Regularity and computations with multihomogeneous or sparse polynomial systems

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Linz - Grapes final open conference and career fair

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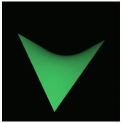
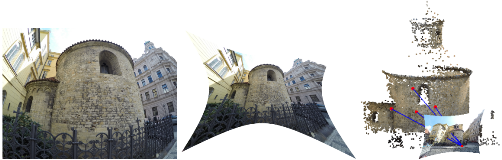
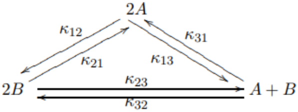
GRAPES

Polynomial systems in applications

Geometric modelling	Computer vision
 [C-Emiris'23]	 [Bhayani-Kukelova-Heikkilä'19]
Find the implicit equation of a parametric curve or surface	Minimal problems in camera geometry

Biology



[Millán-Dickenstein-Shiu-Conradi'11]

Identifying steady states of chemical realction networks

The world is toric

Newton polytopes: sparsity in polynomials

$$F = \sum u_b x^b \quad b \in \mathcal{A} \quad x^b = x_1^{b_1} \cdots x_n^{b_n}$$

$$\mathcal{A} \subset \mathbb{Z}^n \to \text{ Newton polytope } \Delta = conv(\mathcal{A})$$

Exploiting the monomial structure. A toric variety is associated to the polytope.

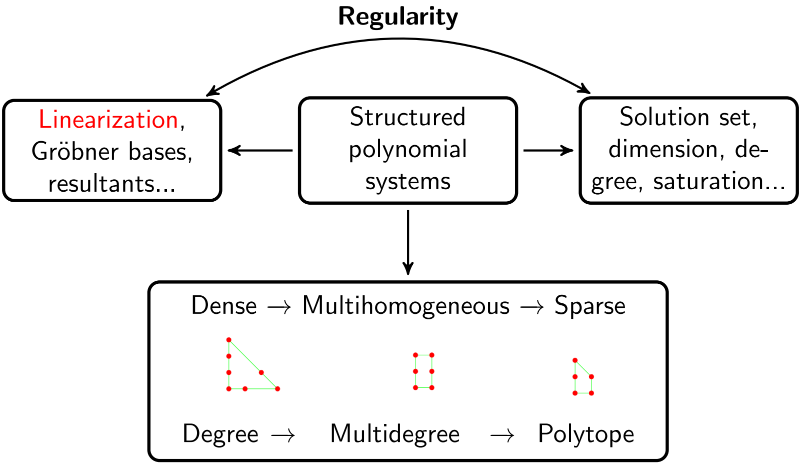
"Hidden" toric structure

$$I_A = \left(\left\langle 2 \times 2 \text{ minors of } \begin{pmatrix} K_1 & K_2 & K_3 & K_4 \\ c_1^3 & c_2^3 & c_3^3 & c_1 c_2 c_3 \end{pmatrix} \right\rangle : (c_1 c_2 c_3)^\infty \right).$$

[Michalek-Sturmfels'21], [Craciun-Dickenstein-Shiu-Sturmfels'07]

A balancing ideal corresponding to a chemical reaction network turns out to be toric (prime and gen. by binomials).

Computational algebra vs. commutative algebra



Motivation: implicitization with elimination matrices

Given a parametric surface $\mathcal{S} \subset \mathbb{R}^3$, construct a matrix $\mathbb{M}(X, Y, Z)$ whose entries depend on (X, Y, Z) such that for $p \in \mathbb{R}^3$

$$p \in \mathcal{S} \iff \text{corank of } \mathbb{M}(p) > 0$$

Example

$$\phi : V \subset \mathbb{R}^2 \to \mathbb{R}^3 \quad (s,t) \to (\phi_1(s,t), \phi_2(s,t), \phi_3(s,t))$$

