UNIVERSITÀ DEGLI STUDI FIRENZE

DIMAI
DIPARTIMENTO DI MATEMATICA
E INFORMATICA "ULISSE DINI"

Funded by the European Union
NextGenerationEU

Italiadomani
PIANO NAZIONALE DI RIPRESA E RESILIENZA

CN1 - Spoke 6

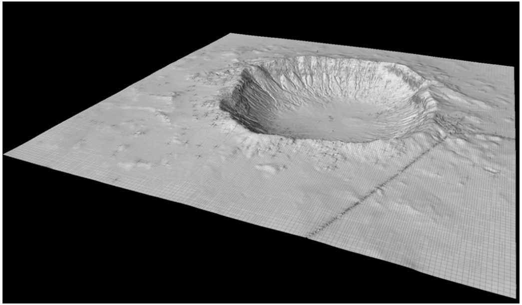
An Algorithmic Introduction to LR B-splines

Francesco Patrizi

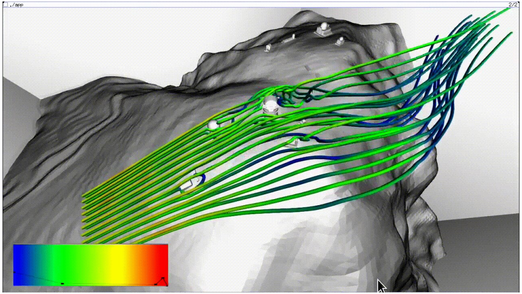
Picture: Acropolis

Athens, Greece

A tool for adaptive approximations



Quasi-Interpolation, Meteor Crater, AZ



Wind streamlines around a telescope

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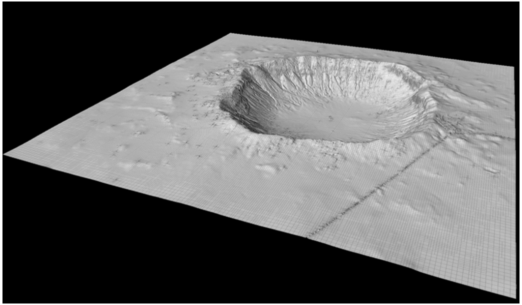

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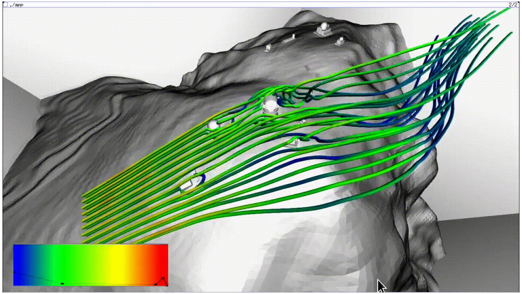

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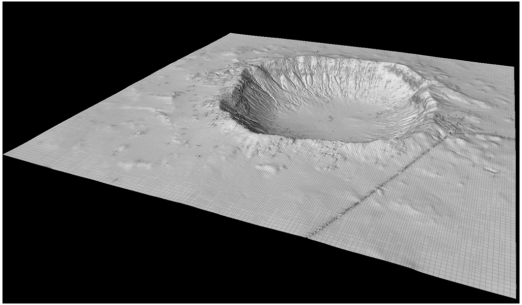

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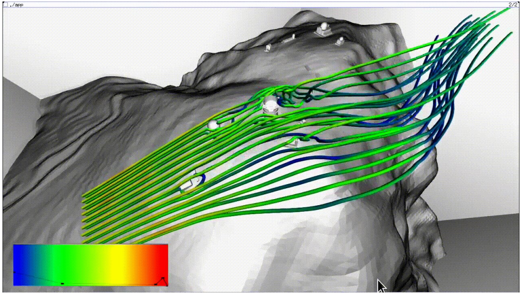

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T. Dokken

T. Lyche

K. F. Pettersen

Recalling univariate B-splines

(Local) Knot Vector: Given a degree p, $\boldsymbol{t} = \boldsymbol{t}_p$ with $|\boldsymbol{t}_p| = p + 2$ with repetitions

$$\underbrace{t_1 = \cdots = t_{m_1}}_{\max p+1 \text{ times}} < \underbrace{t_{m_1+1} = \cdots = t_{m_1+m_2}}_{\max p+1 \text{ times}} < \cdots$$

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$$B[\boldsymbol{t}](t) = \frac{t - t_1}{t_{p+1} - t_1} B[t_1, \ldots, t_{p+1}](t) + \frac{t_{p+2} - t}{t_{p+2} - t_2} B[t_2, \ldots, t_{p+2}](t),$$

where each time a fraction with zero denominator appears, it is taken as zero. The initial B-splines of degree 0 are defined as

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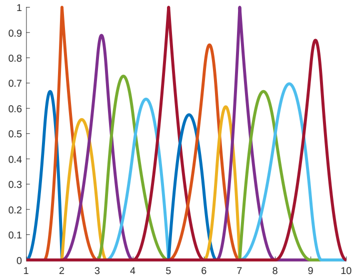

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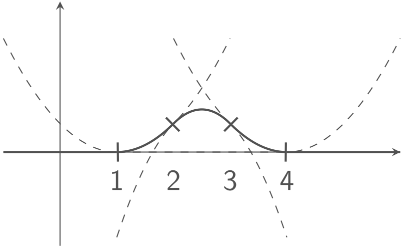

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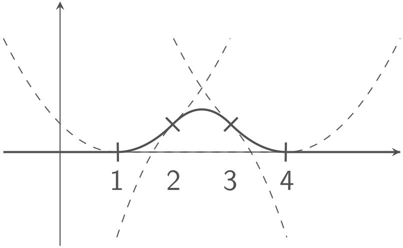

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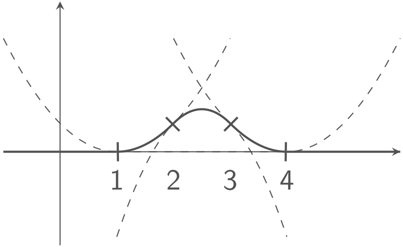

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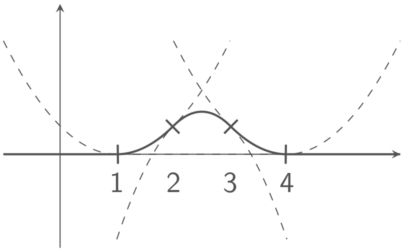

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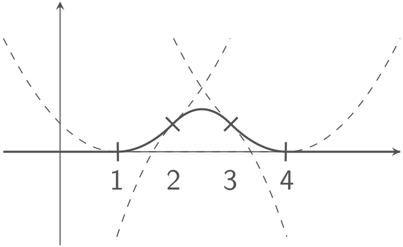

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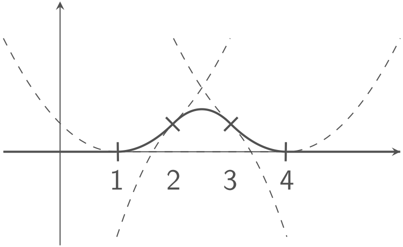

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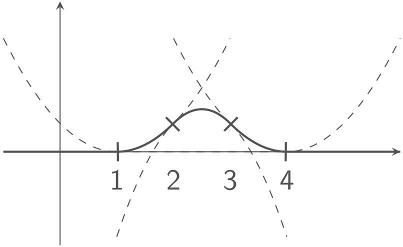

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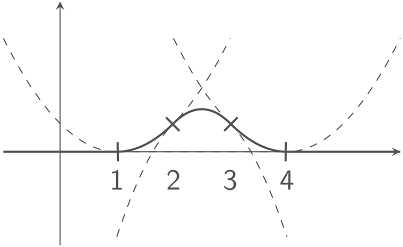

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- form a partition of unity.

Knot Insertion

Knot Insertion: Suppose we insert a knot $\hat{t} \to \boldsymbol{t}$. We obtain two knot vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, considering the first and the last $p+2$ knots respectively in $(t_1, \dots, \hat{t}, \dots, t_{p+2})$. Then

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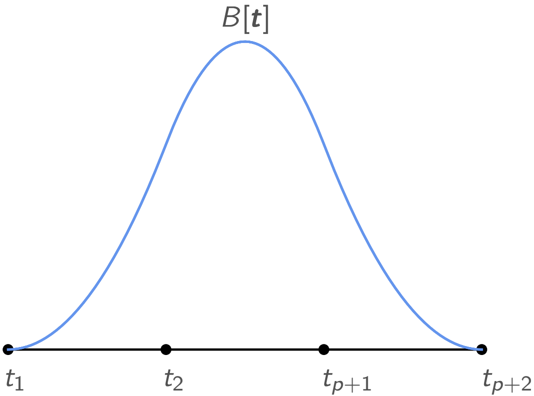

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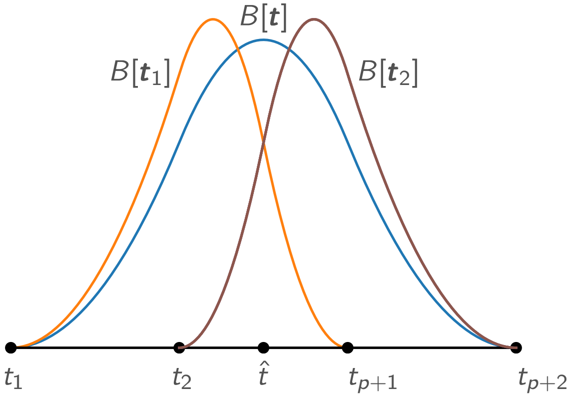
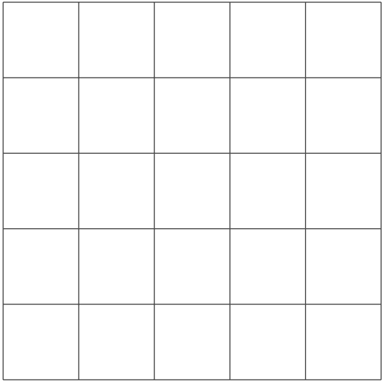


B-splines on Tensor Meshes

Given a tensor mesh $\mathcal{N}$ and a bidegree (p_1, p_2) (for instance $(2, 2)$),



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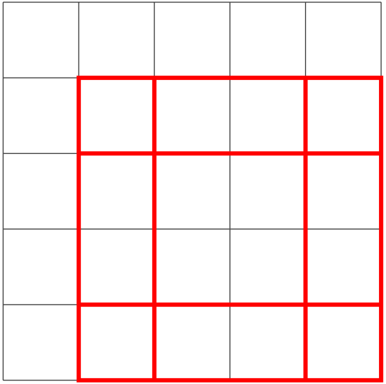
Given a tensor mesh $\mathcal{N}$ and a bidegree (p_1, p_2) (for instance $(2, 2)$),
let $\mathcal{N}_B$ be a subcollection of meshlines in $\mathcal{N}$ forming a sub-grid of $p_1 + 2$ vertical lines and $p_2 + 2$ horizontal lines.



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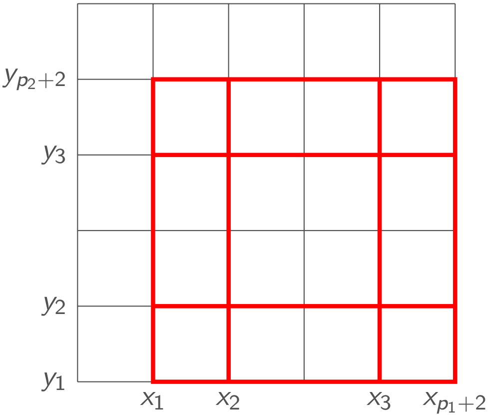

Such vertical and horizontal lines can be parametrized as $\{x_i\} \times [y_1, y_{p_2+2}]$ and $[x_1, x_{p_1+2}] \times \{y_j\}$ with $\boldsymbol{x} := (x_i)_{i=1}^{p_1+2}$ and $\boldsymbol{y} = (y_j)_{j=1}^{p_2+2}$.

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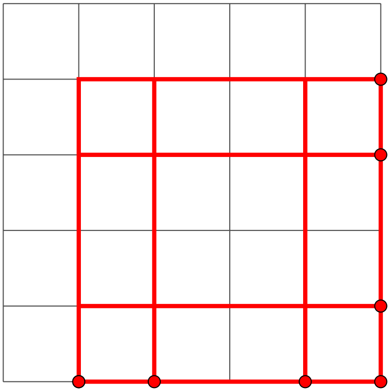
$\boldsymbol{x}$ and $\boldsymbol{y}$ are knot vectors on top on which we define univariate B-splines of degrees p_1 and p_2.



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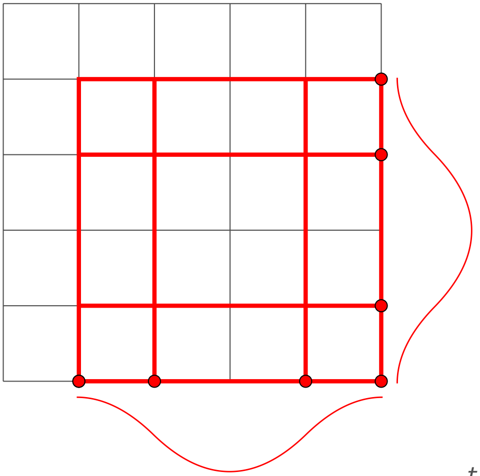

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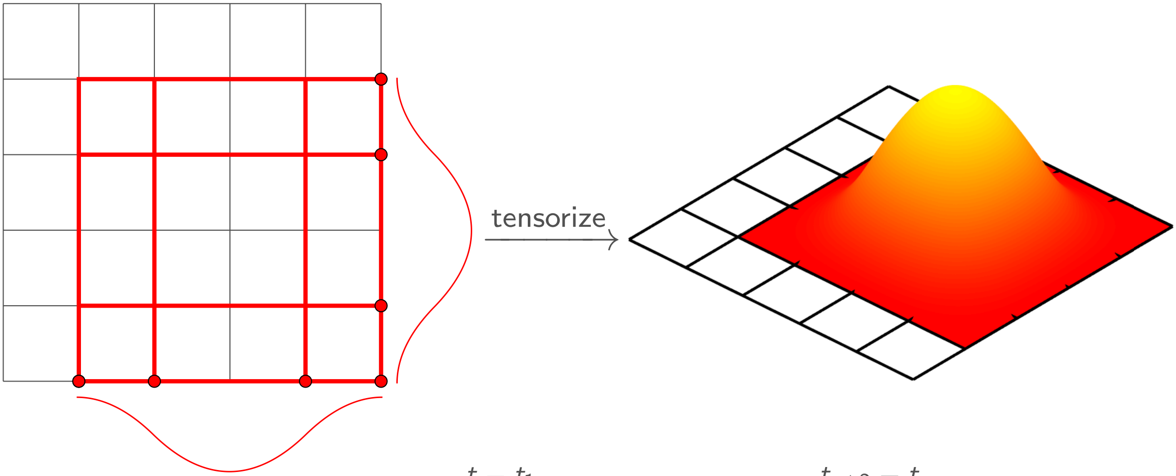

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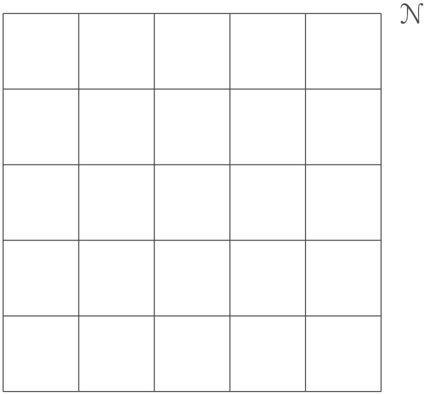
B-splines of Minimal Support

Each $\mathcal{N}_B$ corresponds to a B-spline B defined on $\mathcal{N}$.

If no line in $\mathcal{N} \backslash \mathcal{N}_B$ traverses $\text{int}(\text{supp}\, B)$ then we say that B has minimal support on $\mathcal{N}$.



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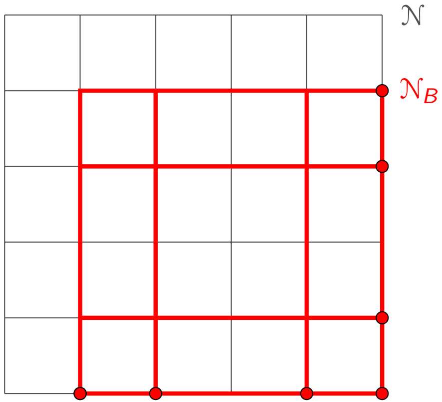

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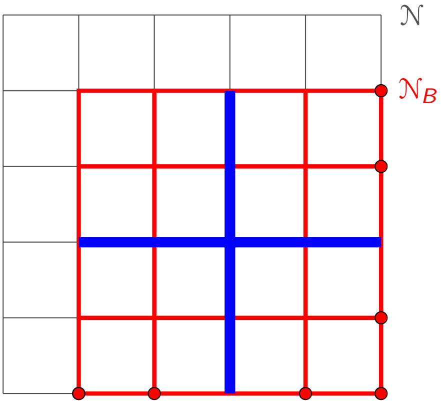

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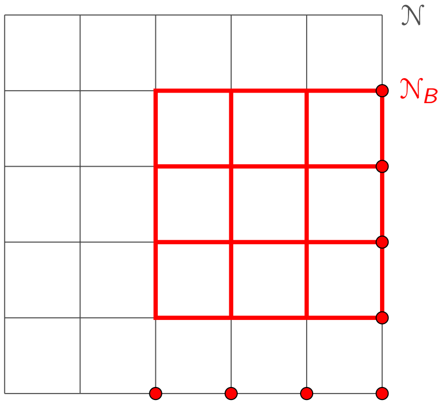

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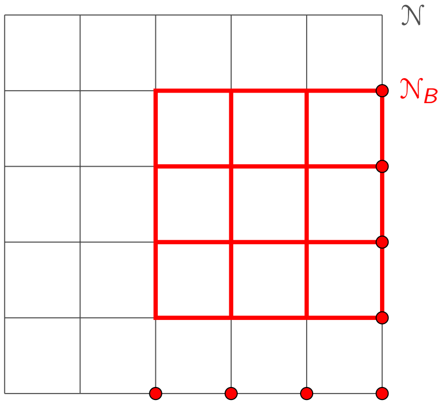
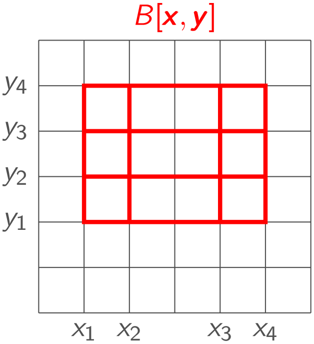


minimal support on $\mathcal{N}$

We call B-spline set on $\mathcal{N}$ the set of all the minimal support B-splines on $\mathcal{N}$.

