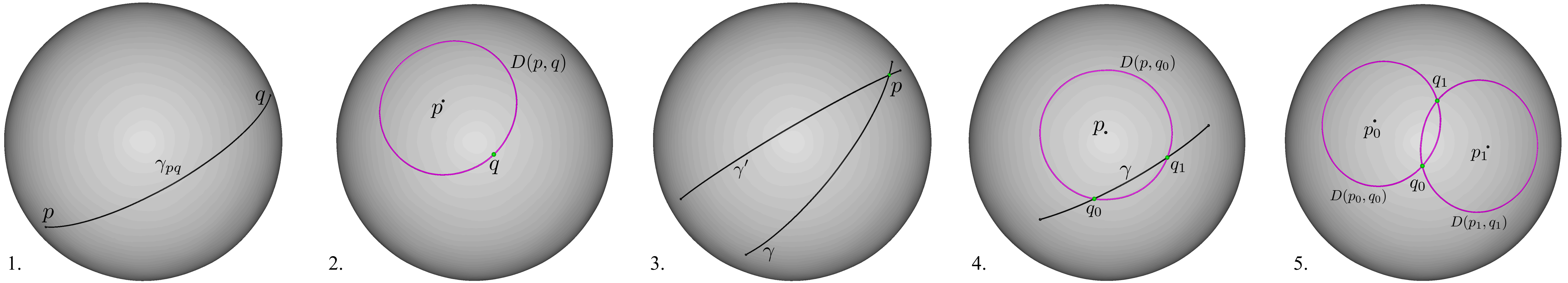
How do we port straightedge and compass constructions on surface?

Draw on the tangent space and map the result to the surface

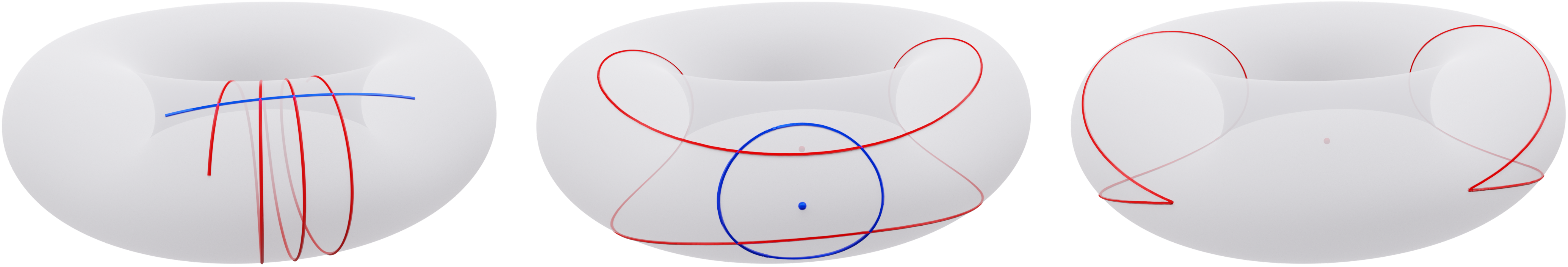
Draw directly on the surface

Straightedge and compass constructions

1. Creating the line through two existing points
2. Creating the circle through one point with center another point
3. Creating the point which is the intersection of two non existing, non parallel lines
4. Creating the one or two points in the intersection of a line and a circle
5. Creating the one or two points in the intersection of two circles



Straightedge and compass constructions



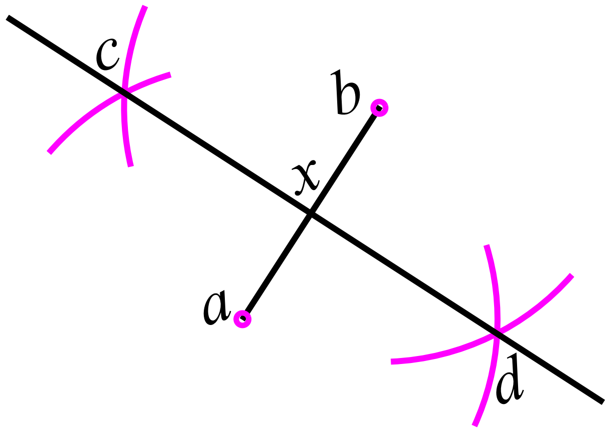
Straightedge and compass constructions

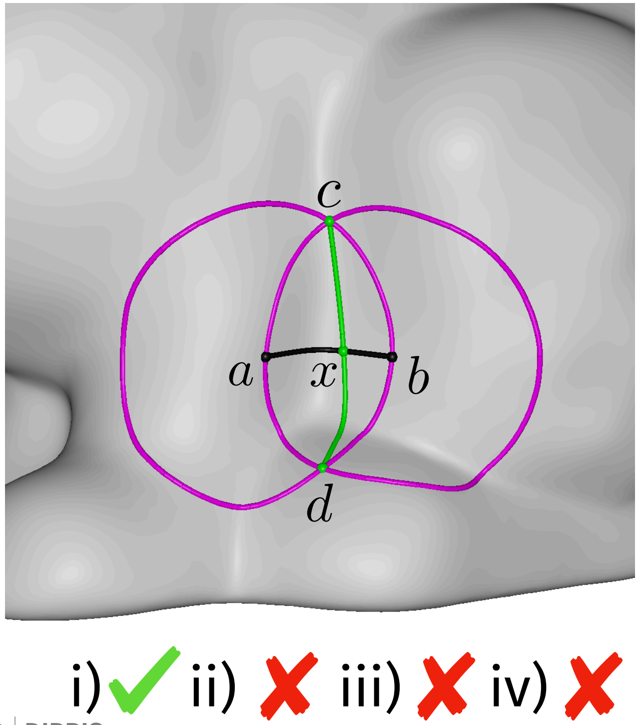
i) the perpendicular bisector is a straight line