$\phi_1 = 1 + 3s - 2st + s^2 t, \phi_2 = -1 - 3s + 4st + 5s^2 t, \phi_3 = -2 + 5s + 4st - 1s^2 t$

$$\begin{array}{c} \\ X-\phi_1 \\ s(X-\phi_2) \\ Y-\phi_1 \\ s(Y-\phi_2) \\ Z-\phi_3 \\ s(Z-\phi_3) \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & s & s^2 & ts & ts^2 & ts^3 \end{array} \\ \begin{pmatrix} X-1 & -3 & 0 & 2 & -1 & 0 \\ 0 & X-1 & -3 & 0 & 2 & -1 \\ Y+1 & 3 & 0 & -4 & -5 & 0 \\ 0 & Y+1 & 3 & 0 & -4 & -5 \\ Z+2 & -5 & 0 & -4 & 1 & 0 \\ 0 & Z+2 & -5 & 0 & -4 & 1 \end{pmatrix} \end{array}$$

The Castelnuovo-Mumford regularity

Given a (homogeneous) polynomial system $I = \langle f_1, \ldots, f_r \rangle$ in $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$. $reg(I)$ is a degree that describes some algebro-geometric properties of I. ■

[Castelnuovo'1896] [Mumford'66].

$reg(I)$ relates with the maximal degree of the monomials in the above matrix construction. *Algebraic curves and surfaces: a history of shapes.* [Buse-Catanese-Postinghel'23].

Monomial orders and Gröbner bases

Consider a total multiplicative order $<$ in the monomials of S.

(Degree) reverse lexicographical monomial order $(x_0 > \cdots > x_n)$.

$$x^a > x^b \iff \text{the last non-zero entry of } a - b \text{ is negative}$$

Leading terms $f = \sum_{a \quad c_a \neq 0} c_a x^a \implies in_<(f) = c_b x^b \quad x^a < x^b$.

Initial ideal $in(I) = (in(f) \quad f \in I)$. The generators of f corresponding to a generators of $in(I)$ are also known as Gröbner basis [Buchberger'65].

Let $reg_0(in(I))$ be the degree of the generators of $in(I)$.

The generators of $in(I)$ and generic coordinates

Theorem $reg(in(I)) \geq reg_0(in(I)) \quad reg(in(I)) \geq reg(I)$

If $reg(in(I)) = reg_0(in(I)) = reg(I)$, then the complexity of Gröbner computations depends on $reg(I)$ (algebra of I).

$$\downarrow$$

Theorem [Galligo'74] For a generic linear change of coordinates $u \in GL(n+1)$, i.e.

$$x_i \to u_{i0}x_0 + \ldots, u_{in}x_n,$$

$in(uI)$ is constant and known as generic initial ideal ($gin(I)$).

No change in the "geometry": dimension, number of solutions, degree...

"A criterion for detecting m-regularity".

D. Bayer, M. Stillman. Inventiones Mathematicae (1987)

Theorem. $reg(I)$ equals $reg(gin(I))$ and coincides with the maximal degree of an element of a minimal Gröbner basis of I, in generic coordinates and using the reverse lexicographical order.

The importance of the Castelnuovo-Mumford regularity

Linearization, Gröbner bases, resultants...

Structured polynomial systems

Solution set, minimal free resolutions, local cohomology...

The regularity might grow double exponentially in the number of variables [Mayr-Meyer'82].

(Multi)-homogeneous polynomial systems

Same motivation

Given a the monomial structure of the parametric surface $\mathcal{S} \subset \mathbb{R}^3$, construct a matrix $\mathbb{M}(X, Y, Z)$ whose entries depend on (X, Y, Z) such that for $p \in \mathbb{R}^3$

$$p \in \mathcal{S} \iff \text{corank of } \mathbb{M}(p) > 0.$$

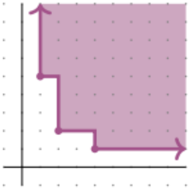
[Khetan-D'Andrea'06], [Botbol'11] - Use toric varieties.

Who takes the role of $reg(I)$?

Two groups of variables

Given a (bi-homogeneous) polynomial system $I = \langle f_1, \dots, f_r \rangle$ in $k[x_0, \dots, x_n, y_0, \dots, y_m]$.

$reg(I)$ describes (using bi-degrees) some algebro-geometric properties of I. ■



[Maclagan-Smith'04], [Botbol-Chardin'12]

Which monomial order?