Knot insertion

Assume $B[\boldsymbol{x}, \boldsymbol{y}]$ no minimal support on $\mathcal{N}$ because of a vertical line at $x = \hat{x}$ with $\hat{x} \notin \boldsymbol{x}$.



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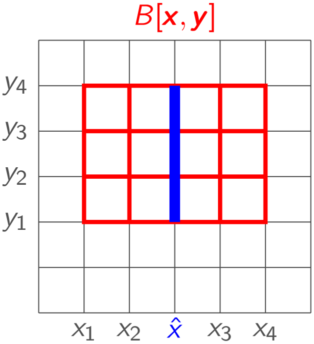

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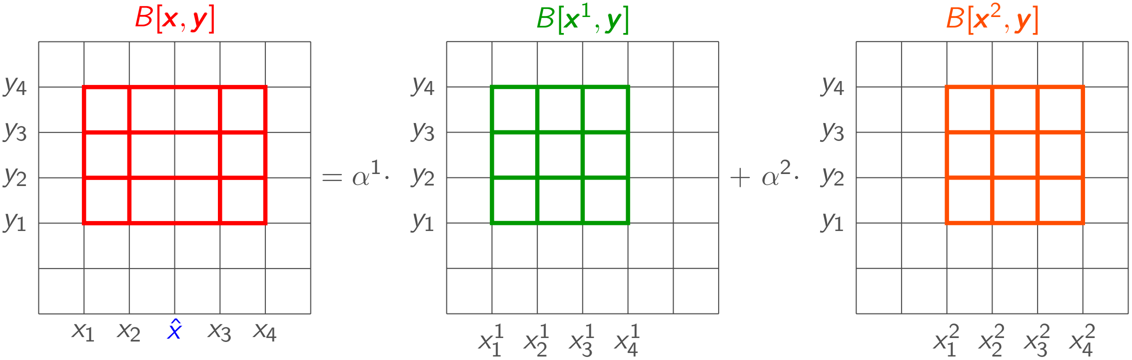

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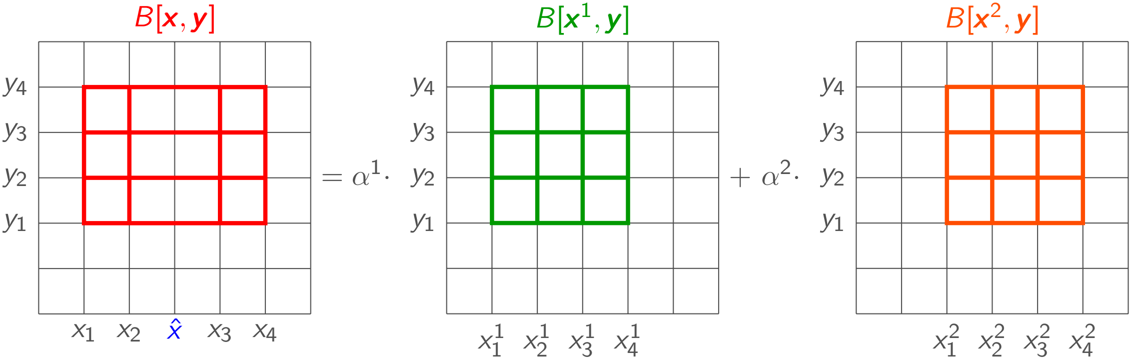
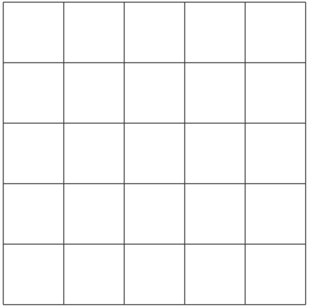


with $\alpha^1, \alpha^2 \in (0,1]$. $B[\mathbf{x},\mathbf{y}]$ is expressed in terms of B-splines of minimal support on $\mathcal{N}$.

LR meshes and LR B-splines

Let $\mathcal{N}$ be a tensor mesh and $\mathcal{B}$ be the set of (minimal support) B-splines on $\mathcal{N}$.



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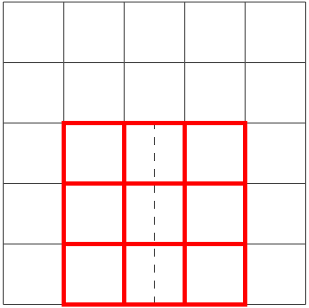
We insert a new line γ, traversing the support of at least one B-spline $B \in \mathcal{B}$.



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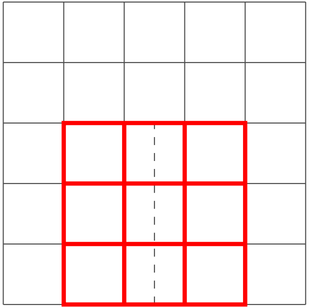
We insert a new line γ, traversing the support of at least one B-spline $B \in \mathcal{B}$.



By construction B has not minimal support on the new mesh $\mathcal{N}' = \mathcal{N} \cup \gamma$.

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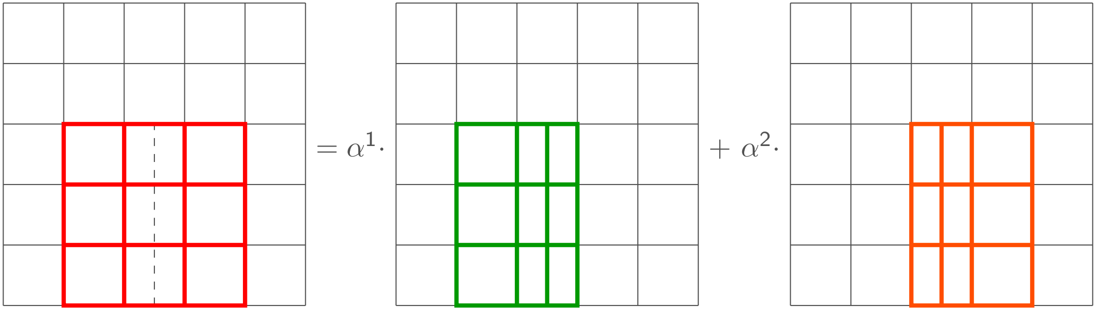

By construction B has not minimal support on the new mesh $\mathcal{N}' = \mathcal{N} \cup \gamma$. By knot insertion we replace B with the B-splines B^1 and B^2 of minimal support on $\mathcal{N}'$. This operation creates a new set $\mathcal{B}'$ of minimal support B-splines on $\mathcal{N}'$.

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LR meshes and LR B-splines

Partition of Unity Weights: sum of the knot insertion coefficients

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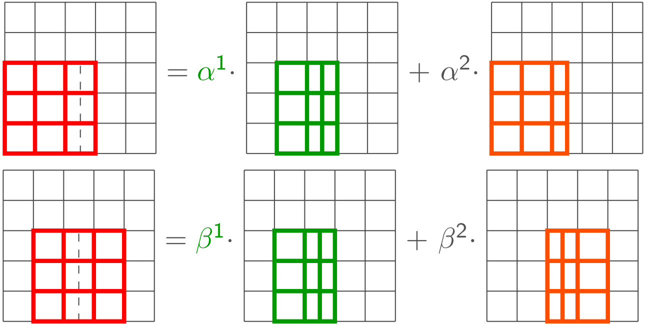
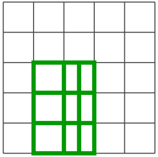


$= \alpha^1 \cdot \quad + \alpha^2 \cdot$

$= \beta^1 \cdot \quad + \beta^2 \cdot$

The PoU weight for

$B =$ 

is $w_B = \alpha^1 + \beta^1$.

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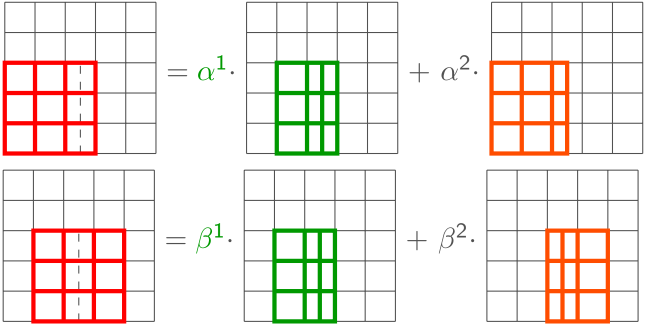

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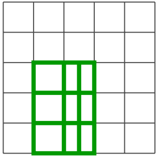
$B =$ 

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Remark: If not specified otherwise, we consider internal meshlines of multiplicity 1 and boundary meshlines of multiplicity $p_k + 1$, for $k = 1, 2$, for vertical and horizontal meshlines respectively.

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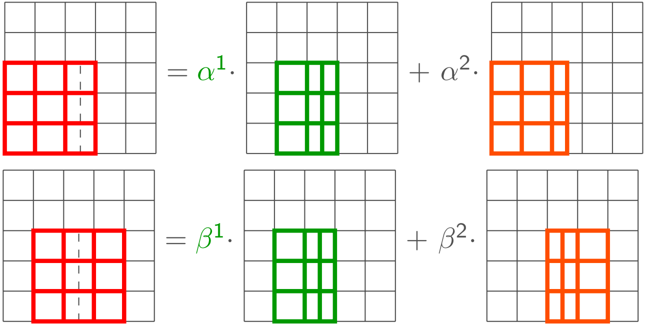

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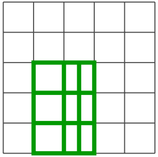

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Remark: Meshline insertion ordering can often be changed. However, the final LR B-spline set is well defined because independent of such insertion ordering.

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Remark: Not all meshes with local lines are LR meshes

LR mesh: $\begin{cases} \mathcal{N}_{i+1} = \mathcal{N}_i \cup \gamma_i & \gamma_i \text{ new line, traversing at least one support} \\ \mathcal{N}_0 & \text{tensor mesh,} \end{cases}$

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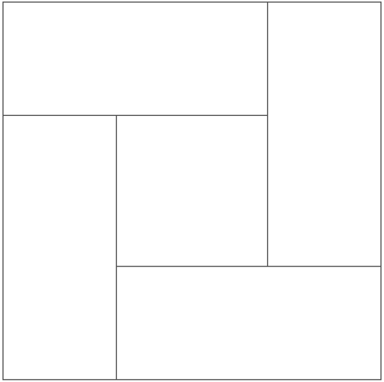
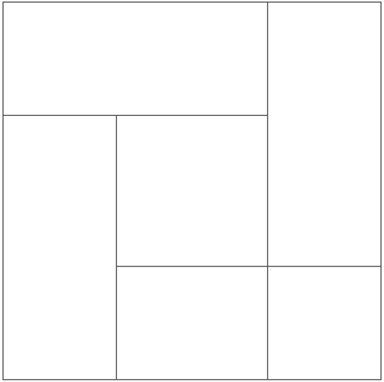


Not LR mesh



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Remark: Secondary splits may be needed.

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with $\mathcal{K}$ all the refinements via knot insertion needed to have minimal supports.

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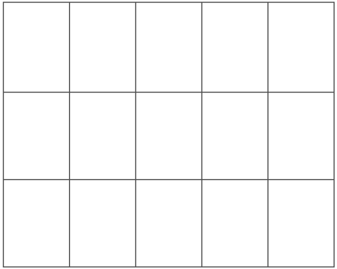
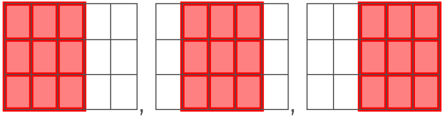


(a) current mesh



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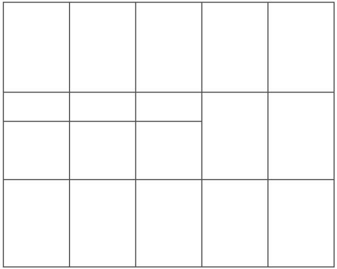

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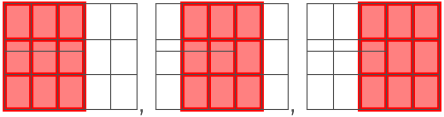

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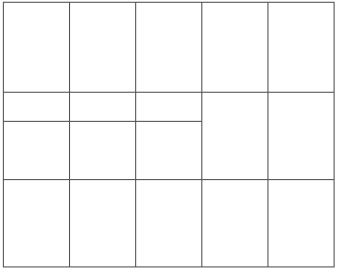

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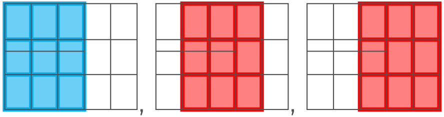

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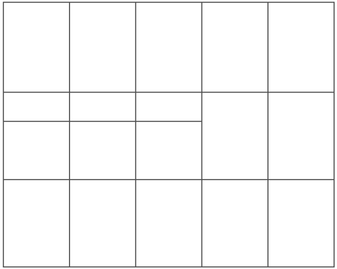

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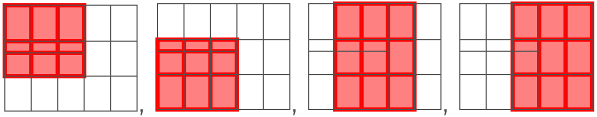

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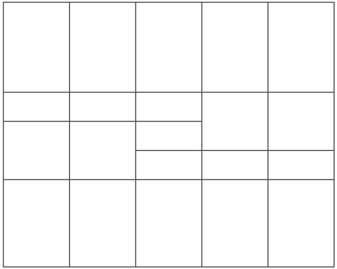

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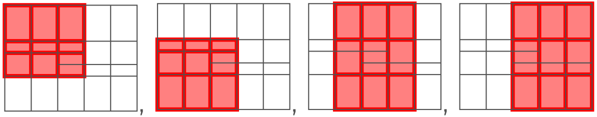

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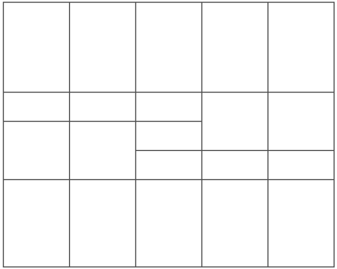

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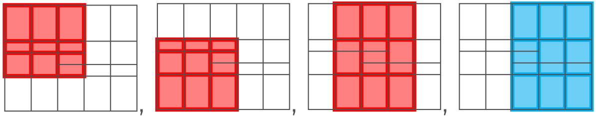

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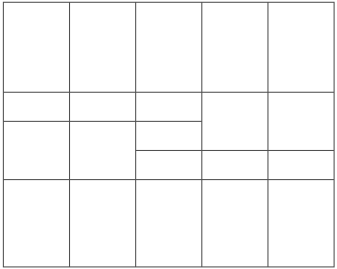

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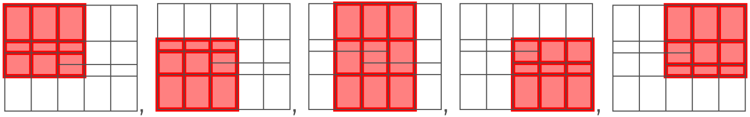

(b) $\mathcal{B}_2$

LR meshes and LR B-splines

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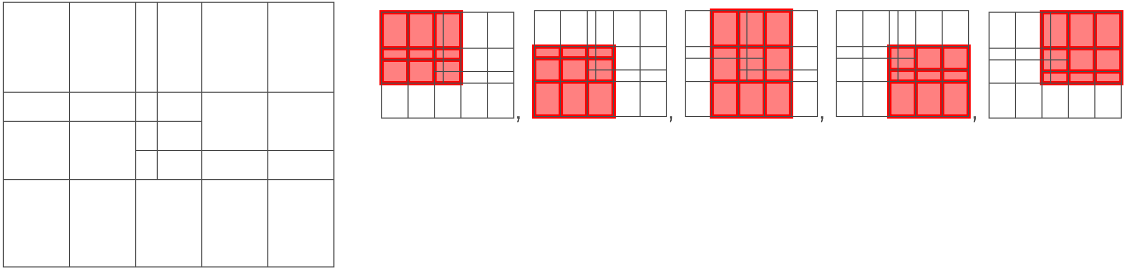

(a) current mesh

(b) $\mathcal{B}_2$

LR meshes and LR B-splines

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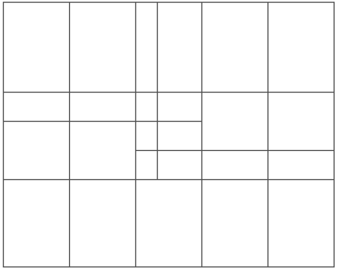
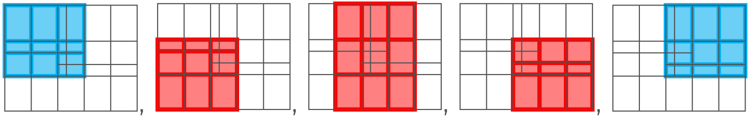


(a) current mesh



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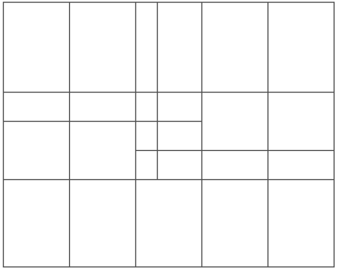
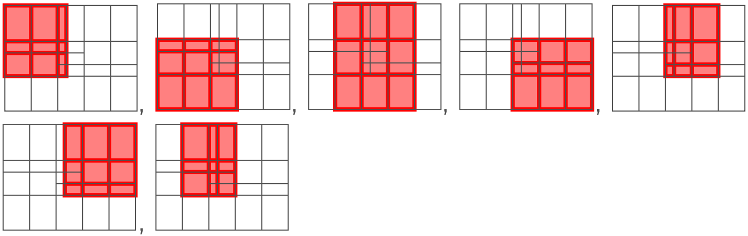
LR B-splines: $\begin{cases} \mathcal{B}_{i+1} = (\mathcal{B}_i \backslash \mathcal{B}_i(\gamma_i)) + \mathcal{K}(\mathcal{B}_i(\gamma_i)) & \mathcal{B}_i(\gamma_i) := \text{ B-splines in } \mathcal{B}_i \text{ traversed by } \gamma_i \\ \mathcal{B}_0 & \text{tensor B-splines on } \mathcal{N}_0 \end{cases}$

with $\mathcal{K}$ all the refinements via knot insertion needed to have minimal supports.