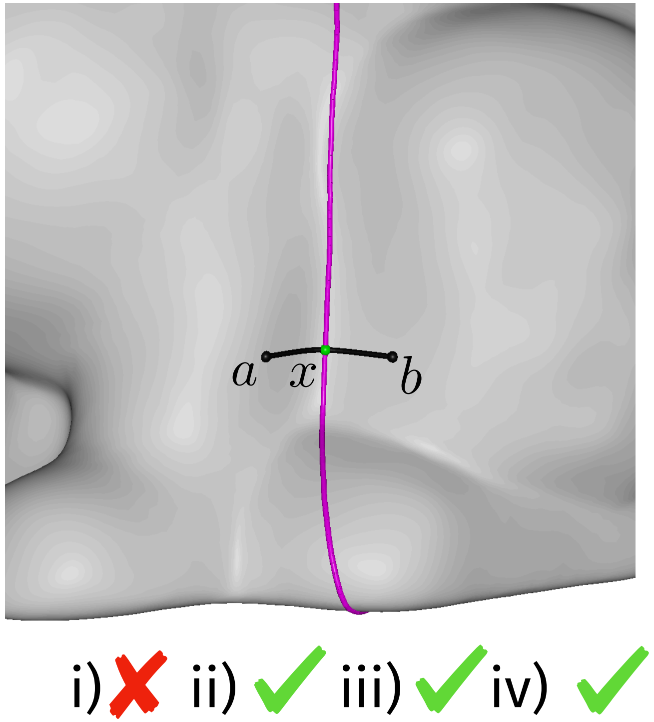
ii) x is the midpoint of the segment

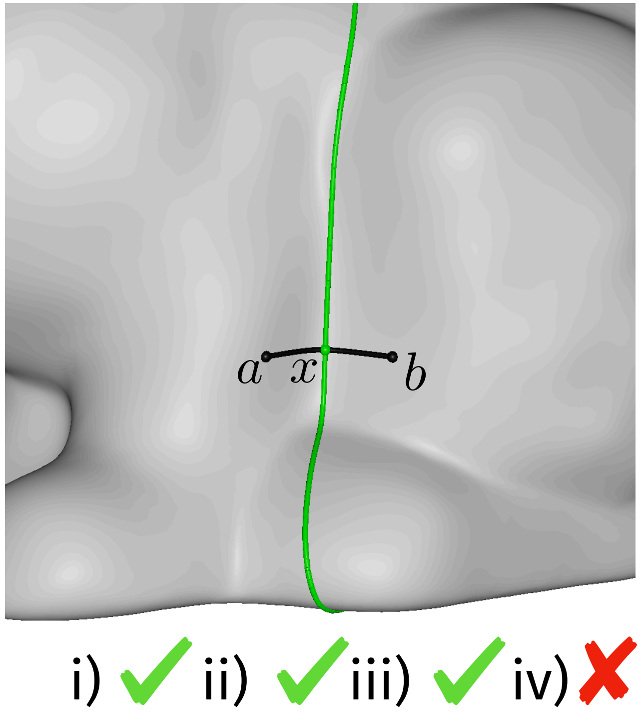
iii) the bisector is orthogonal to the segment

