Classification of Dupin cyclidic orthogonal coordinate systems

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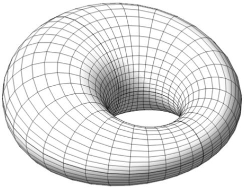
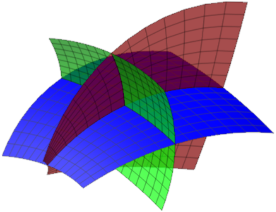
Outline

Dupin cyclidic (DC) systems

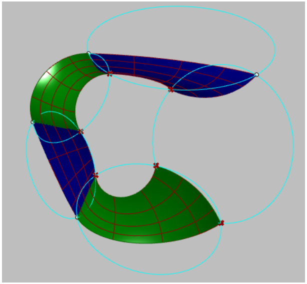
- A DC system is a 3-orthogonal coordinate system that has circles or straight lines as coordinate lines.
- The coordinate surfaces of a DC system are necessarily Dupin cyclides by Dupin's Theorem (1822): the intersection curve of two surfaces from different families must be a curvature line of both surfaces.
- DC systems are distinct from the broader systems studied by G. Darboux in the second half of 19th century.

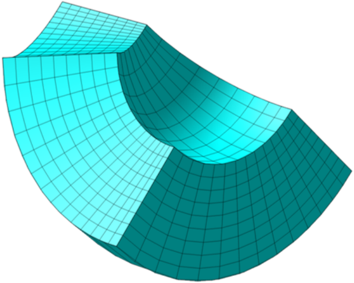




Motivation

- DC systems can be used for variable separation in the Laplace's equation [Moon and Spencer 1988], which is used in engineering. Avoiding singularity ensures the stability and validity of the solutions.
- DC cubes (cut out of DC systems) are used for modeling application as smooth-boundary joined pieces, generalizing the standard cyclidic splines based on principal patches. Singularity must be avoidable here.





Quaternions and inversions

- The Euclidean space $\mathbb{E} = \mathbb{R}^3$ is identified to the space of imaginary quaternions $\operatorname{Im}\mathbb{H} = \{q \in \mathbb{H} \mid \operatorname{Re}(q) = 0\}$,

$$\mathbb{H} = \{q = r + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mid r, x, y, z \in \mathbb{R},\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}.$$

- An inversion Inv_q^r with respect to a sphere of center q and radius r can be written explicitly as $\operatorname{Inv}_q^r(p) = q - r^2(p-q)^{-1}$, $p \in \operatorname{Im}\mathbb{H}$.
- The group generated by inversions coincides with the group of Möbius transformations (circles and angle preserving transformations) in $\mathbb{R}^3$.
- Actually, Möbius transformations are generated by Euclidean similarities and the unit inversion $p \mapsto -p^{-1}$.

Quaternionic representation

A DC system can be equivalently defined as a 3-linear rational quaternionic mapping to the compactified imaginary quaternions, i.e.,

$$\begin{aligned} F: \ & (\mathbb{R}P^1)^3 \to \mathrm{Im}\,\mathbb{H} \cup \{\infty\} \equiv \mathbb{R}^3 \cup \{\infty\} \\ & (s,t,u) \mapsto F_1(s,t,u)\mathbf{i} + F_2(s,t,u)\mathbf{j} + F_3(s,t,u)\mathbf{k}, \end{aligned}$$

such that

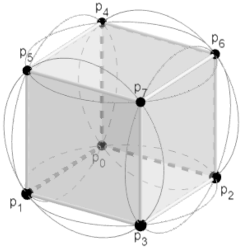
- F can be written as $F = UW^{-1}$, where $U, W \in \mathbb{H}[s,t,u]$ trilinear;
- all three partial derivatives $\partial_s F$, $\partial_t F$, $\partial_u F$ are mutually orthogonal;
- the Jacobian $\mathrm{Jac}(F) = |\partial_s F \ \ \partial_t F \ \ \partial_u F| \neq 0$ at least in one point.

Quaternionic Bézier representation

A QB form associated to a DC system $F = UW^{-1}$ is a collection of *control points* $p_0, p_1, \ldots, p_7 \in \operatorname{Im}\mathbb{H} \cup \{\infty\}$ and *quaternionic weights* $w_0, w_1, \ldots, w_7 \in \mathbb{H}$ such that

$$\begin{pmatrix} U(s,t,u) \\ W(s,t,u) \end{pmatrix} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} \begin{pmatrix} p_{ijk} w_{ijk} \\ w_{ijk} \end{pmatrix} B_i^1(s) B_j^1(t) B_k^1(u),$$

where $B_0^1(\tau) = 1 - \tau$, $B_1^1(\tau) = \tau$ are linear Bernstein polynomials and $000 = 0$, $010 = 1$, $001 = 2$, $011 = 3, \ldots$, are binary form indices.