(a) current mesh



(b) $\mathcal{B}_3$

LR meshes and LR B-splines

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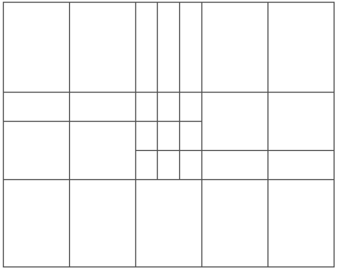
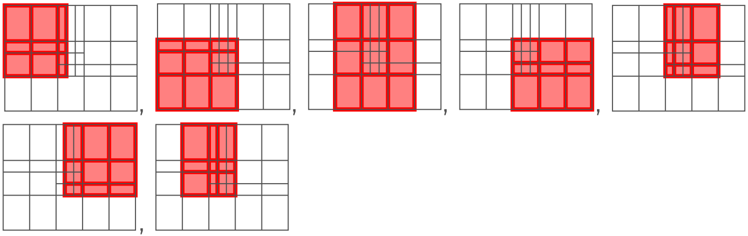


(a) current mesh



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LR meshes and LR B-splines

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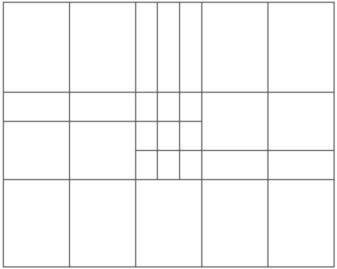

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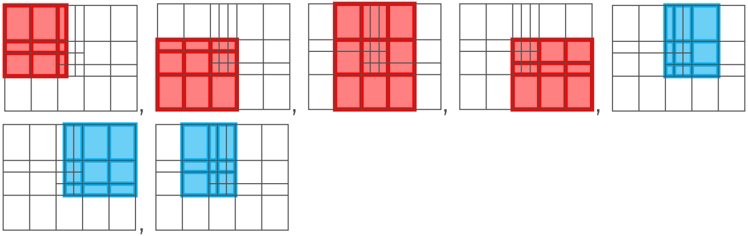

(b) $\mathcal{B}_3$

LR meshes and LR B-splines

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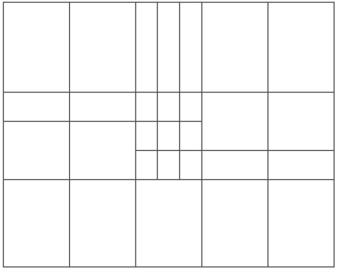

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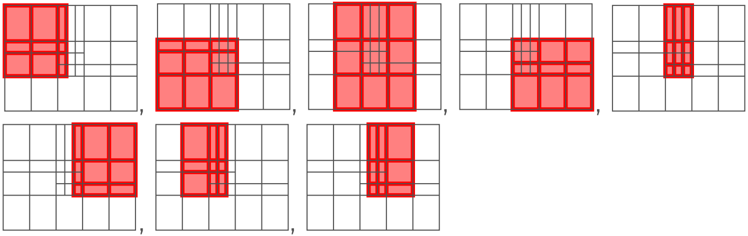

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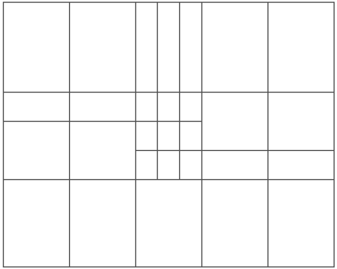

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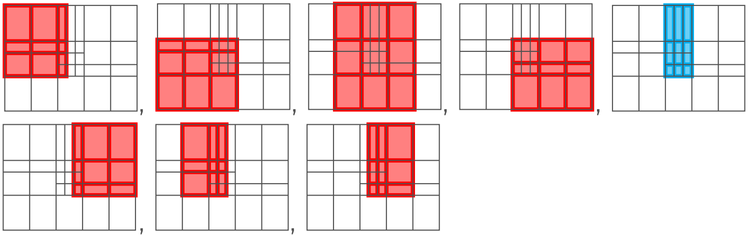

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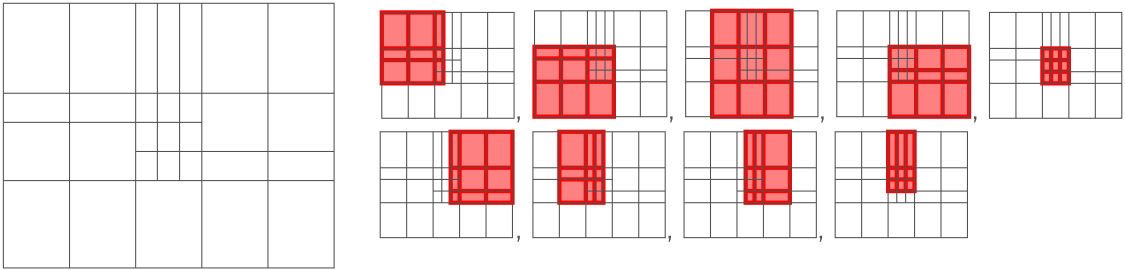

(a) current mesh **(b)** $\mathcal{B}_4$

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Remark: LR B-spline set $\neq$ Minimal Support B-spline set.

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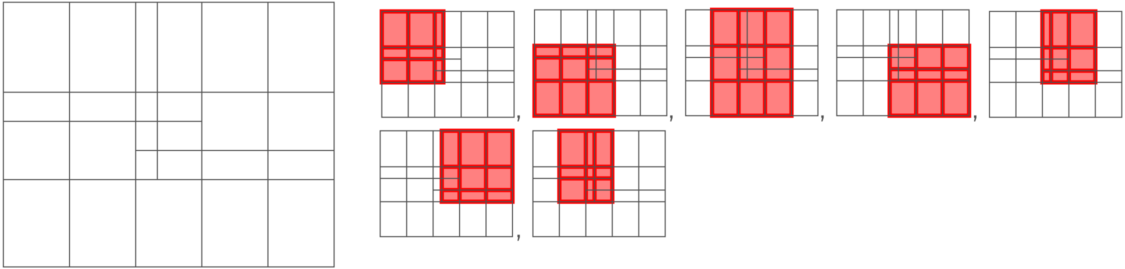

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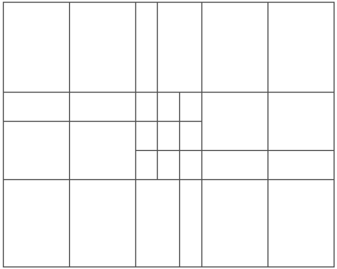

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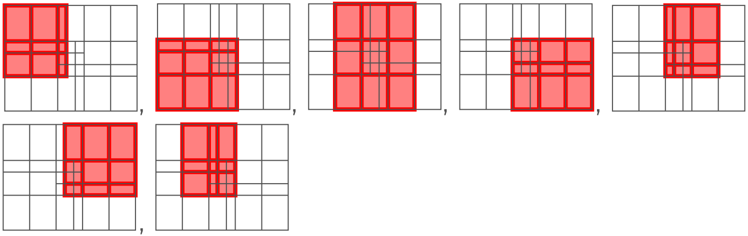

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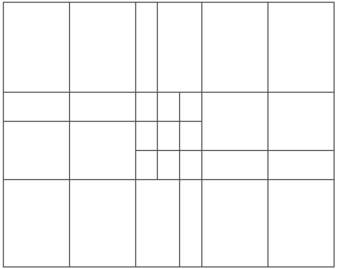
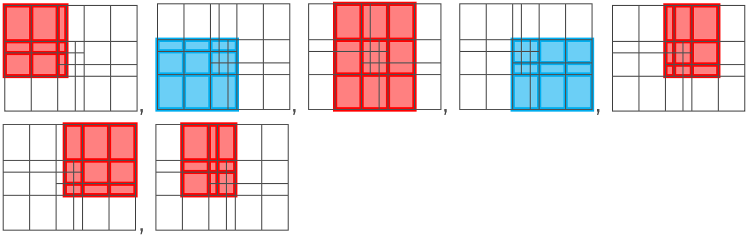


(a) current mesh



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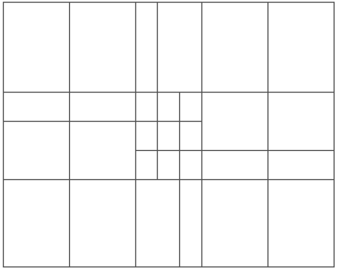

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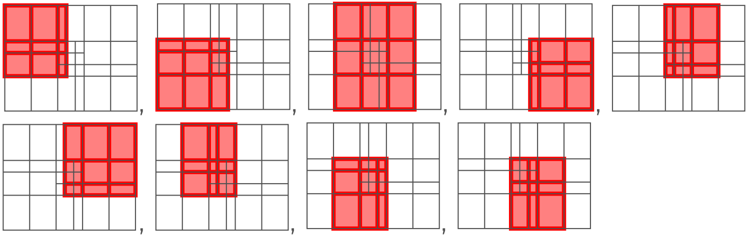

(b) $\mathcal{B}_4$

LR meshes and LR B-splines

Remark: LR B-spline set $\subsetneq$ Minimal Support B-spline set.

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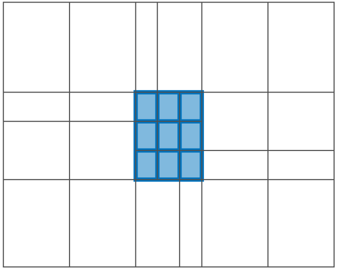

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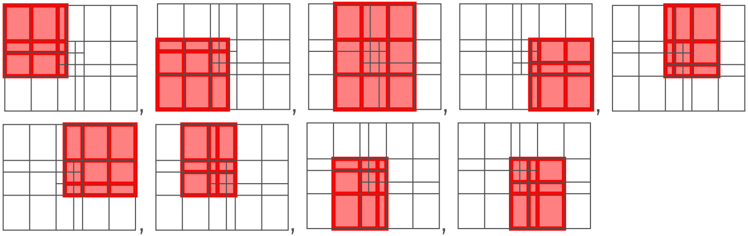

(b) $\mathcal{B}_4$

Minimum Span Refinement strategy

Input: Bunch of boxes where a larger error is committed is some sense.
For each of such boxes

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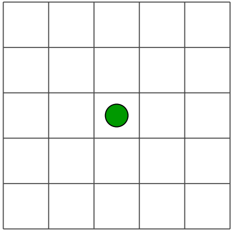

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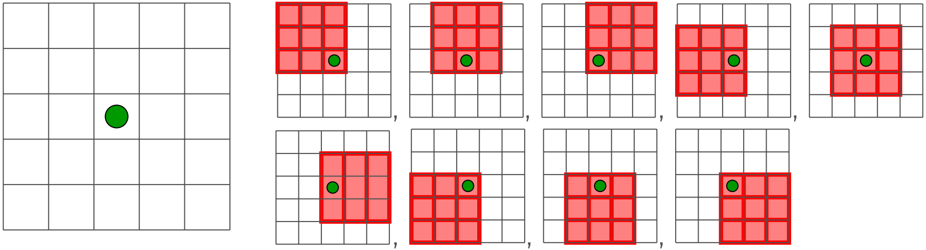

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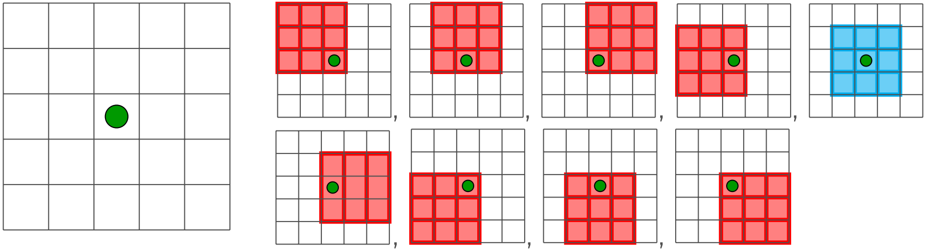

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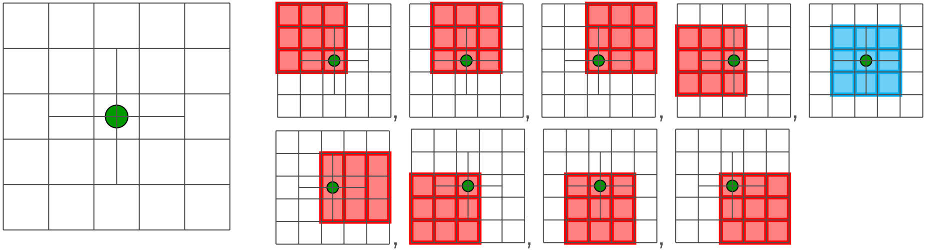
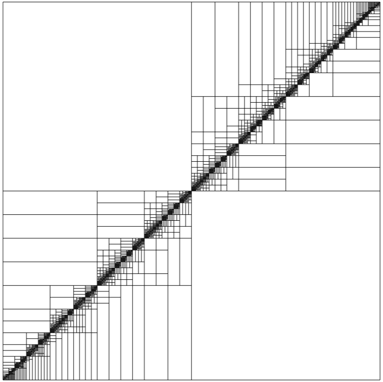
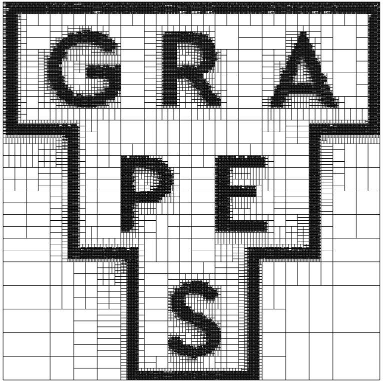
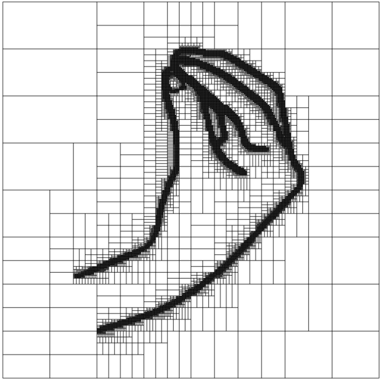


Minimum Span Refinement strategy: Examples







Full Span Refinement strategy

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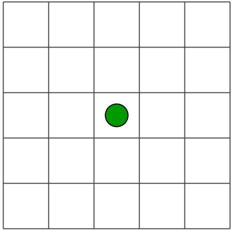

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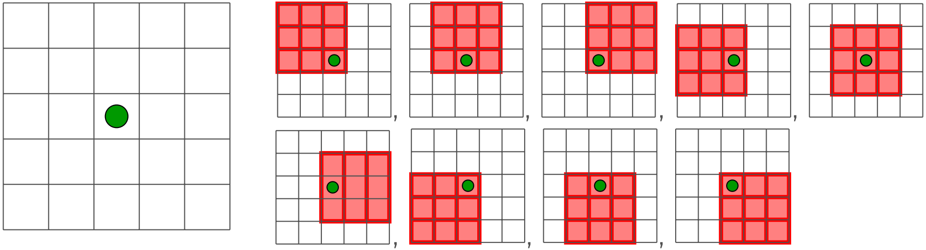

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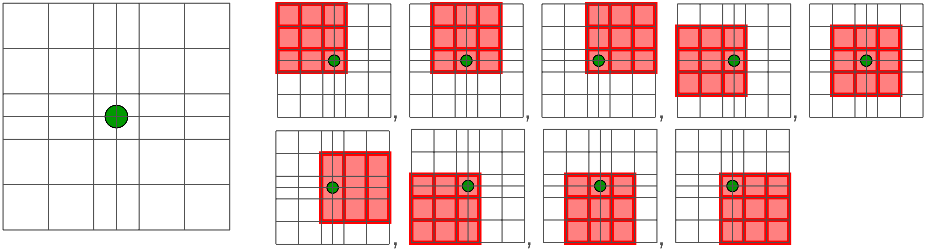

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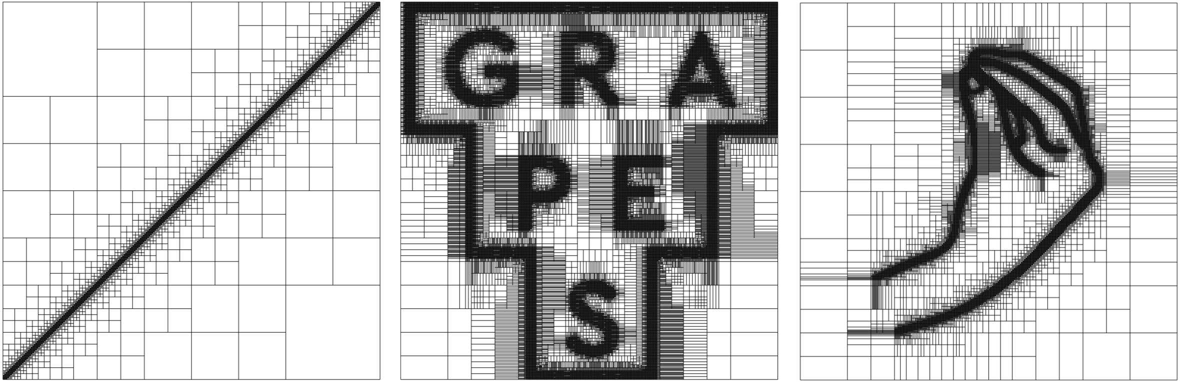

Structured Mesh Refinement strategy

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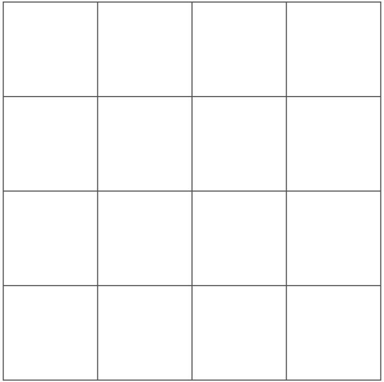

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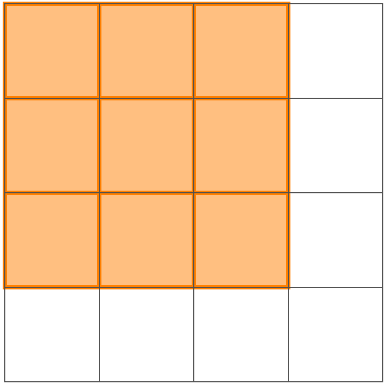