iv) all the points on the bisector have the same distance from a and b

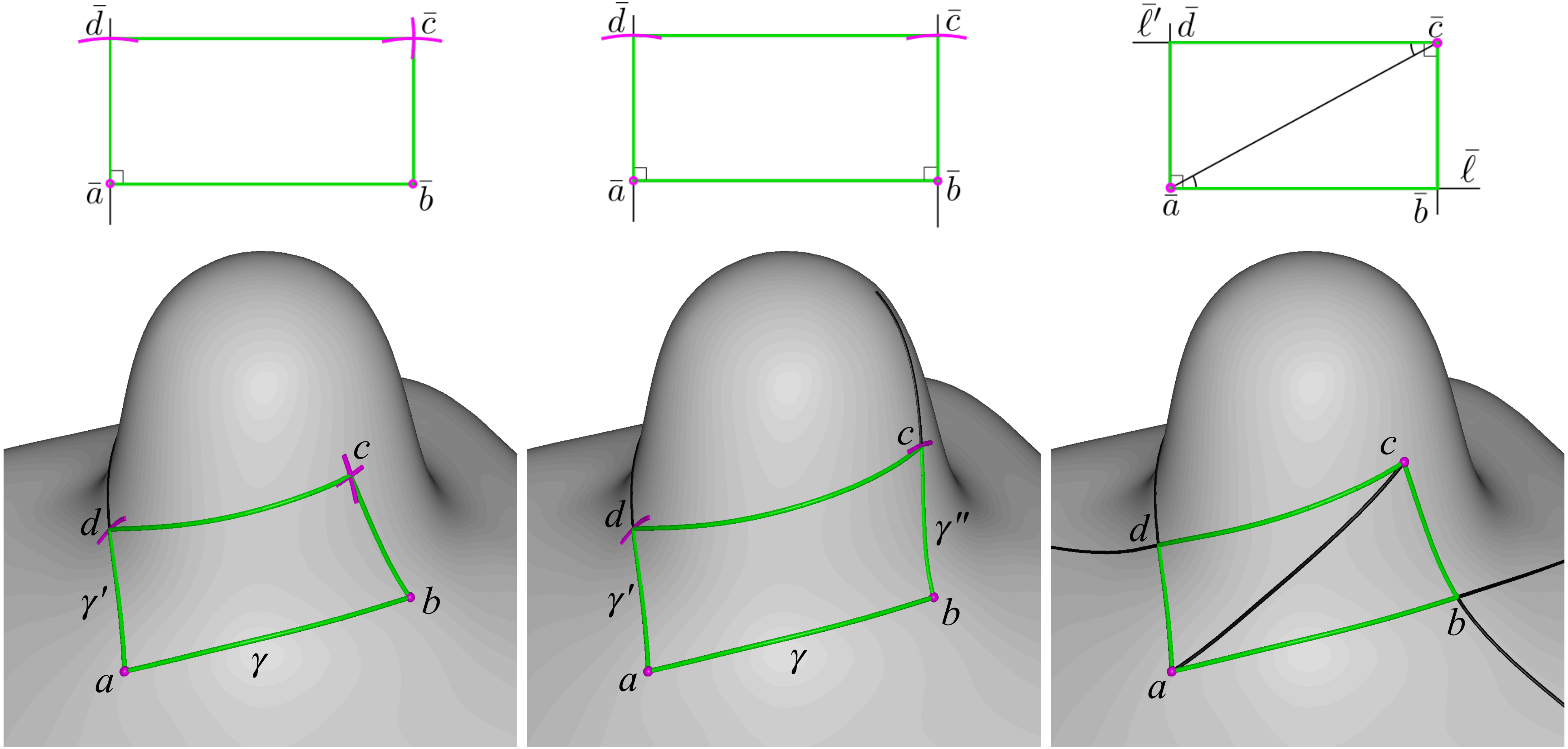








Straightedge and compass constructions



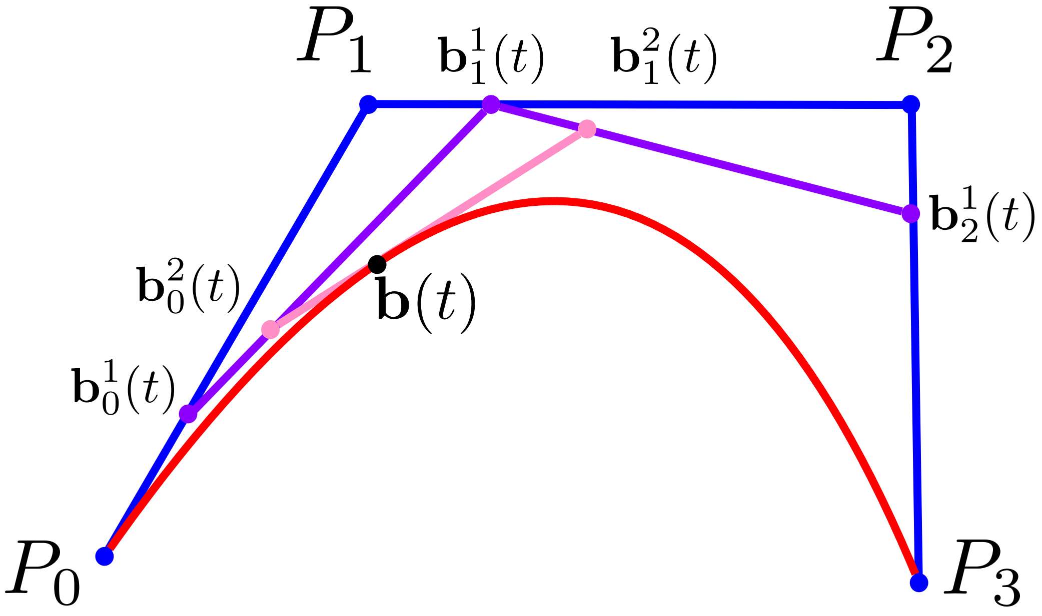
Bézier Curves

Weighted Averages (Bernstein)

$$\mathbf{b}^k(t) = \sum_{i=0}^{k} B_i^k(t) P_i \qquad t \in [0,1]$$

de Casteljau (recursive algortihm)

$$\mathbf{b}_i^0(t) = P_i$$
$$\mathbf{b}_i^r(t) = (1-t)\mathbf{b}_i^{r-1}(t) + t\mathbf{b}_{i+1}^{r-1}(t)$$



37

Bézier Curves

Weighted Averages (Bernstein)

Euclidean setting

$$\mathbf{b}^k(t) = \sum_{i=0}^{k} B_i^k(t) P_i \qquad t \in [0,1]$$

Manifold setting

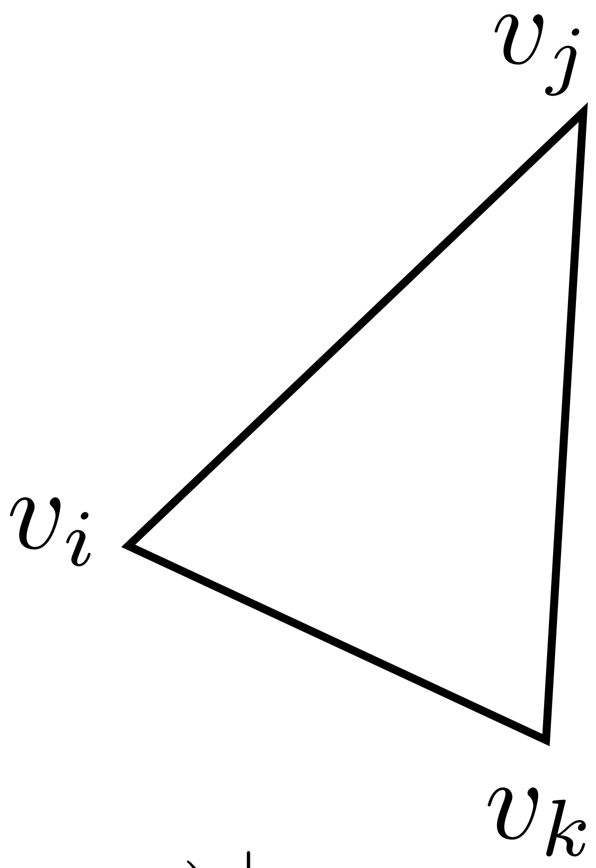
$$\mathbf{b}^k(t) = \arg\min_{P \in M} \sum_{i=0}^{k} B_i^k(t) d^2(P, P_i)$$