Example

$$f_1 = x_1y_0 + x_0y_1 = 0 \quad f_2 = x_0y_1 - x_1y_0 = 0$$

Let $in_x(I)$ and $in_y(I)$ be the reverse lexicographical order $x_0 < x_1 < y_0 < y_1$ (resp. $y_0 < y_1 < x_0 < x_1$).

$$in_x(I) = (x_0y_0, x_0y_1, y_0^2x_1) \quad in_y(I) = (x_0y_0, x_1y_0, x_0^2y_1)$$

(Bi)-generic initial ideals

If we change the variables as before, we lose the structure.

We structure the changes of coordinates.

$x_0 \to x_0 + 12x_1 \quad x_1 \to x_0 - 2x_1 \quad y_0 \to y_0 + 3y_1 \quad y_1 \to y_0 + -5y_1$

The (bi)-generic initial $bigin(I)$ ideals exist.

In the example

$$bigin_x(I) = (x_0y_0, x_0y_1, y_0^2x_1) \quad bigin_y(I) = (x_0y_0, x_1y_0, x_0^2y_1)$$

Remark The (bi)-degrees still depend on the order of the variables.

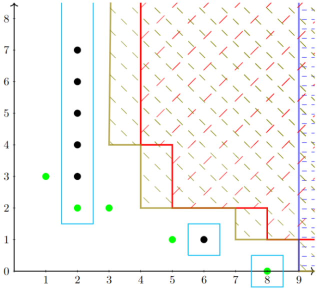
Our results

If we are using the *DRL* order:

$$x_0 < \cdots < x_n < y_0 < \cdots < y_m$$

Theorem [Bender-Busé-C-Tsigaridas]. There is $1 \leq s \leq n+1$ such that:

$$(a,b) \in reg(I) \implies \text{No generator of } bigin(I) \text{ has degree } \geq (a+s,b)$$



The regions $reg(I) + (s,0)$, in $xreg(I) + (1,0)$ and a third region in terms of the Betti numbers [Aramova-Crona-DeNegri'00,Römer'01]. We also certify where there are generators of $bigin(I)$.

Sparse polynomial systems and toric varieties.

The Newton polytopes and the Macaulay matrix.

The Newton polytopes of the polynomials of $f_1, \dots, f_r$ we consider correspond to "degrees" (nef Cartier divisors) in the homogeneous coordinate ring R of a toric variety X_Σ.

$$M_d : \oplus_{i=1}^r S_{d-d_i} \to S_d \quad (g_1, \dots, g_r) \to \sum_{i=1}^r g_i f_i$$

Big question:

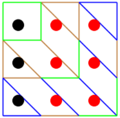
Does multigraded regularity govern these computations?

Sparse resultants

When $r = n + 1$, choose minor $\mathcal{M}, \mathcal{E}$ of M_d such that:

$$Res_{\mathcal{A}} = \frac{det(\mathcal{M})}{det(\mathcal{E})}$$

where $Res_{\mathcal{A}}$ is a polynomial in the coefficients of $f_1, \ldots, f_r$ that vanishes when there are solutions.



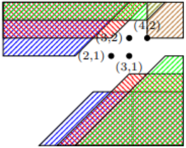
The combinatorial construction of a minor of M_d based on a mixed subdiviison of $\Delta = \sum_{i=0}^{n} \Delta_i$.

The construction [Canny-Emiris'93], the proof of the formula [D'Andrea-Jerónimo-Sombra'21], smaller matrices (under some conditions) [C-Emiris'22] (with an tropical result on mixed subdivisions).

Sylvester forms

Construction of a (universal) basis $\{s_\mu\}$ of $(I^{sat}/I)_d$ for some d.

$$M_d : \oplus_{i=1}^{r} S_{d-d_i} \oplus (I^{sat}/I)_d \to S_d \quad (g_1, \ldots, g_r, l_\mu) \to \sum_{i=1}^{r} g_i f_i + \sum_{\mu \in \mathcal{I}} s_\mu l_\mu$$



Example in the Hirzebruch surface $\mathcal{H}_1$. The dots represent the "degrees" d where a basis of $(I^{sat}/I)_d$ is given.

The classical case [Jouanolou'97], the multihomogeneous case [Busé-Chardin-Nemati'22], a smooth toric variety [Busé-C'23].

The investigation continues.

Gràcies! Gracias! Merci! Thanks! ευχαριστώ! Tak! Danke! Grazie!