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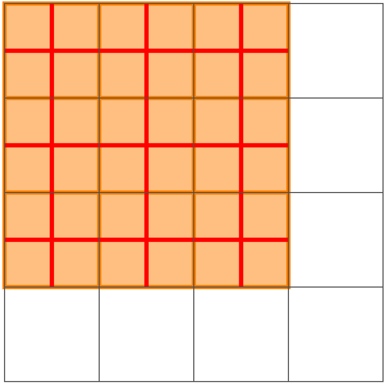

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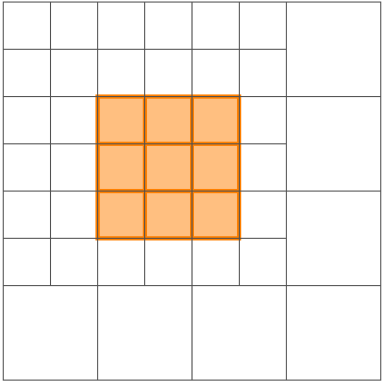

Structured Mesh Refinement strategy

Input: Bunch of LR B-splines where a larger error is committed is some sense.
For each of such LR B-splines

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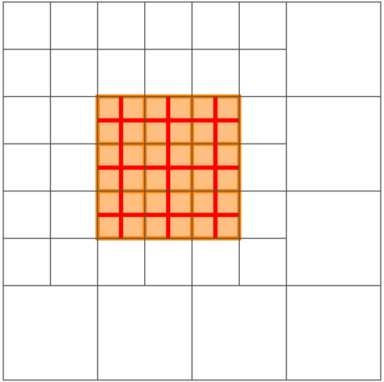
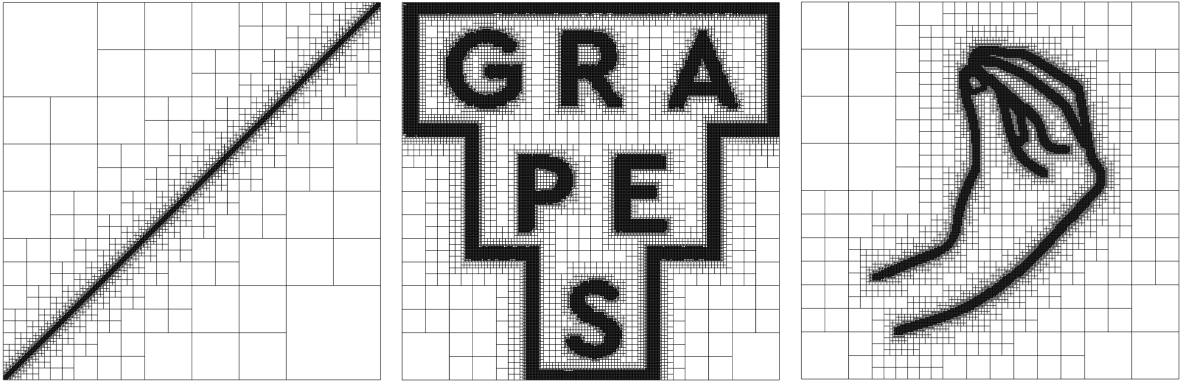


Structured Mesh Refinement strategy: Examples



The Linear Dependence Problem

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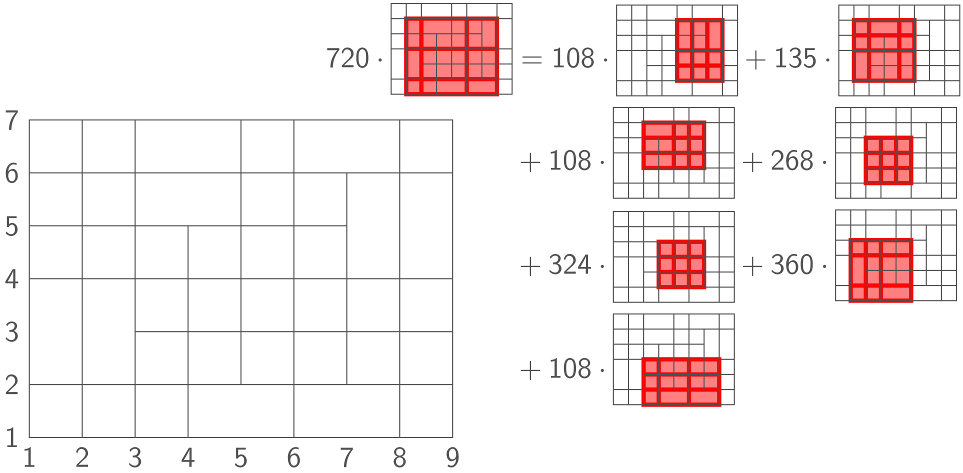

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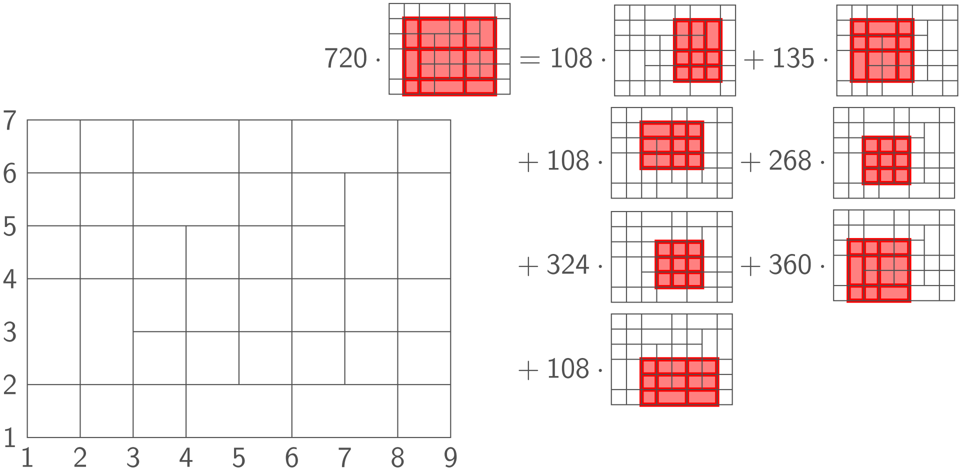

Minimum Span, Full Span and Structured Mesh may have linear dependence

Conjecture: the latter only for $(p_1, p_2) \geq (4, 4)$.

Seek and Destroy Linear Dependence: Peeling Algorithm

Remark: In every box we span the polynomial space $\Pi_{\boldsymbol{p}}$ $\Rightarrow$ each box is in at least $(p_1 + 1)(p_2 + 1) = \dim \Pi_{\boldsymbol{p}}$ LR B-spline supports.

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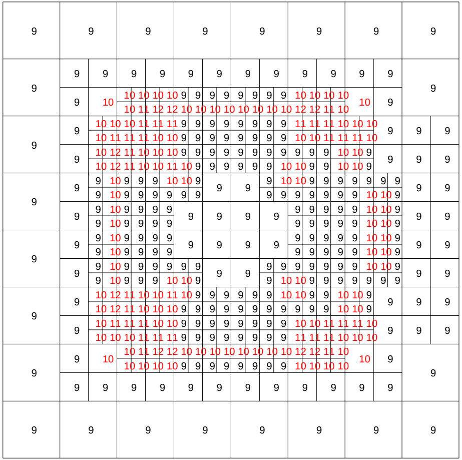

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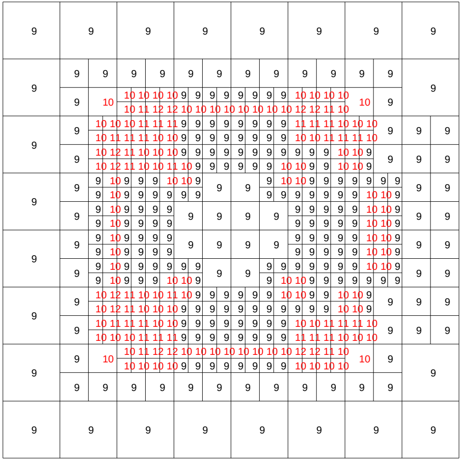

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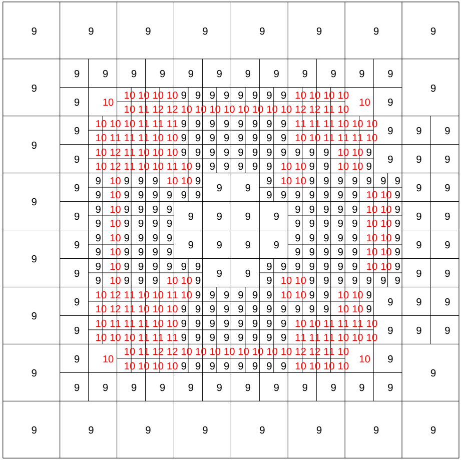

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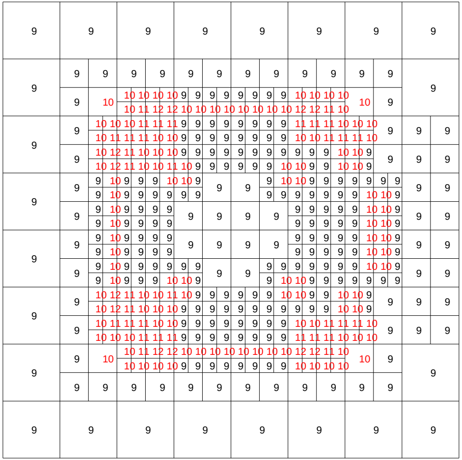

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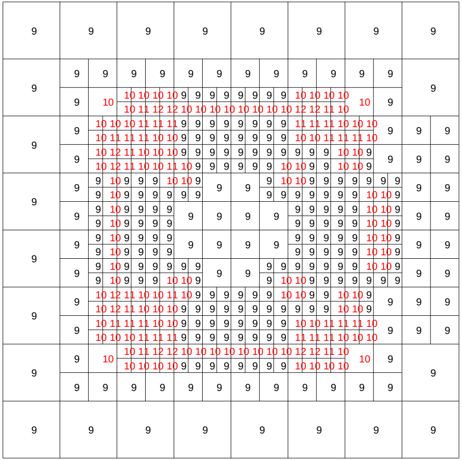

Overloaded LR B-spline: all the boxes in its support are overloaded.

- Only overloaded LR B-splines can be in a linear dependence relation.
- A linear dependence relation needs at least 2 functions.

$\Rightarrow$ If a box is just in one overloaded LR B-spline B, then B cannot be part of a linear dependence relation.

Seek and Destroy Linear Dependence: Peeling Algorithm

Peeling Algorithm

1 Create the set $\mathcal{B}^O$ of overloaded LR B-splines;
2 Let $\mathcal{E}^O$ be the boxes of the LR B-splines in $\mathcal{B}^O$;
3 **for** *every box in* $\mathcal{E}^O$ **do**
4   Identify the set $\mathcal{B}_1^O \subseteq \mathcal{B}^O$ of those having a box covered by no other LR B-splines in $\mathcal{B}^O$;
5 **if** $\mathcal{B}^O \backslash \mathcal{B}_1^O = \emptyset$ **then**
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7 **else**
8   **if** $\mathcal{B}_1^O = \emptyset$ **then**
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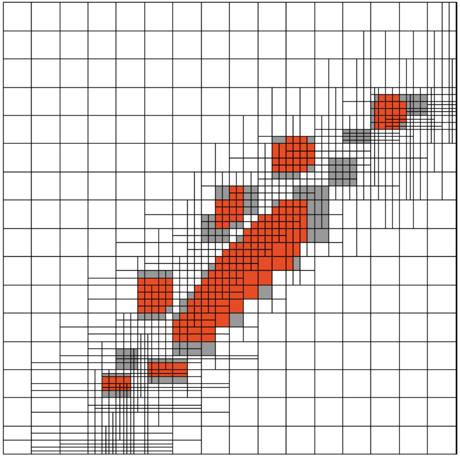

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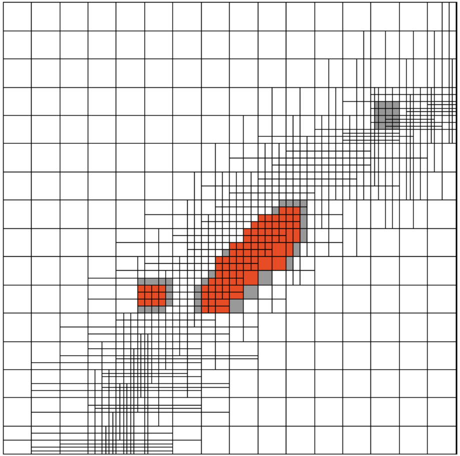

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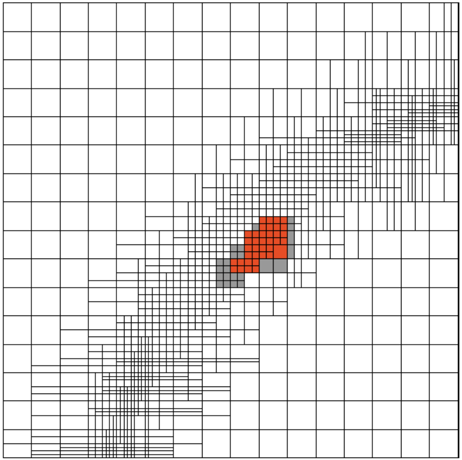

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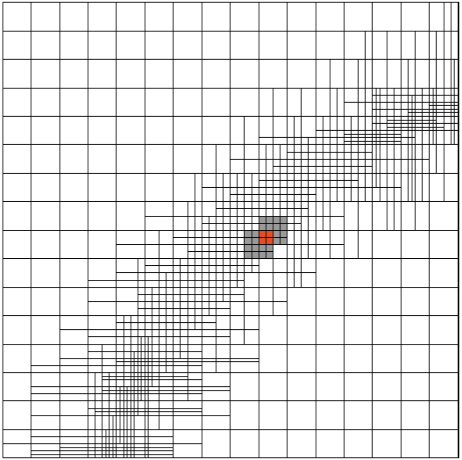