Excursus: gradient of a scalar field

$$f = \{f_1, f_2, \ldots, f_n\}$$

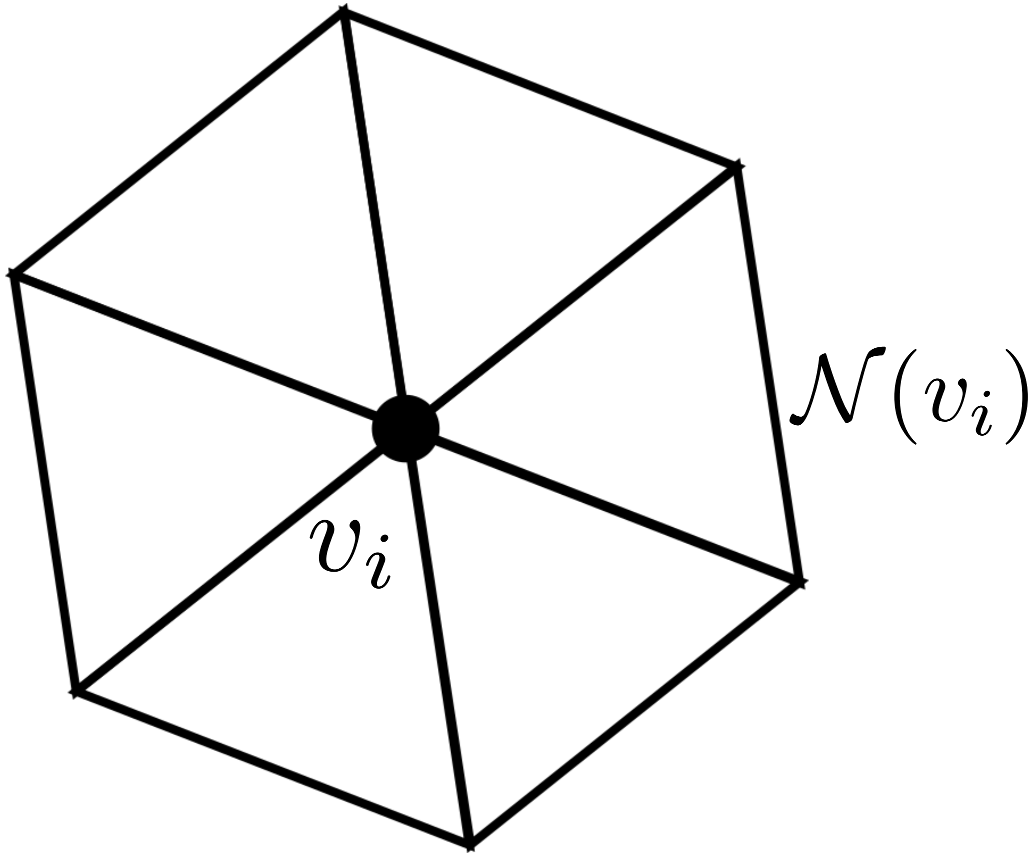
$$\nabla f_t \approx \frac{1}{A_t} \int_{A_t} \nabla f \, d\mathbf{a} = \frac{1}{A_t} \int_{\partial A_t} f \mathbf{n} \, d\ell$$

$$\nabla f_t = (f_j - f_i) \frac{(v_i - v_k)^{\perp}}{2A_t} + (f_k - f_i) \frac{(v_j - v_i)^{\perp}}{2A_t}$$



Excursus: gradient of a scalar field

$$\nabla f_i = \frac{1}{\sum_{t_j \in \mathcal{N}(v_i)} A_{t_j}} \sum_{t_j \in \mathcal{N}(v_i)} A_{t_j} P_{t_j, v_i}(\nabla f_{t_j})$$