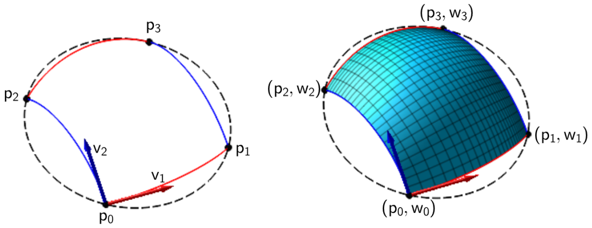


Dupin cyclide principal patches

Theorem

The DC patch defined by cocircular points p_0, p_1, p_2, p_3 and orthogonal tangent vectors v_1 and v_2 at p_0 can be parametrized using the homogeneous control points $(p_i w_i, w_i)$, $i = 0, 1, 2, 3$ such that

$$w_0 = 1,\ w_1 = (p_1 - p_0)^{-1} v_1,\ w_2 = (p_2 - p_0)^{-1} v_2,$$
$$w_3 = (p_3 - p_0)^{-1} \left((p_1 - p_0)^{-1} - (p_2 - p_0)^{-1} \right) v_1 v_2.$$



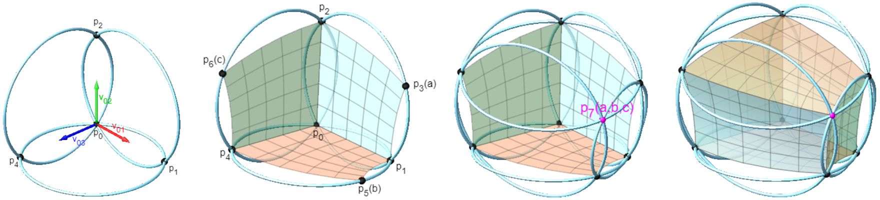
DC system construction steps

Theorem

A DC system is uniquely determined by its 3 adjacent DC patch faces intersecting orthogonally at a point.

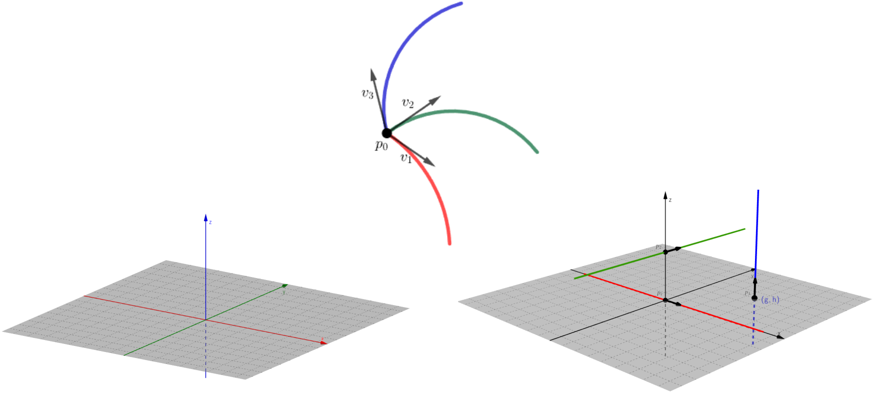
Sketch proof.

The existence of the 8th control point p_7 follows from the circularity conditions and the Miquel's Theorem. There are 3 formulas for w_7 from the 3 adjacent patches at p_7. The existence and uniqueness of w_7 making the parametrization compatible follows from reparametrization ideas. □



Canonicalization

We consider the Möbius class of the DC cube construction based on 3 circles coordinate lines meeting orthogonally at a regular point p_0 in space.