Seek and Destroy Linear Dependence: Peeling Algorithm

Peeling Algorithm

```
1  Create the set 𝓑^O of overloaded LR B-splines;
2  Let 𝓔^O be the boxes of the LR B-splines in 𝓑^O;
3  for every box in 𝓔^O do
4  |   Identify the set 𝓑_1^O ⊆ 𝓑^O of those having a box
   |     covered by no other LR B-splines in 𝓑^O;
5  if 𝓑^O \ 𝓑_1^O = ∅ then
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10 |   𝓑^O ← 𝓑^O \ 𝓑_1^O;
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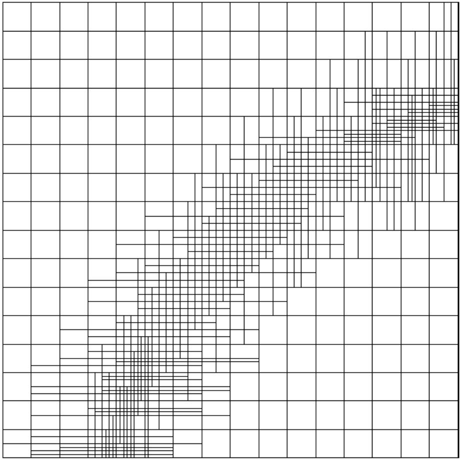

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4 **for** *every box in* $\mathcal{E}^O$ **do**
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6 **for** *every T-vertex* $\boldsymbol{v}$ *in* $\mathcal{V}^O$ **do**
7   If $\boldsymbol{v}$ is only in one $B \in \mathcal{B}^O$, add B to $\mathcal{B}_1^O$;
8 **if** $|\mathcal{B}^O \backslash \mathcal{B}_1^O| < 8$ **then**
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11   **if** $\mathcal{B}_1^O = \emptyset$ **then**
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Conjecture: Peeling Algorithm++ sorts out all cases, if $\mathcal{B}_1^O = \emptyset$ then there is a linear dependece relation.

Local linear independence and N_2S property

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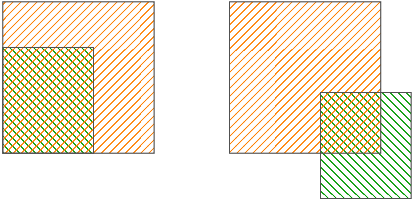
Non-Nested-Support (N_2S) property of the mesh:

☹ ☺



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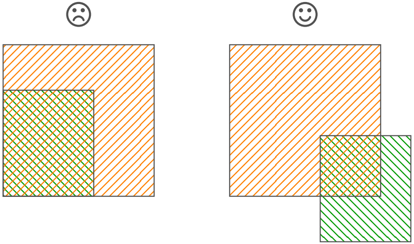

$\nexists\, B^1, B^2$ such that $\operatorname{supp} B^2 \subseteq \operatorname{supp} B^1$.

How do we build LR meshes with the N_2S property?

Non-Nested Support Structured (N_2S_2) Mesh Refinement

1. select the LR B-spline contributing more to the approximation error (in some sense),

Non-Nested Support Structured (N_2S_2) Mesh Refinement

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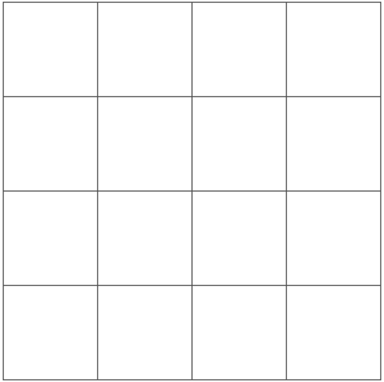

bidegree (2,2)

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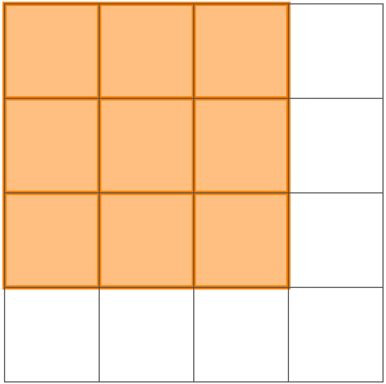

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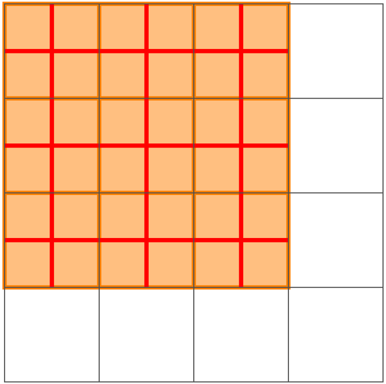

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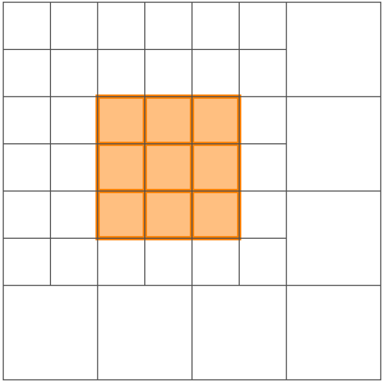

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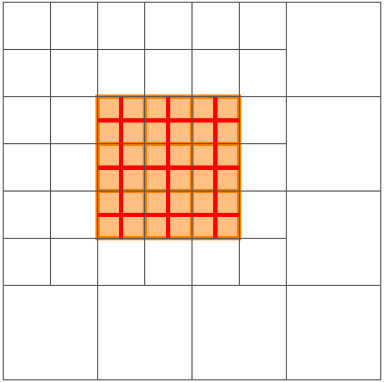

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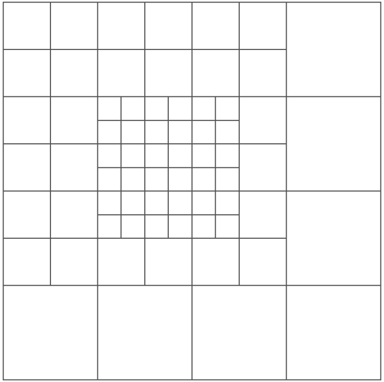

final mesh

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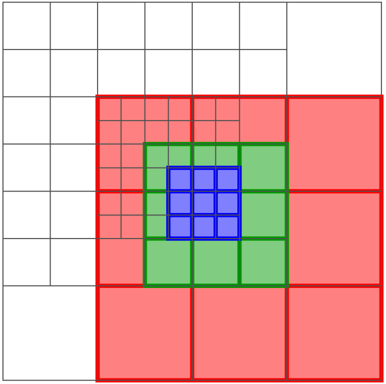
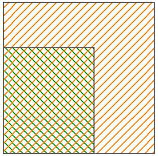


final mesh (no N_2S property)

recall, NOT OK:



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⇒ The LR B-splines defined in these zones of the mesh behave like the standard B-splines, and therefore are locally linearly independent.

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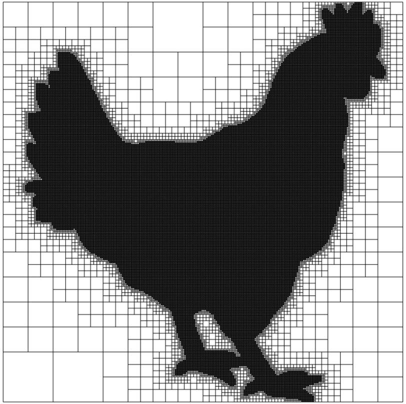
- B-splines defined on a plain tensor mesh are locally linearly independent. The meshes generated with 1.–2. are locally tensor meshes far from the boundary of the region where the refinement is applied.

$\Rightarrow$ The LR B-splines defined in these zones of the mesh behave like the standard B-splines, and therefore are locally linearly independent.



Non-Nested Support Structured (N_2S_2) Mesh Refinement

1. select the LR B-spline contributing more to the approximation error (in some sense),
2. split in 4 all the boxes in their supports (i.e., insert new meshlines),
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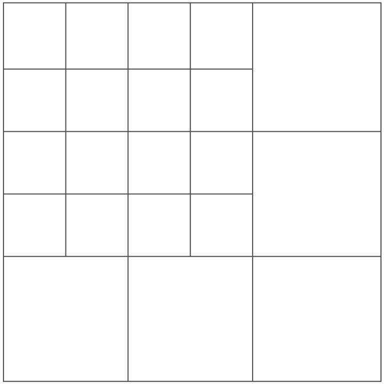

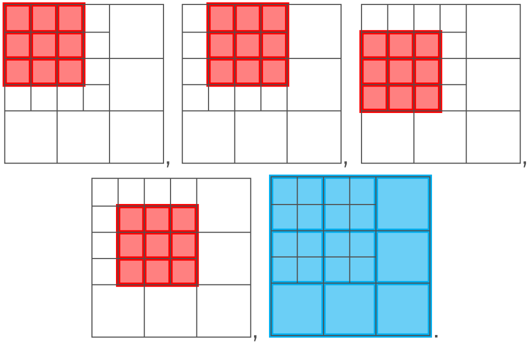

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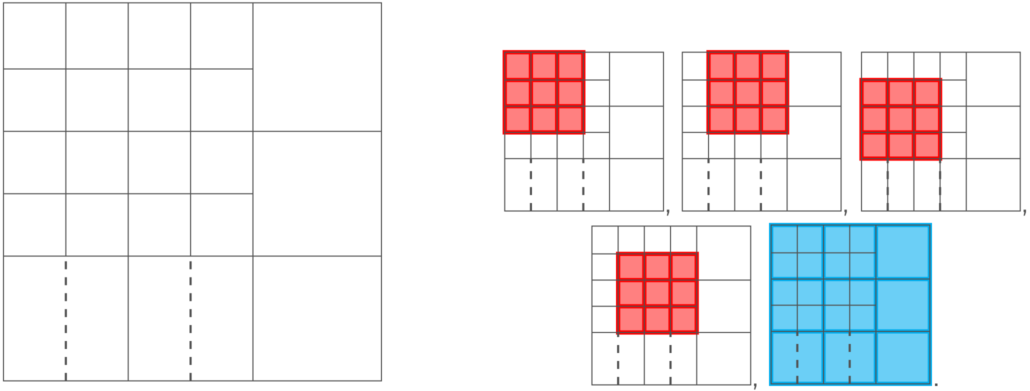

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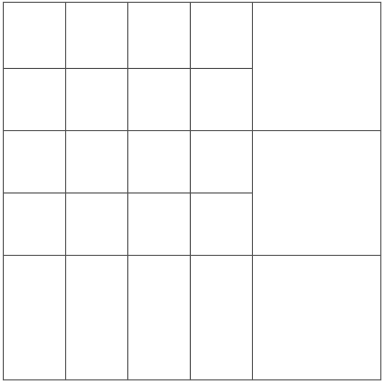

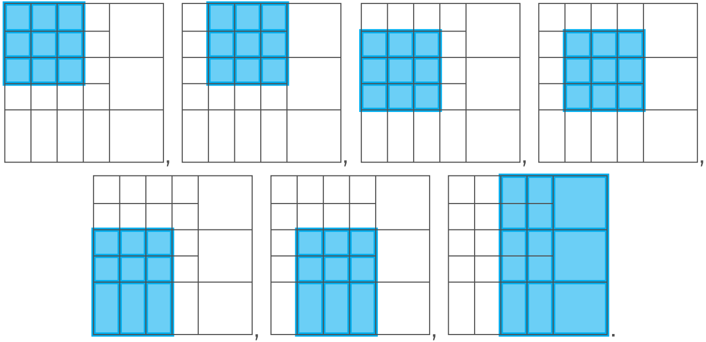