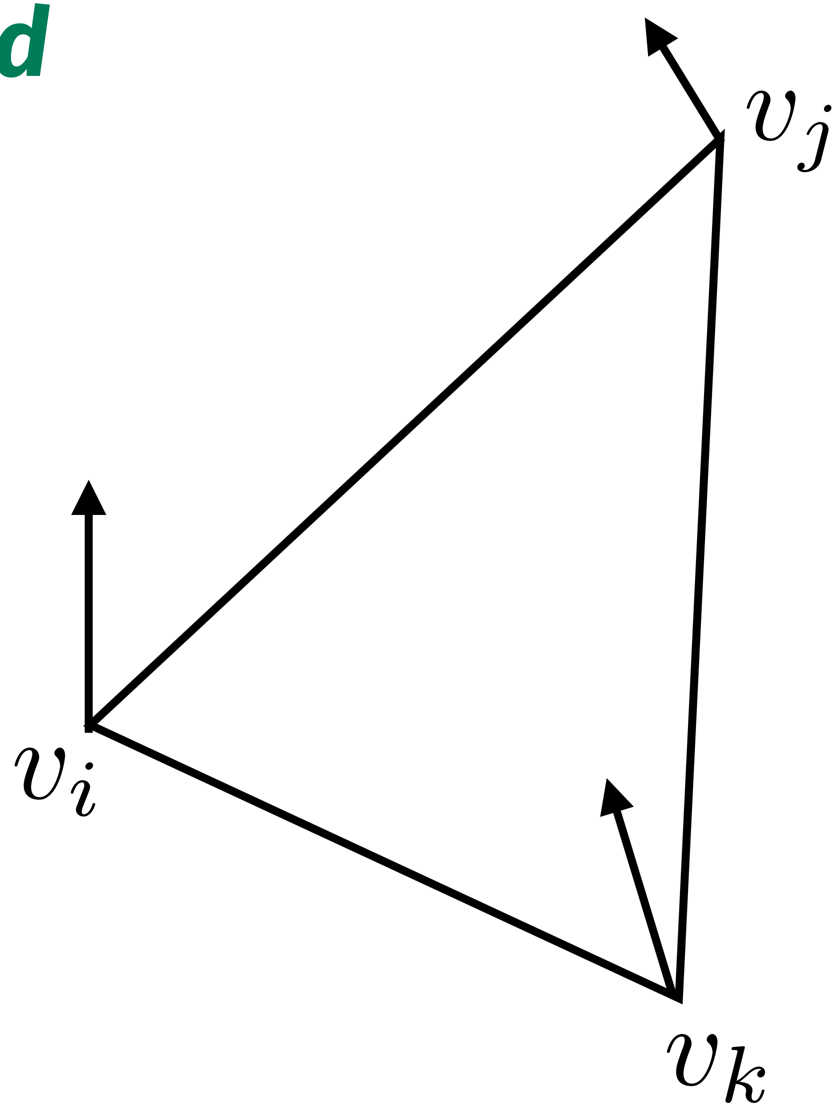


Excursus: Hessian of a scalar field

$$V_i := \nabla f_i$$

$$X_t := (1 - \alpha - \beta)V_i + \alpha V_j + \beta V_k$$

$$\mathrm{Hess}\, f_t = J(X_t)$$



Solving an optimization problem on a triangle mesh

$$U(x) := \sum_{i=0}^{k} w_i d^2(x, P_i) \longrightarrow \nabla U(x) = 0$$

Algorithm 1: Computation of the RCM with Newton's algorithm

Input: Mesh M, Control points P_i, Weights w_i, Warm start Q

Output: Riemannian center of mass $\bar{P}$

if $Q \neq$NONE **then**

  $\bar{P} \leftarrow Q$

else

  $\mathsf{k} \leftarrow \texttt{argmax}\ w_i$

  $\bar{P} \leftarrow P_{\mathsf{k}}$

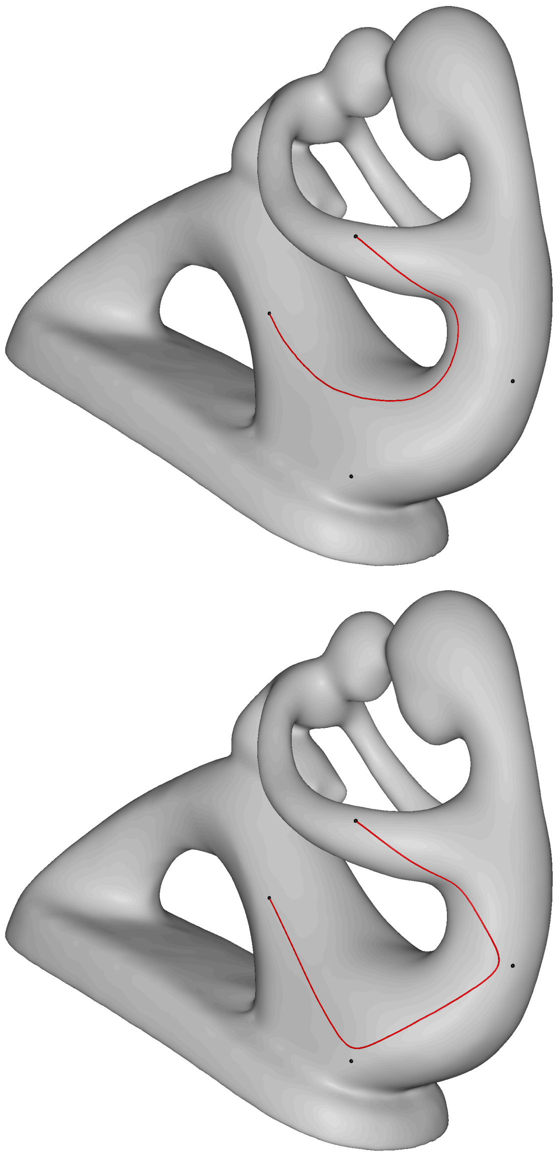
while $\|\nabla U(\bar{P})\| > \varepsilon$ **do**

  solve the Newton equation $\mathrm{Hess}U(\bar{P})\mathbf{s} = -\nabla U(\bar{P})$