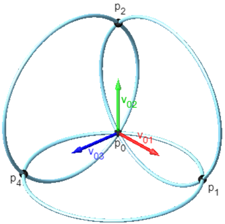
DC systems with 3 planes of symmetry

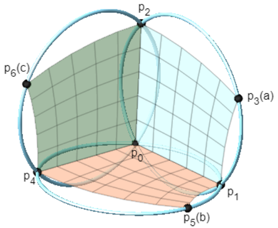
- Here we assume that the x-axis, y-axis and z-axis are already coordinate lines. We build the 3-parameter family of DC systems based on that constraint:

$$\begin{pmatrix} 0 & \mathbf{i} & \mathbf{j} & \mathbf{i}+\mathbf{j} & \mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{j}+\mathbf{k} & a+b+c+\mathbf{i}+\mathbf{j}+\mathbf{k} \\ 1 & 1 & 1 & 1-a\mathbf{k} & 1 & 1-b\mathbf{j} & 1-c\mathbf{i} & 1-c\mathbf{i}-b\mathbf{j}-a\mathbf{k} \end{pmatrix}.$$

- The corresponding numerator and denominator in the DC system are simply:

$$U(s,t,u) = (a+b+c)stu + s\mathbf{i} + t\mathbf{j} + u\mathbf{k},$$
$$W(s,t,u) = 1 - ctu\mathbf{i} - bsu\mathbf{j} - ast\mathbf{k}.$$





Detecting spheres/planes in the coordinate surfaces

- There are 3 quadratic polynomial equations $f_1(s) = 0$, $f_2(t) = 0$, $f_3(u) = 0$ that detect degeneration to spheres/planes of coordinate surfaces in all 3 directions.
- In the (a, b, c) case, we have

$$f_1(s) = -\frac{s(a+c)}{2}, \quad f_2(t) = \frac{t(a+b)}{2}, \quad f_3(u) = \frac{u(b+c)}{2}.$$

- One of the quadratic polynomials is identically zero if $(a+b)(a+c)(b+c) = 0$. Here, at least in one direction, the coordinate surfaces are composed of spheres/planes. This type of DC system is called *spherical*.
- The roots of f_i, $i = 1, 2, 3$ correspond to the xy, xz and yz-planes and those planes are planes of symmetry of the DC system.

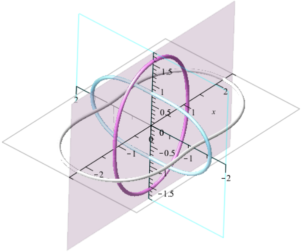
Singular locus

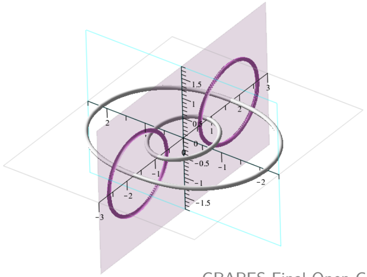
The singular locus of this DC system, where the Jacobian vanish, is arrangement of 3 bicircular quartics located on the 3 degeneration planes:

$$BQ_1 : z = abc(x^2 + y^2)^2 + (ab - cd)x^2 + (bd - ac)y^2 - d = 0,$$
$$BQ_2 : y = abc(x^2 + z^2)^2 + (ad - bc)x^2 + (ac - bd)z^2 - d = 0,$$
$$BQ_3 : x = abc(y^2 + z^2)^2 + (bc - ad)y^2 + (cd - ab)z^2 - d = 0,$$

where $d = a + b + c$.

It is interesting that each family of coordinate surfaces has a common intersection curve in one of those bicircular quartics.





Classification of DC systems

Theorem

There are four classes of DC systems:

(S) *Spherical DC system;*

(O) *Offset DC system, with two spheres and a zero sphere of symmetry;*

(A) *DC systems with three spheres of symmetry;*

(B) *DC systems with two real and one imaginary spheres of symmetry.*

Remark

- *The singular locus in the case (O) can be reduced to focal ellipse/hyperbola or two focal parabolas on orthogonal planes.*
- *The singular locus in the case (B) can be reduced to arrangement of two 2-oval bicircular quartics on orthogonal planes.*

Conclusion

- A natural generalization of the principal Dupin cyclide patch to a volume object, a Dupin cyclidic (DC) cube is rationally parametrized by the fraction of 3-linear quaternionic polynomials.
- This construction parametrizes any triply orthogonal coordinate system having coordinate lines circles or straight lines, which we call a DC system.
- The full classification of such DC systems up to Möbius transformations in space $\mathbb{R}^3$ is presented in the form of four big classes:
  - (S) spherical, with a family of coordinate surfaces, composed of spheres;
  - (O) offsets, constructed as offsets of quartic and cubic Dupin cyclides;
  - (A) systems with 3 spheres of symmetry;
  - (B) systems with 2 real spheres and one imaginary sphere of symmetry.