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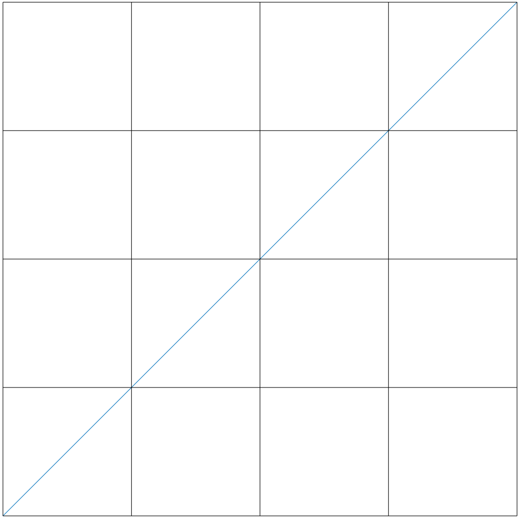

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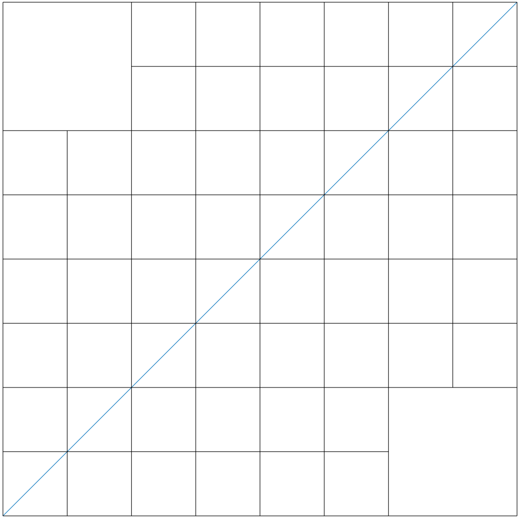

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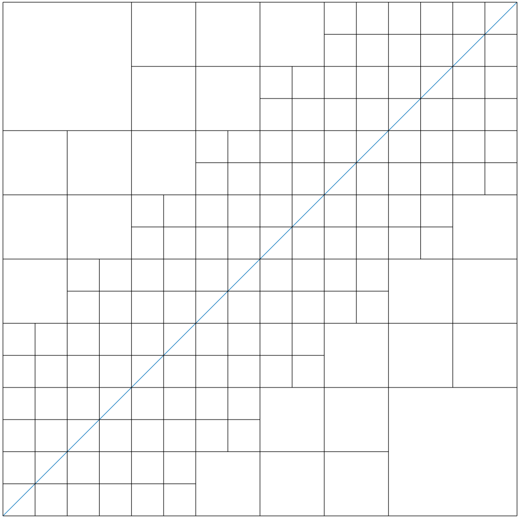

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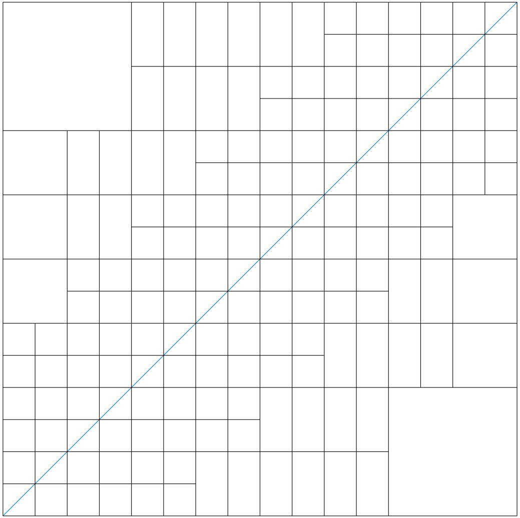

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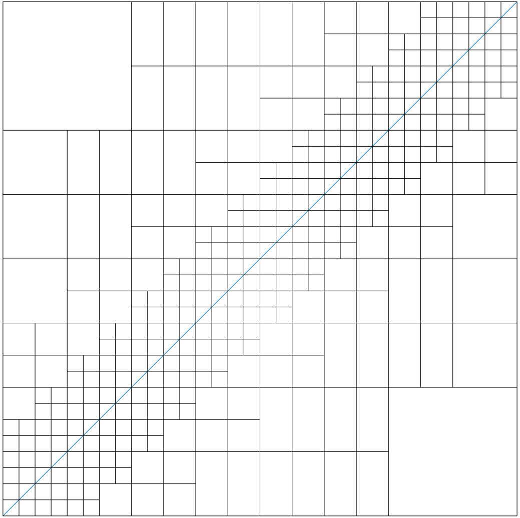

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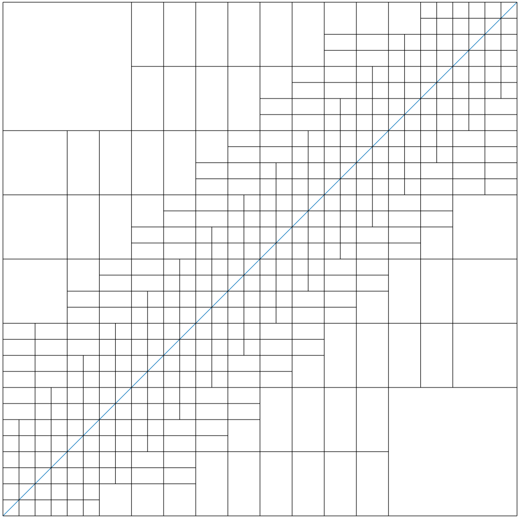

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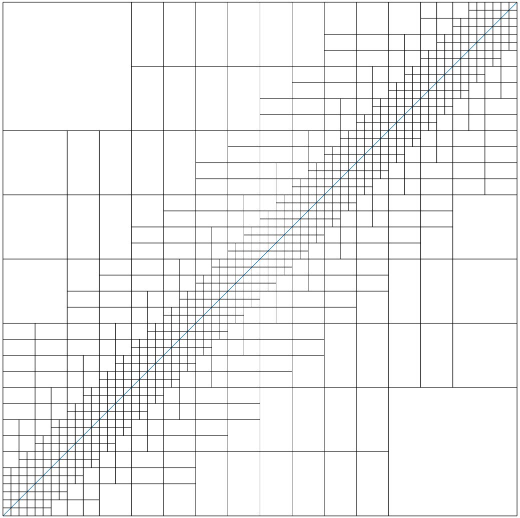

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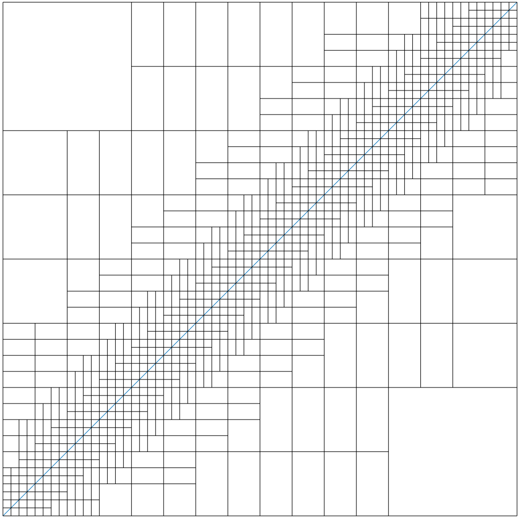

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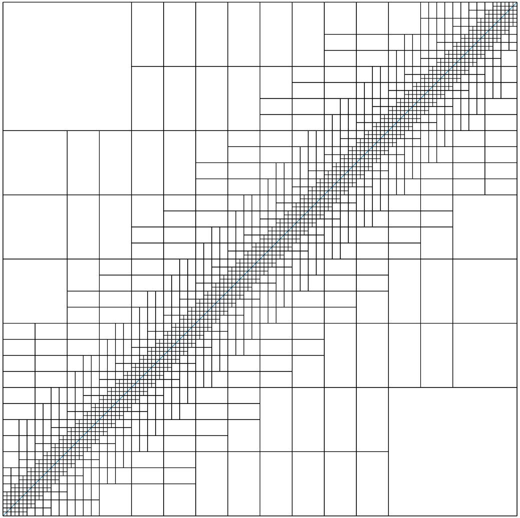

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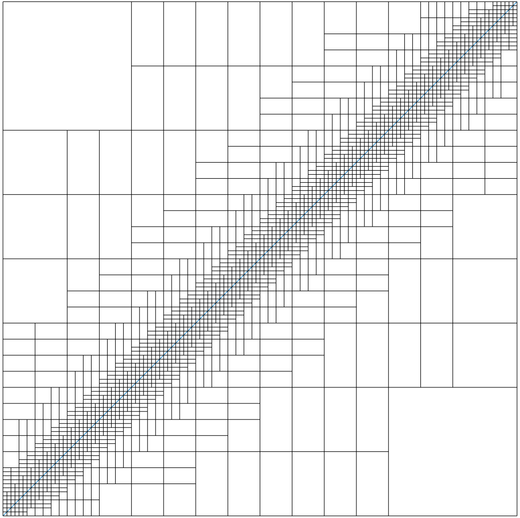

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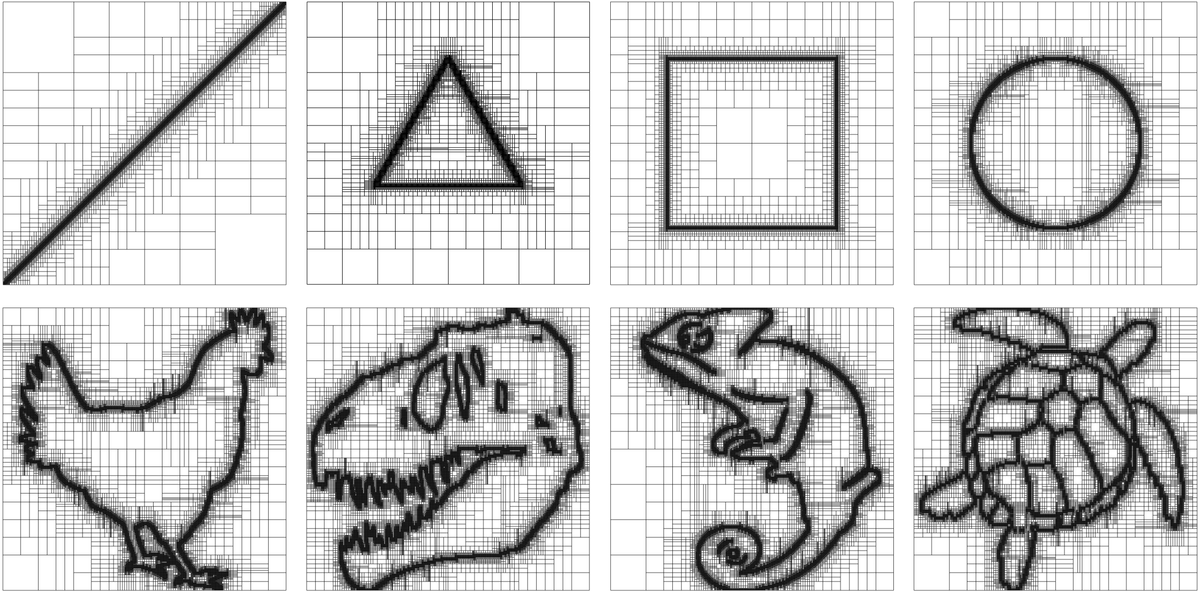

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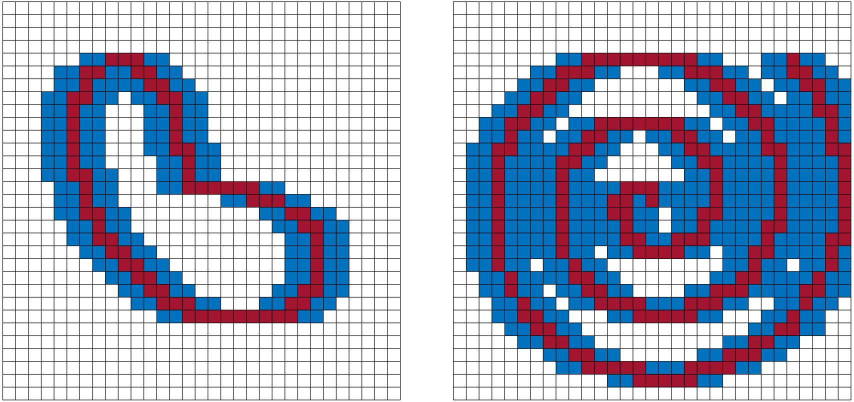

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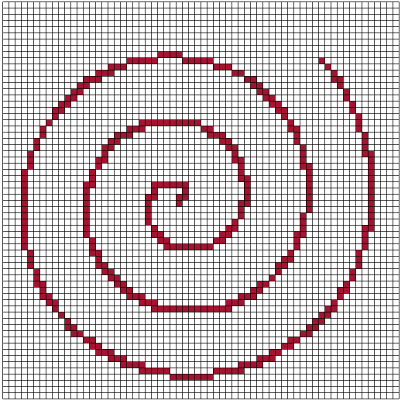

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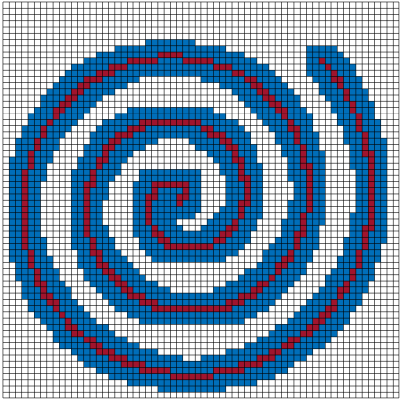

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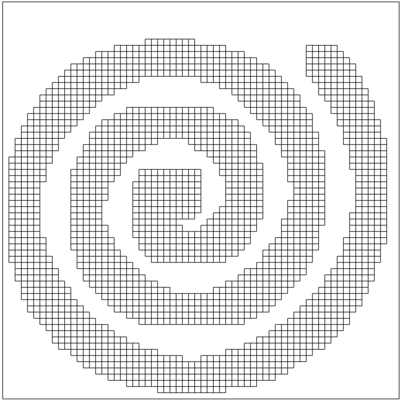

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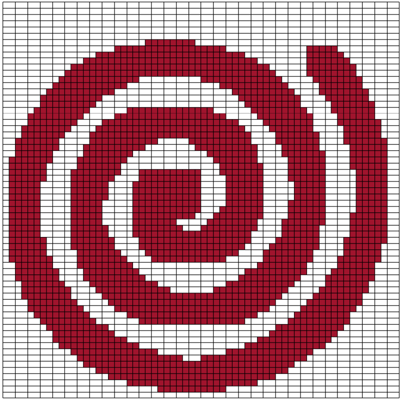

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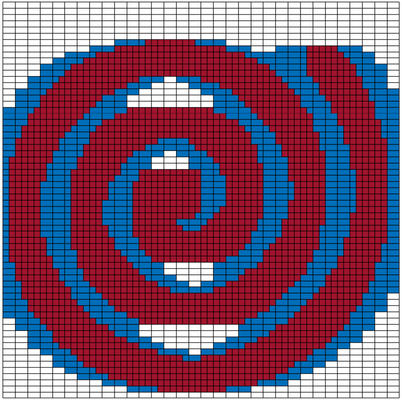

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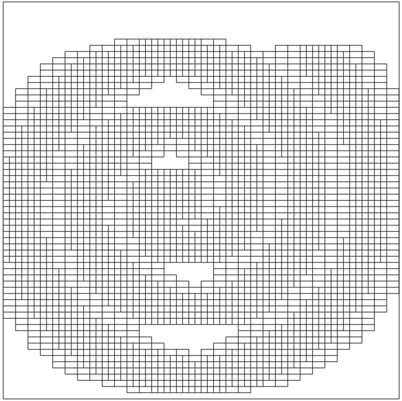

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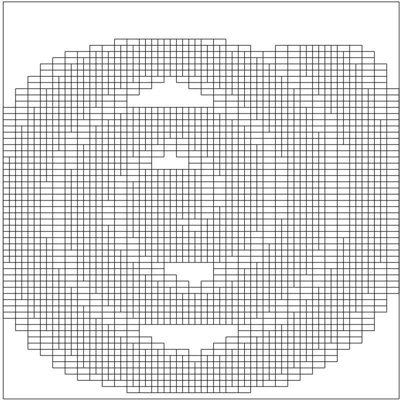

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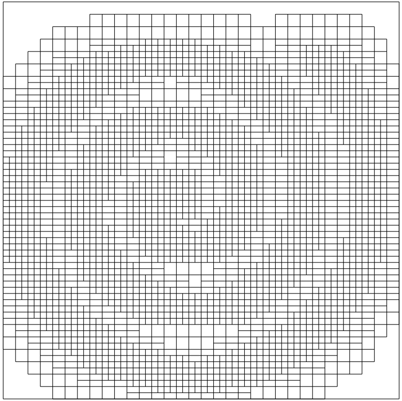

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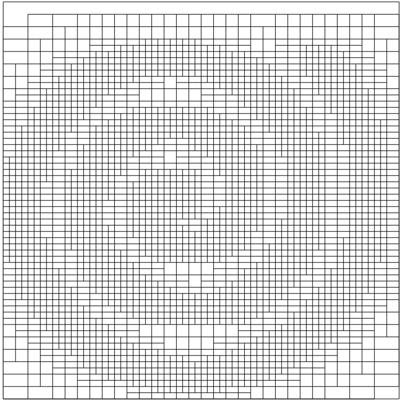

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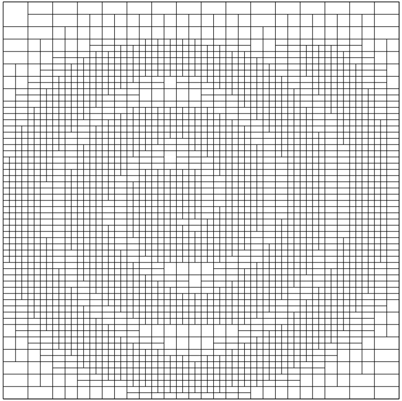

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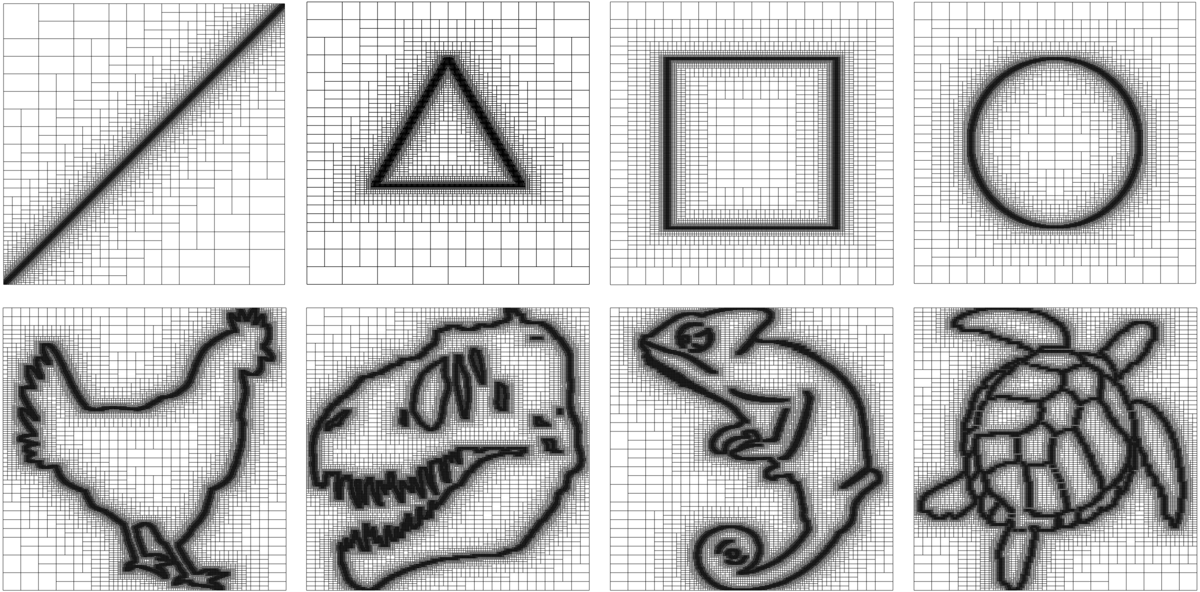

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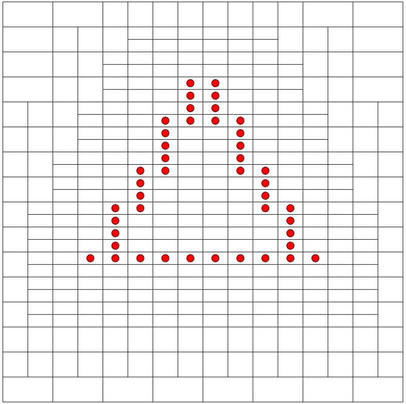

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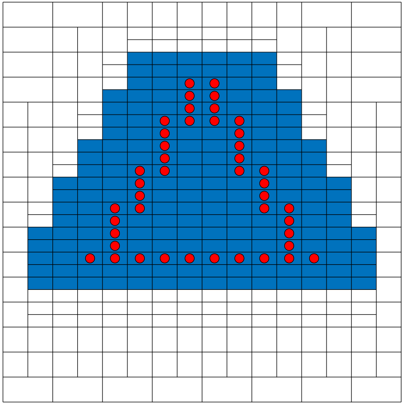

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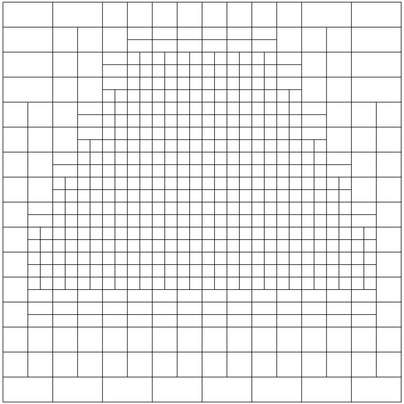

Effective Grading Refinement Strategy

Refinement macro-step:

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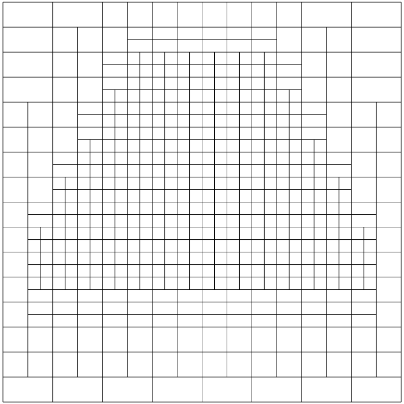

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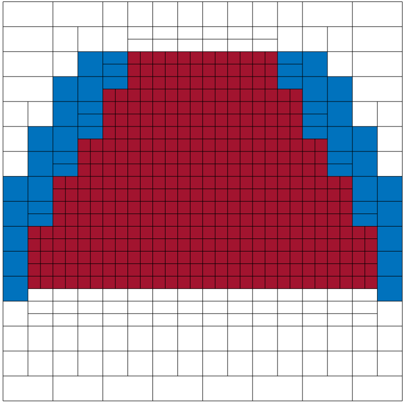

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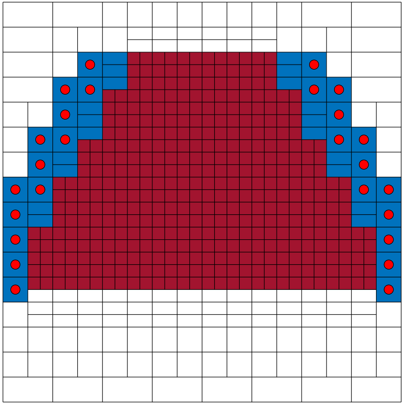

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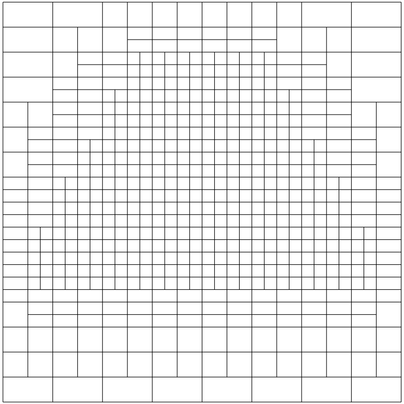

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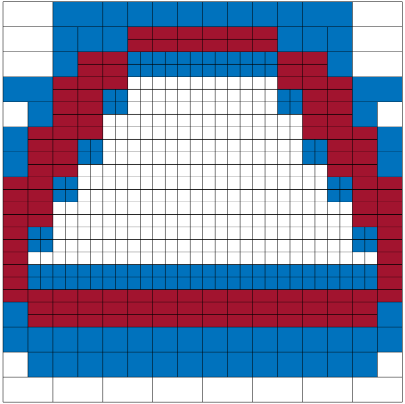

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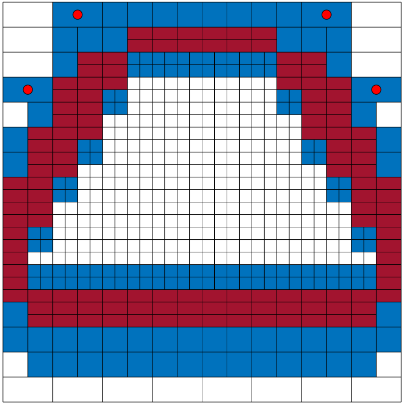

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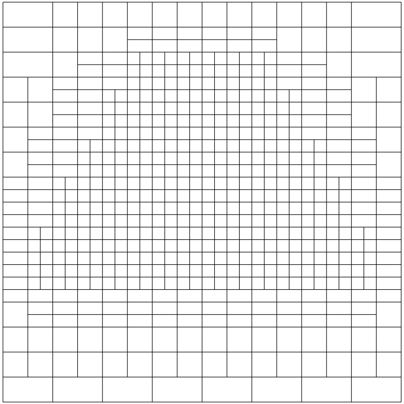
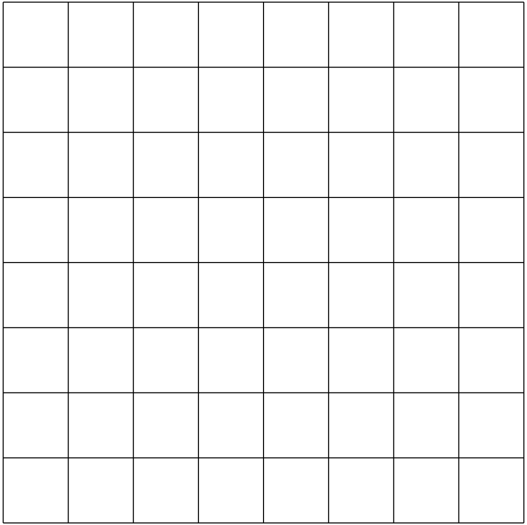


Effective Grading Refinement Strategy

Example: degree $(2, 2)$, Triangle $\rightarrow$ Circle $\rightarrow$ Square



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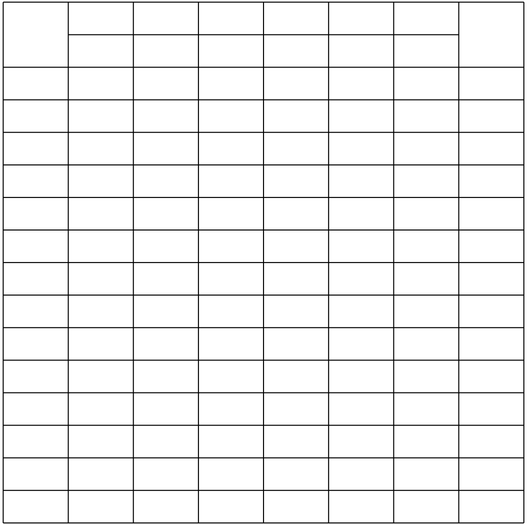

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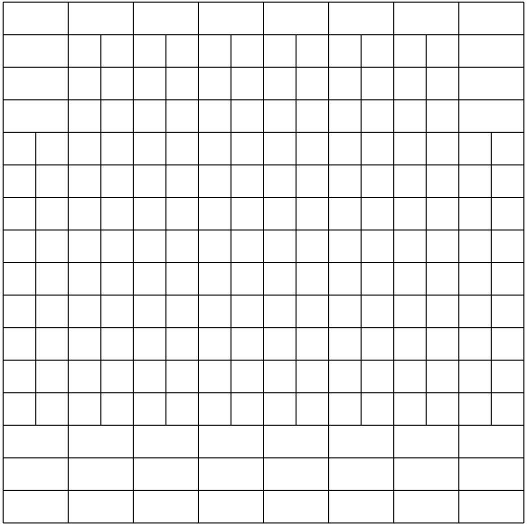

Effective Grading Refinement Strategy

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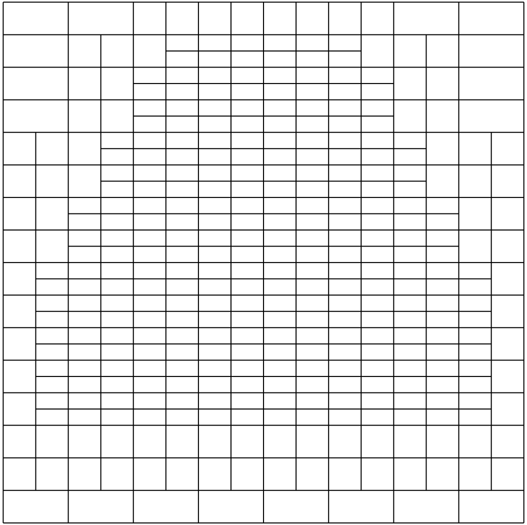

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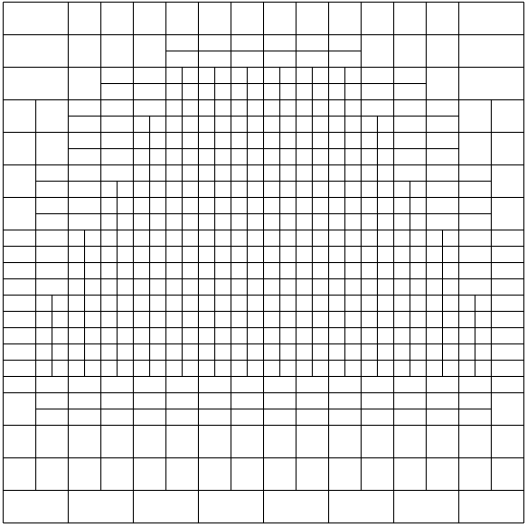

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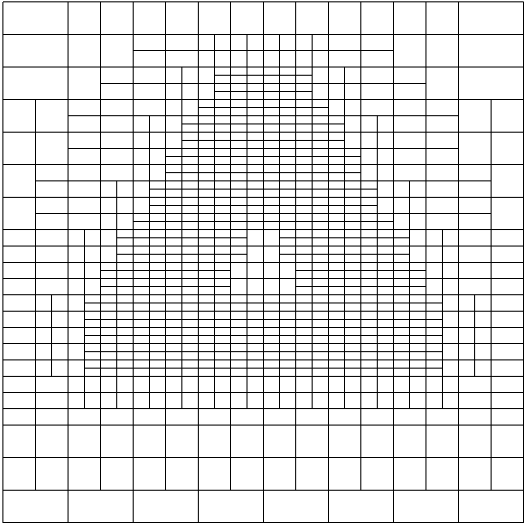

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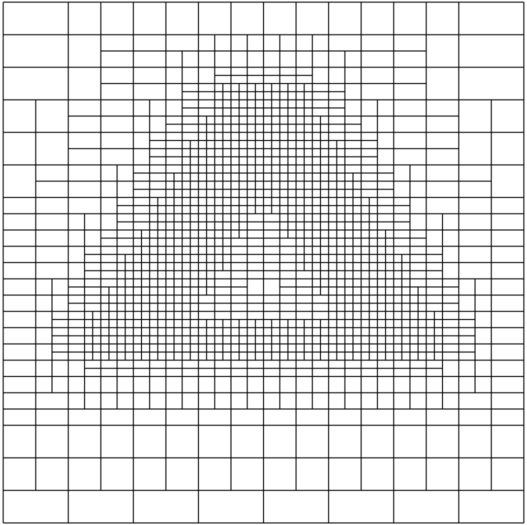