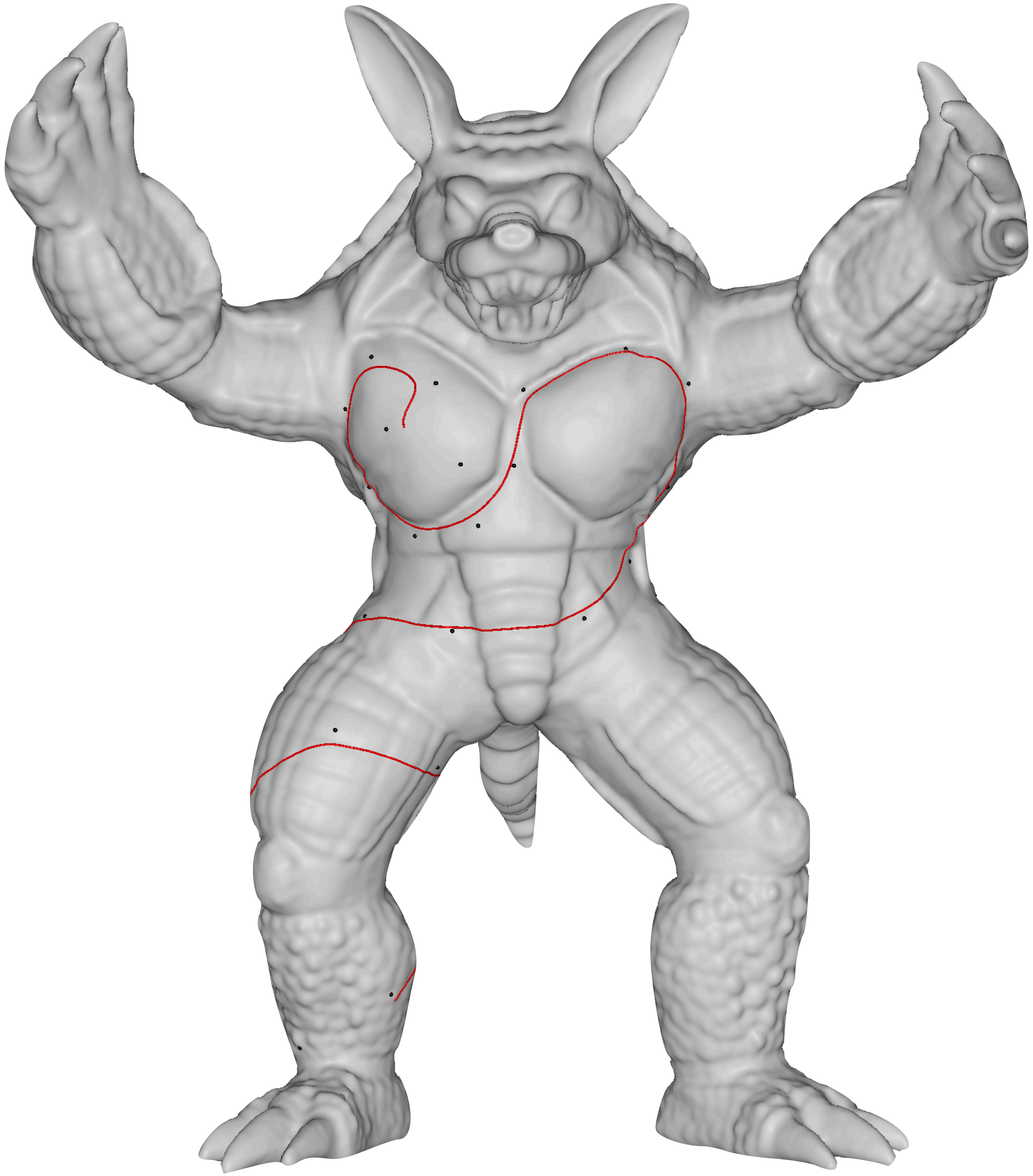
  $\gamma \leftarrow \texttt{trace\_geodesic}(\bar{P}, \mathbf{s})$