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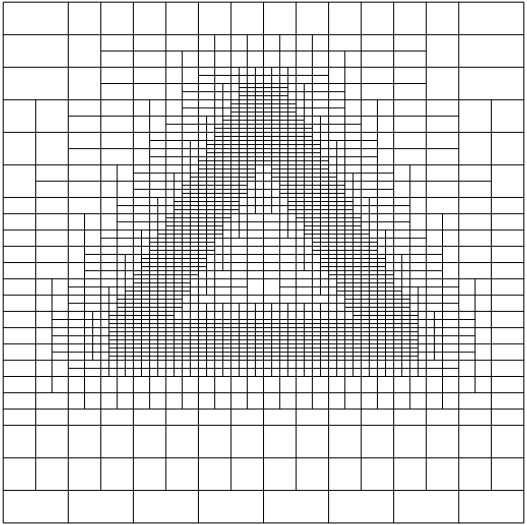

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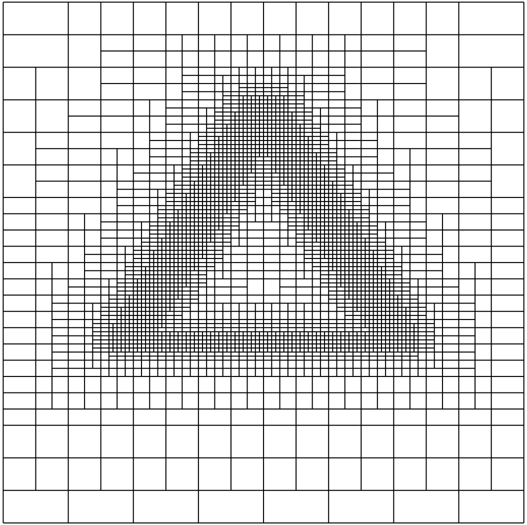

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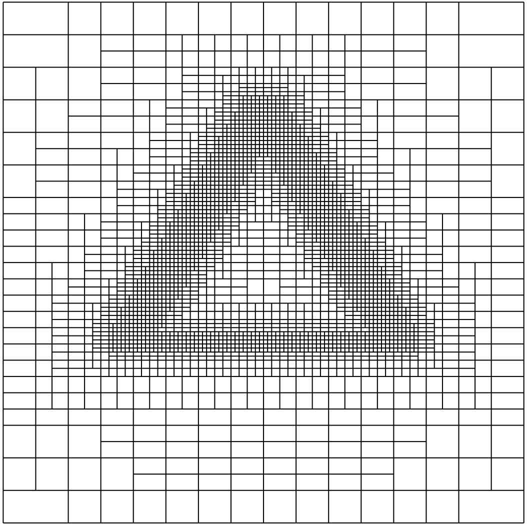

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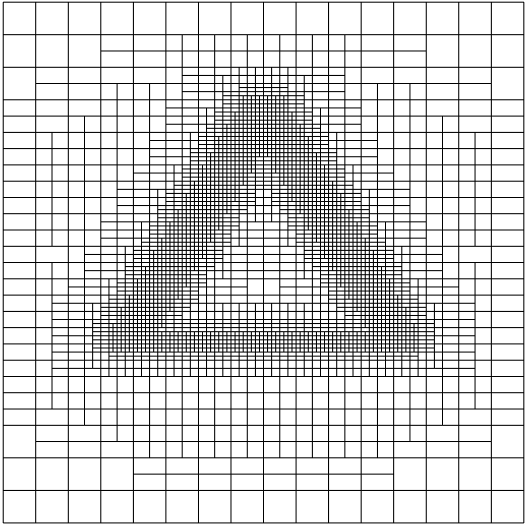

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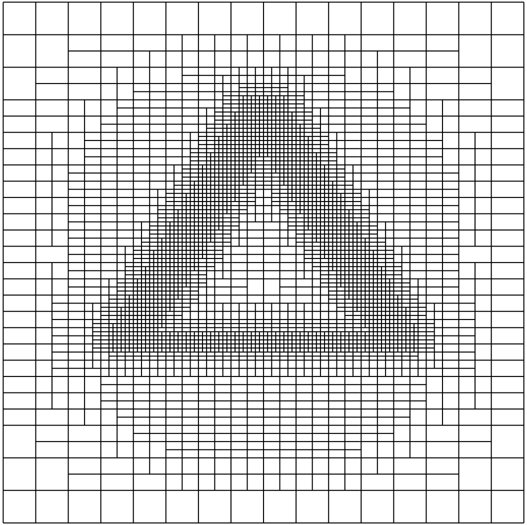

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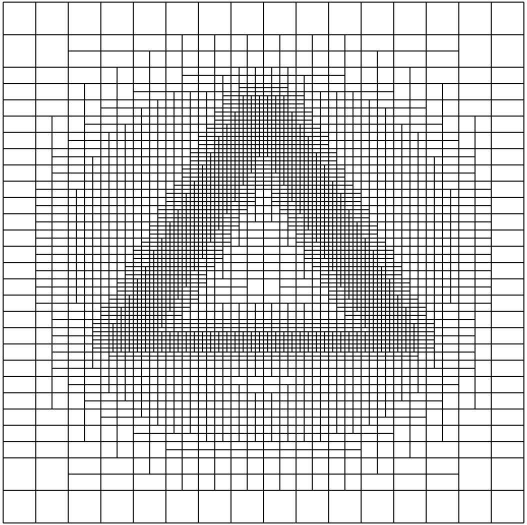

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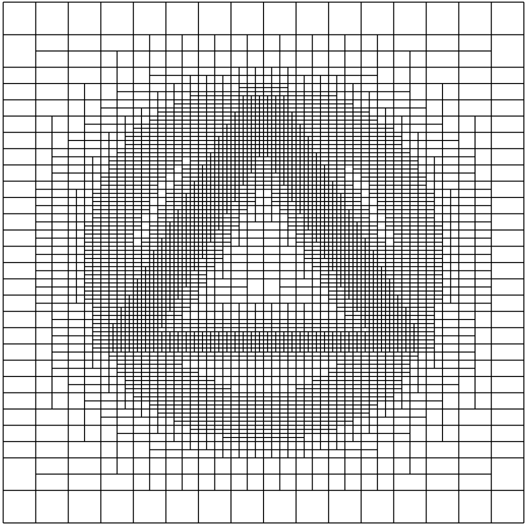

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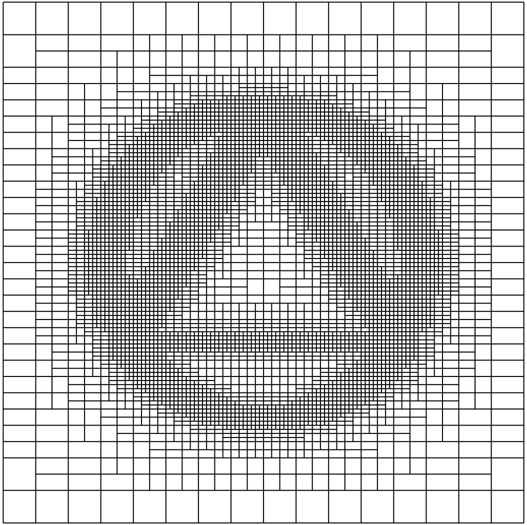

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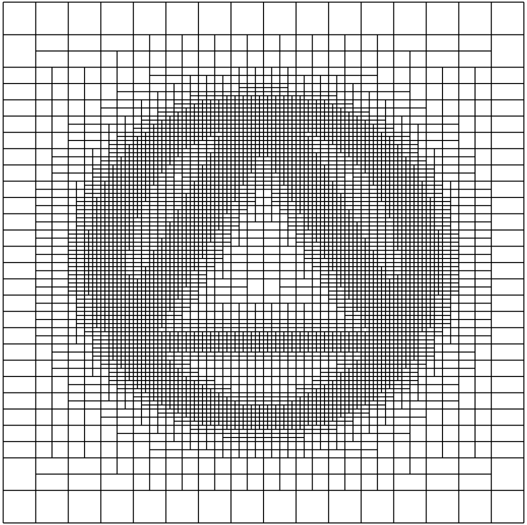

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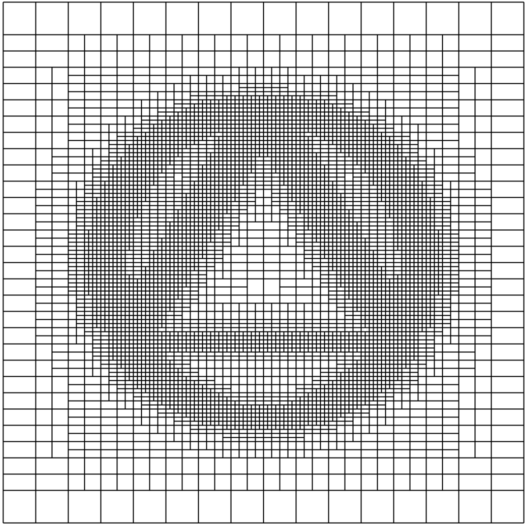

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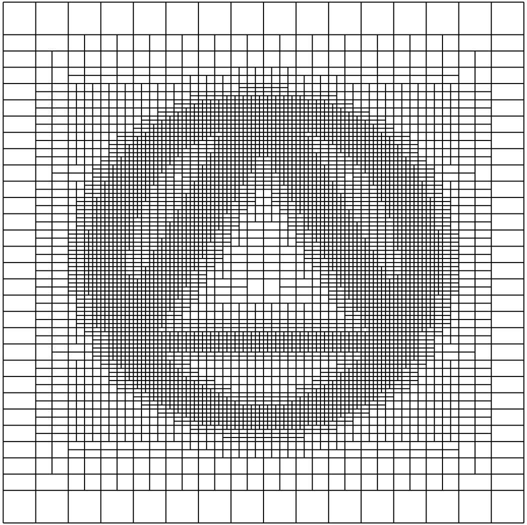

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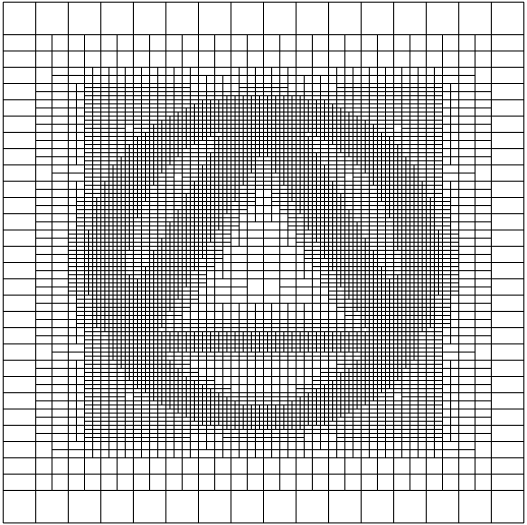

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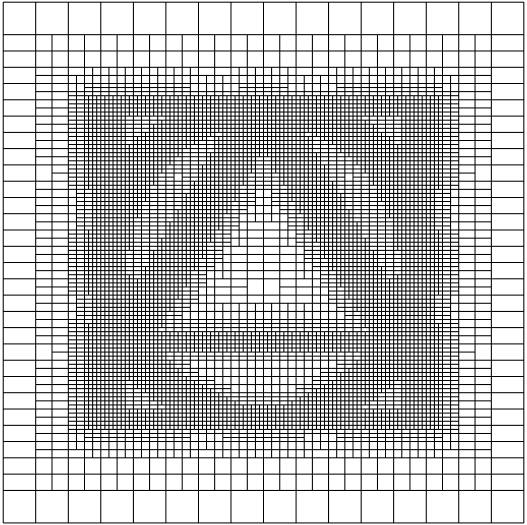

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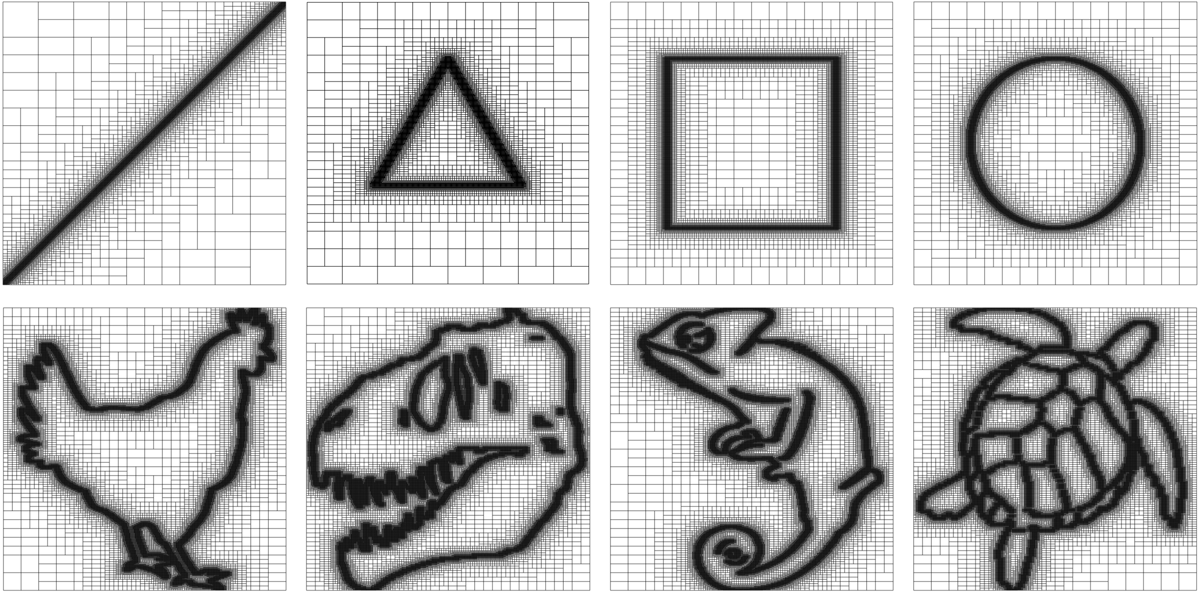
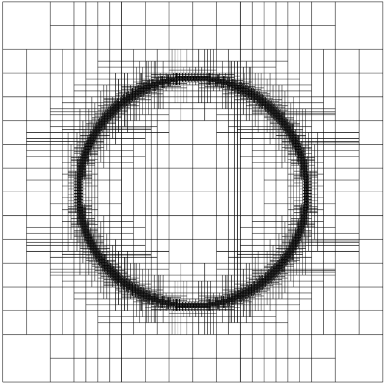
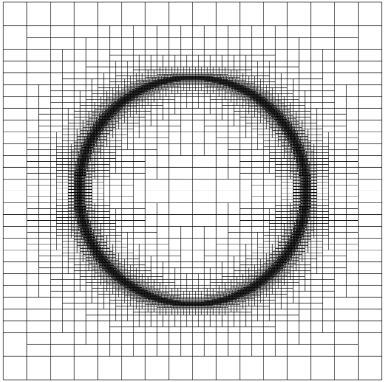


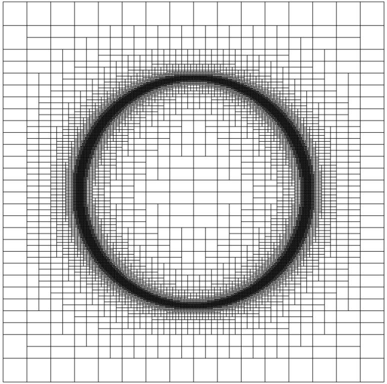
Comparison



N_2S_2 10818 LR B-splines



HLR 12374 LR B-splines



EG 15238 LR B-splines

One word on the spline space

$\mathcal{N}$ mesh with local insertions (not necessarily LR mesh)

$$\mathbb{S}(\mathcal{N}) := \left\{ \begin{aligned} f : \mathbb{R}^2 \to \mathbb{R} \ : & \operatorname{supp} f \subseteq \Omega, \\ & f|_\beta \text{ is a polynomial of bidegree } \boldsymbol{p} \text{ in any } \beta \text{ box of } \mathcal{N}, \\ & f \in C^{p_{3-k}-\mu(\gamma)}\text{-continuous across } \gamma \in \mathcal{N} \text{ in the } k\text{th direction.} \end{aligned} \right\}.$$

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Dimension formula: In general combinartorial part + homological part.

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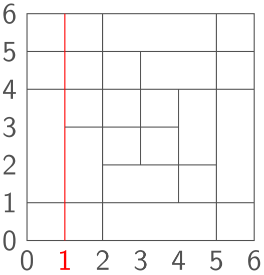
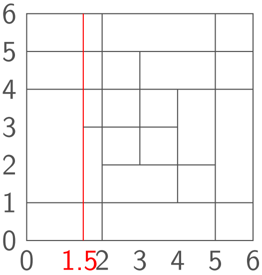
Dimension formula: In general combinartorial part + homological part. The homological part makes it parametrization-dependent ⇒ Unstable ☹

$\dim \mathbb{S}(\mathcal{N}) = 36$





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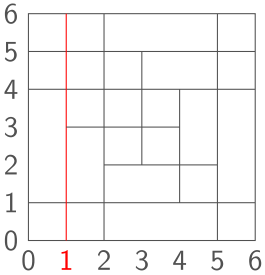

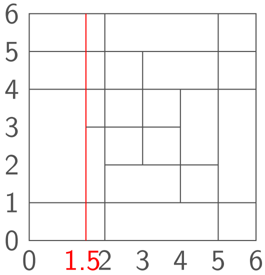

$\dim \mathbb{S}(\mathcal{N}) = 36 + 1$

On LR meshes only combinatorial ☺.

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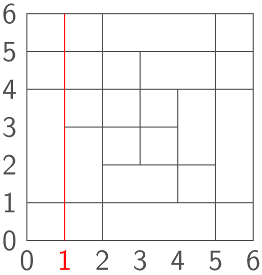
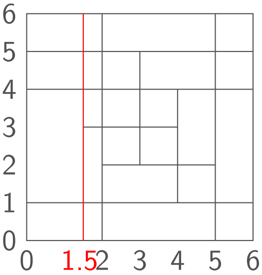




$\dim \mathbb{S}(\mathcal{N}) = 36 + 1$

On LR meshes only combinatorial ☺. HB, THB, LR, $\ldots \subseteq \mathbb{S}(\mathcal{N})$.

Comparison

Adaptivity:

Comparison

Adaptivity: Local Refinement

Comparison

Adaptivity: Local Refinement + Change Region at any time

Comparison

Adaptivity: Local Refinement + Change Region at any time

Grading:

Comparison

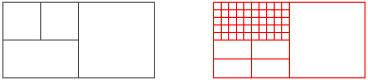
Adaptivity: Local Refinement + Change Region at any time

Grading: Shape Regularity: No skinny elements

Comparison

Adaptivity: Local Refinement + Change Region at any time

Grading: Shape Regularity: No skinny elements + Local Quasi Uniformity:

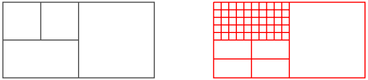


no large boxes side by side small boxes

Comparison

Adaptivity: Local Refinement + Change Region at any time

Grading: Shape Regularity: No skinny elements + Local Quasi Uniformity:



no large boxes side by side small boxes

Completeness:

Comparison

Adaptivity: Local Refinement + Change Region at any time