  $\bar{P} \leftarrow \texttt{argmin}_{P \in \gamma} \nabla U(P)$

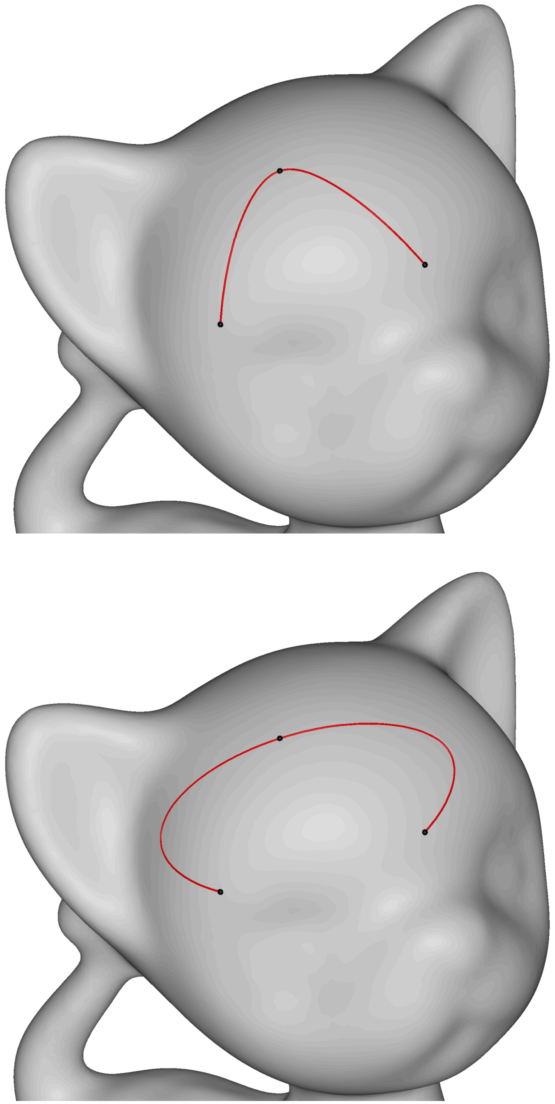
return $\bar{P}$



Rational Bézier



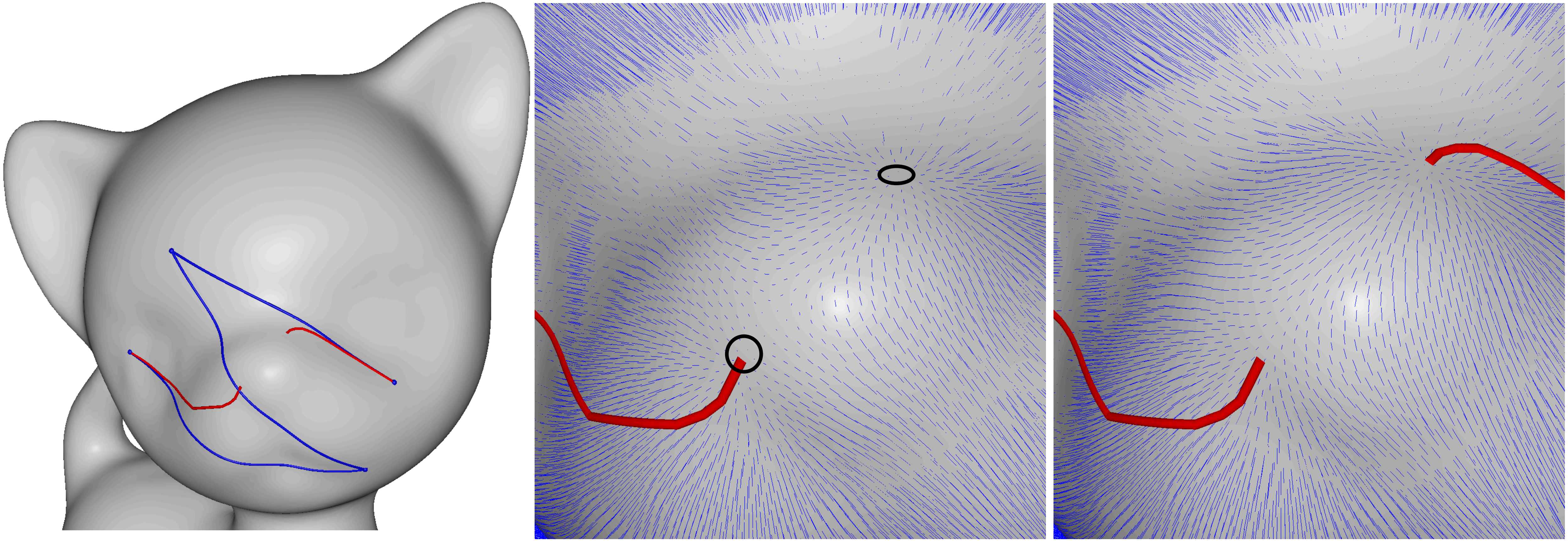
NURBS



[Ramantantoanina and Hormann, 2021]

Limitations

We need convexity!!



Bézier Curves

de Casteljau on Manifolds

We define the *manifold average operator* between two points as

$$\mathcal{A} : M \times M \times [0,1] \longrightarrow M; \quad (P, Q; w) \mapsto \gamma_{P,Q}(w)$$

Always well defined except at the cut locus

$$\mathbf{b}_i^0(t) = P_i$$
$$\mathbf{b}_i^r(t) = \mathcal{A}(\mathbf{b}_i^{r-1}(t), \mathbf{b}_{i+1}^{r-1}(t), t)$$

[Park and Ravani, 1995]

Bézier Curves

Subdivision Schemes

Approaches based on subdivision schemes can be used to solve the problem of broken curves (no free lunch!)

- Recursive de Casteljau (RDC) [Morera et al., 2008]
- Open Lane-Riesenfield (OLR) [Mancinelli et al., 2022]

Bézier Curves

Subdivision Schemes

Approaches based on subdivision schemes can be used to solve the problem of broken curves (no free lunch!)

- Recursive de Casteljau (RDC) [Morera et al., 2008]
- Open Lane-Riesenfield (OLR) [Mancinelli et al., 2022]

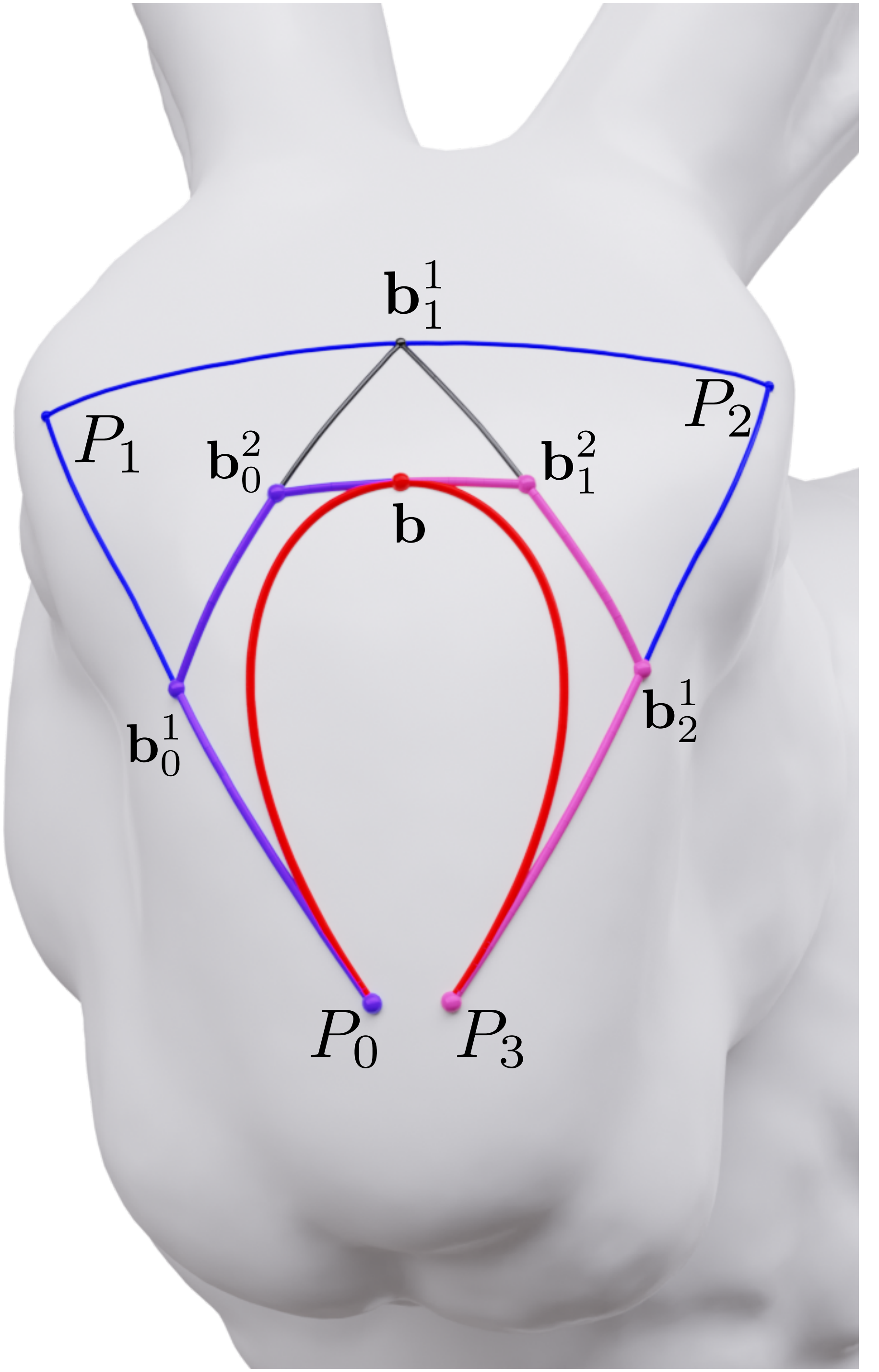
Bézier Curves

RDC

At each iteration, we split every control polygon into two sub-polygons using de Casteljau's algorithm.

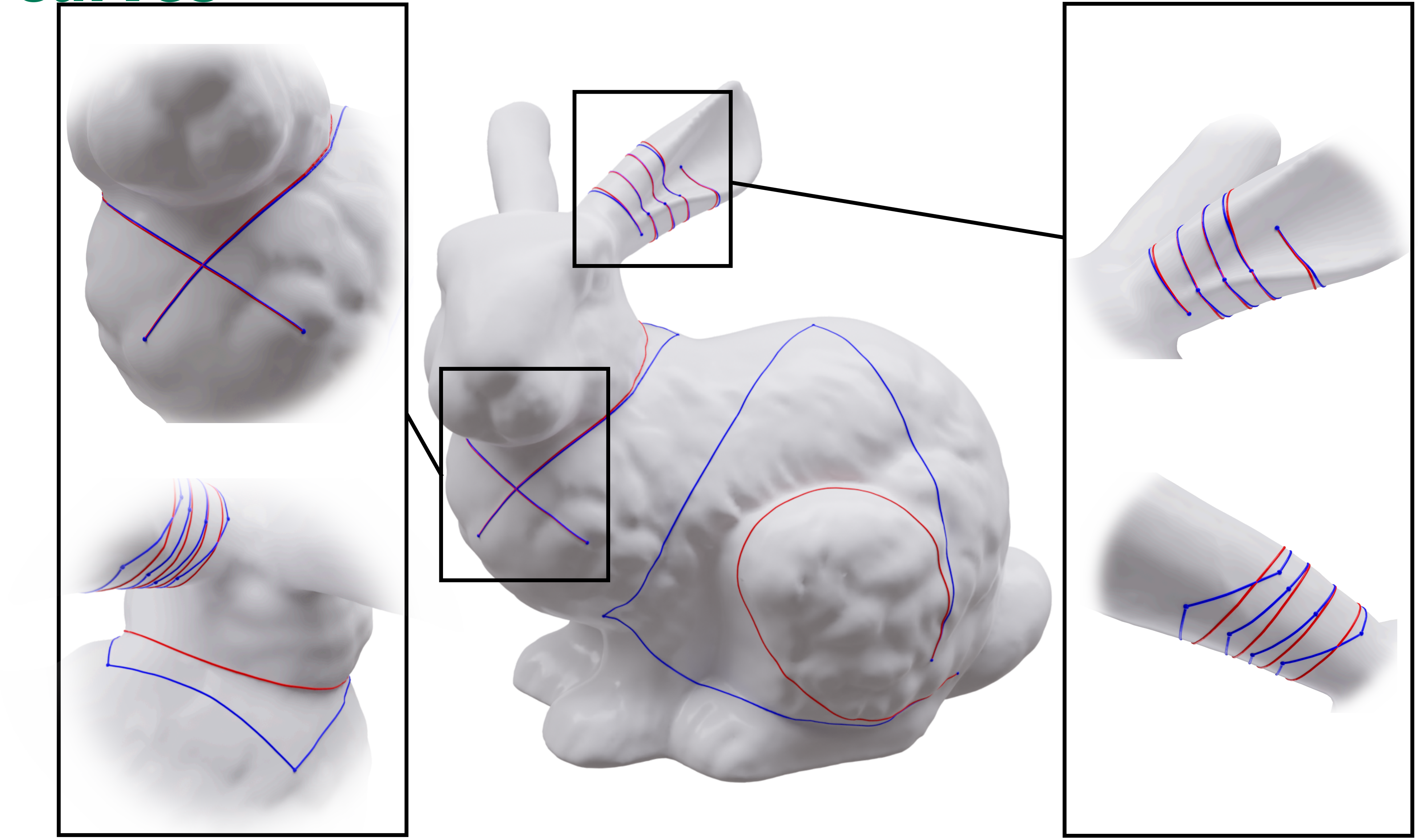
↓

Converges to a $\mathcal{C}^1$ curve
[Noakes, 1998; Mancinelli et al., 2022]



Bézier Curves

RDC



Bézier Curves
RDC

- 2056 curves on the 394k Pumpkin
- 289 ms to trace all of them


UniGe | DIBRIS