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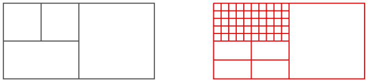


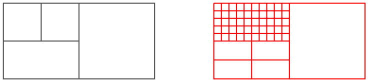
no large boxes side by side small boxes

Completeness: The LR B-splines span the full ambient spline space $\mathbb{S}(\mathcal{N})$.

Comparison

Adaptivity: Local Refinement + Change Region at any time

Grading: Shape Regularity: No skinny elements + Local Quasi Uniformity:



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Marking:

Comparison

Adaptivity: Local Refinement + Change Region at any time

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no large boxes side by side small boxes

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Marking: ▶ function-based: refine those LR B-splines contributing more to the error,

Comparison

Adaptivity: Local Refinement + Change Region at any time

Grading: Shape Regularity: No skinny elements + Local Quasi Uniformity:

no large boxes side by side small boxes

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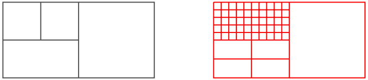
Comparison

Adaptivity: Local Refinement + Change Region at any time

Grading: Shape Regularity: No skinny elements + Local Quasi Uniformity:



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Comparison

Adaptivity: Local Refinement + Change Region at any time

Grading: Shape Regularity: No skinny elements + Local Quasi Uniformity:

no large boxes side by side small boxes

Completeness: The LR B-splines span the full ambient spline space $\mathbb{S}(\mathcal{N})$.

Marking:
- function-based: refine those LR B-splines contributing more to the error,
- box-based: refine those boxes in which a larger error is committed.

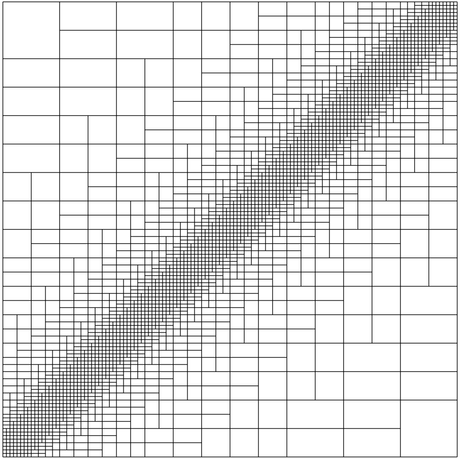
	(Loc.) Lin. Ind.	Adaptivity	Grading	Completeness	Marking
N_2S_2 strategy	✔	✔	✘	?	function-based
Hierarchical	Under Assumptions*	✘	✔	✔	box-based
Effective Grading	✔	✔	✔	✔	box-based

*fix maximal resolution and region of refinement *a priori*

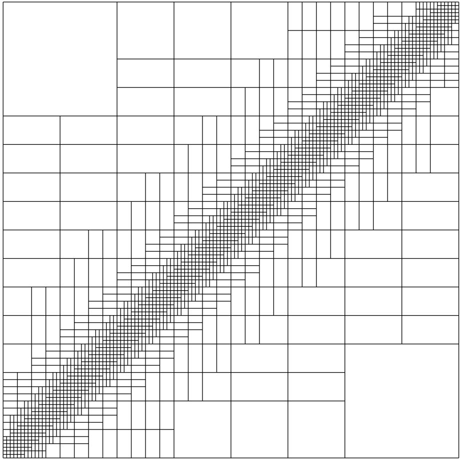
Conjecture: Local Linear Independence $\Rightarrow$ Completeness.

Comparison of Adaptivity with and without Grading

From a refinement localized on a diagonal we switch to the other diagonal to form an "X".



EG: Adaptivity & Grading



N_2S_2: Adaptivity but no Grading

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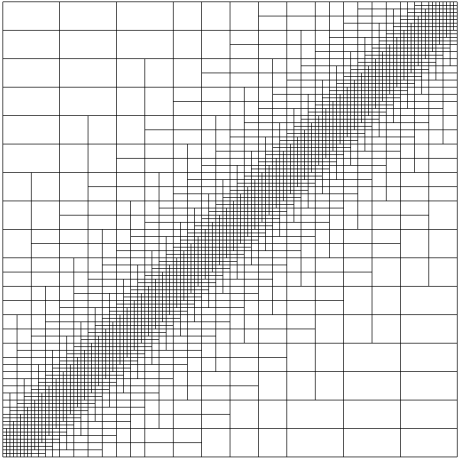

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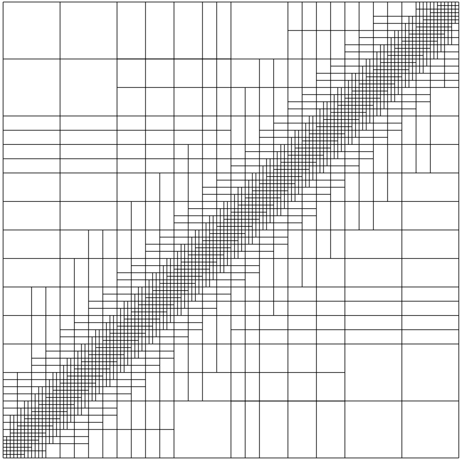

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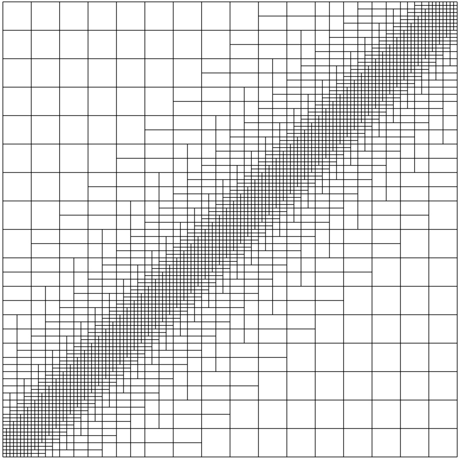

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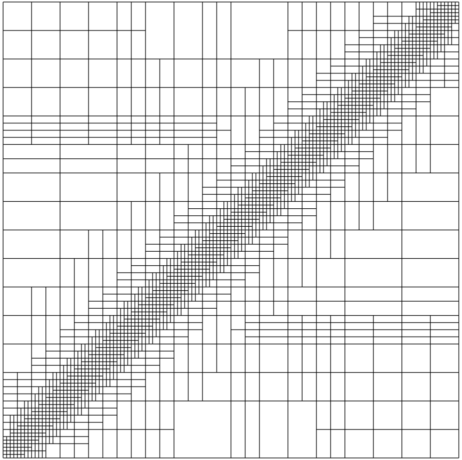

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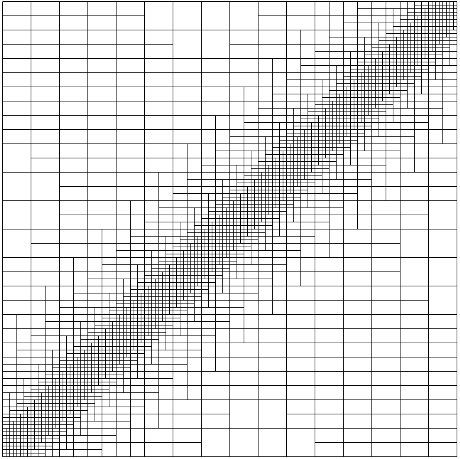

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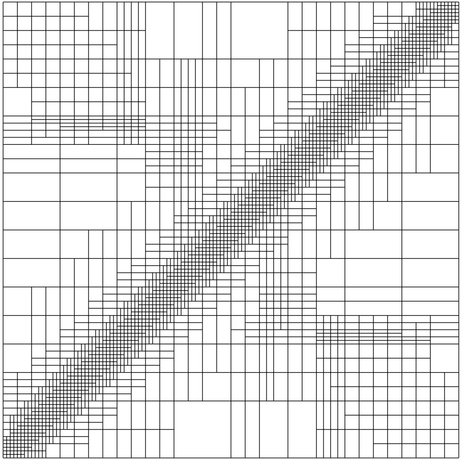

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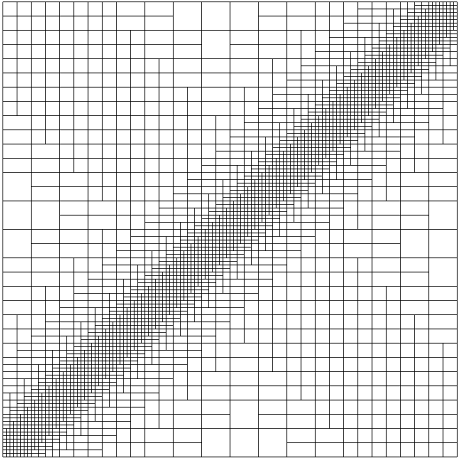

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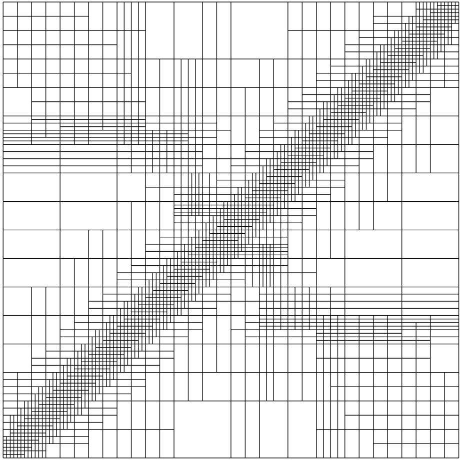

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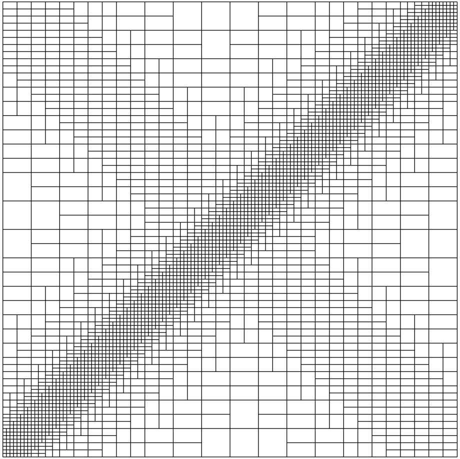

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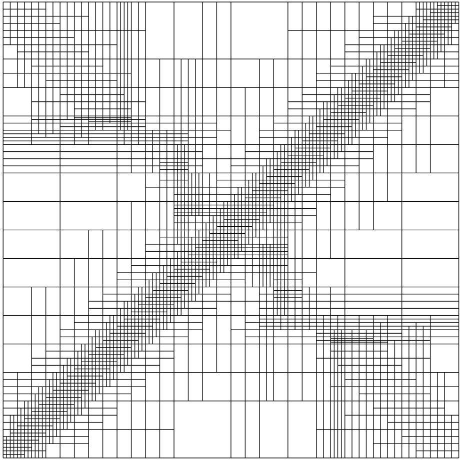

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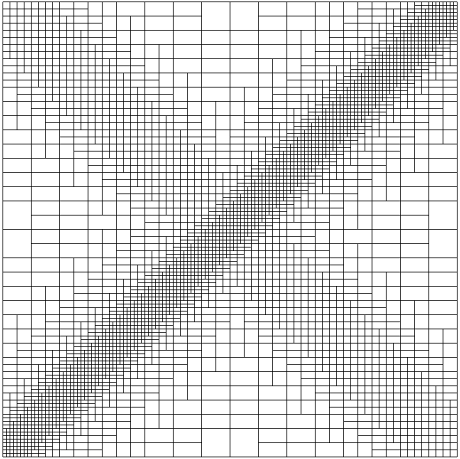

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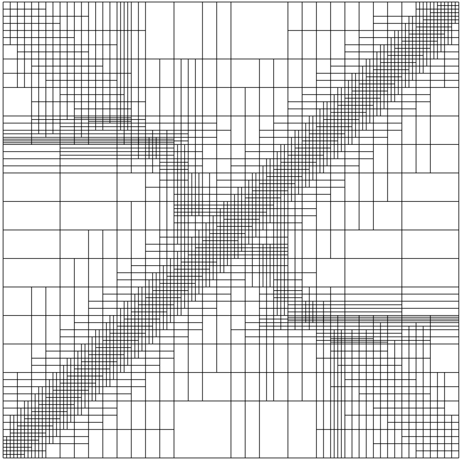

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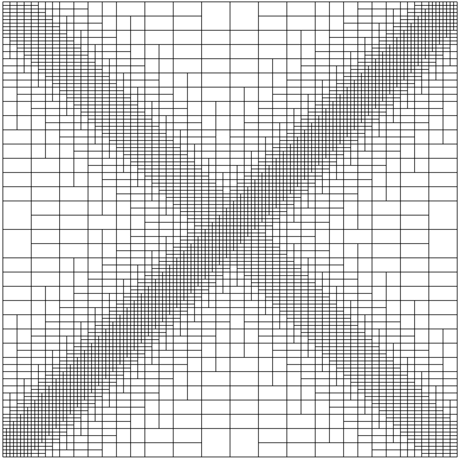

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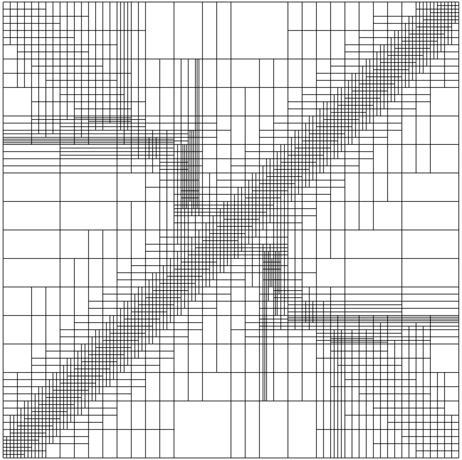

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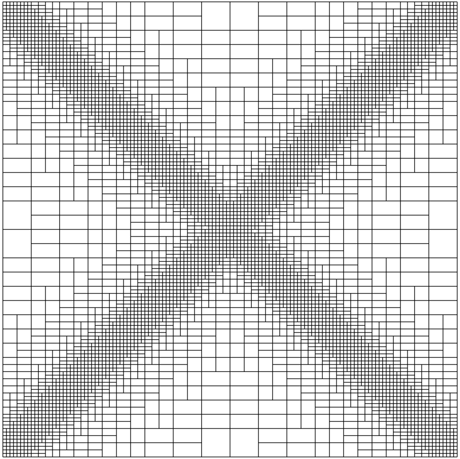

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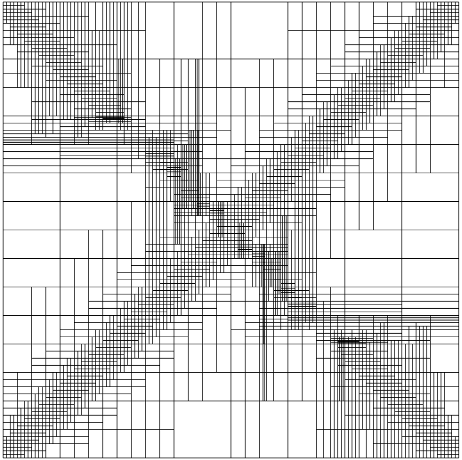

N_2S_2: Adaptivity but no Grading

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