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THESIS

Multihomogeneous and sparse polynomial systems: resultants, Gröbner bases and regularity.

Carles Checa
PhD.20.X001

Επιβλέπων:
Ioannis Z. Emiris

Περίληψη

Η παρούσα διατριβή πραγματεύεται πολυομογενή και αραιά πολυωνυμικά συστήματα από θεωρητική και υπολογιστική άποψη. Αυτή η εργασία περιέχει δύο διαφορετικούς τρόπους συσχέτισης πινάκων με πολυωνυμικά συστήματα με τρόπο που εκμεταλλεύεται τη δομή της αραιότητας τους. Πρώτον, δίνουμε μια οικογένεια μικτών υποδιαιρέσεων που ικανοποιούν τον τύπο Canny-Emiris για τον υπολογισμό της αραιής απαλοίφουσας και παρέχει μια οικογένεια πινάκων των οποίων το μέγεθος μπορεί να μειωθεί χρησιμοποιώντας την άπληστη προσέγγιση σε αυτόν τον τύπο. Δεύτερον, επεκτείνουμε την κατασκευή των μορφών Sylvester στην περίπτωση μιας ομαλής τορικής ποικιλότητας που ικανοποιεί μια ορισμένη ιδιότητα. Τέλος, μελετάμε τη σχέση μεταξύ της πολυ-ομογενούς κανονικότητας Castelnuovo-Mumford και των βάσεων Gröbner με τρόπο που μας επιτρέπει να κατανοήσουμε ποιες είναι οι ελάχιστοι βαθμοί στις βάσεις Gröbner ενός ιδεώδους, σε γενικές συντεταγμένες χρησιμοποιώντας την αντίστροφηβαθμού λεξικογραφική διάταξη μονωνυμων. Προσθέτουμε παραδείγματα σε εφαρμογές όπως σε Υπολογιστική Οραση και Γεωμετρική Σχεδίαση, μαζί με τον κώδικα JULIA ορισμένων από τις υλοποιήσεις.

Abstract

This thesis deals with multihomogeneous and sparse polynomial systems from the theoretical and computational point of view. This work contains two different ways of associating matrices to polynomial systems in a way that exploits their sparsity structure. Firstly, we give a family of mixed subdivisions that satisfy the Canny-Emiris formula for the computation of the sparse resultant, providing a family of matrices whose size can be reduced using the greedy approach to this formula. Secondly, we extend the construction of Sylvester forms to the case of a smooth toric variety satisfying a certain property. Finally, we study the relation between the multigraded Castelnuovo-Mumford regularity and Gröbner bases in a way that allows us to understand which are the minimal bi-degrees in a Gröbner bases of an ideal, in generic coordinates using the degree reverse lexicographical monomial order. We add examples in applications such as Computer Vision and Geometric Design, together with `JULIA` code of some of the implementations.

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Y, como no puede ser de otra forma, viva el Betis manque pierda y viva Palestina libre ;)

Contents

Chapter 1

Introduction

Systems of polynomial equations appear everywhere in science and engineering. Making computations with them, finding their zeros or handling the different possible representations of their solution set is a vast area for research, which we can comprise under the name of *computational algebraic geometry* [CLO98; MS21; Sch03]. A smooth communication between the theoretical mathematics coming from commutative algebra and the knowledge of the insights of each application has been producing a vast number of advances over the last centuries. We can list some of the areas where these advances have been relevant.

Figure 1.1: Implicit surface of a map given by three bilinear forms.

- **Geometric modelling:** Transforming polynomial maps into implicit equations is a fundamental problem of computer-aided design (see Figure 1.1). This problem can be seen from the point of view of sparsity [BCD03]. One might also want to compute directly with the parametric form and study its topology [Kat+22]. On the other hand, Bezier surfaces play a central role in the manipulation of the algebraic objects and they can also be seen from the perspective of toric geometry [CC20; Kra02].

- **Biology:** In chemical reaction network analysis, parameter-dependent systems arise when modeling the steady states of dynamical systems associated with the networks [Fel20]. Studying these parameters can be helpful to determine the number of steady states, especially when deciding whether the system has a unique solution over the real positive numbers [BDG18; FHPE23; Con+17]. These systems can also exhibit some toric structure [Cra+09].

- **Kinematics:** Polynomial systems are often used to parametrize the movements of a robotic arm. In this area, the typical tasks involve eliminating the parameters [Emi+06] or seeking geometric descriptions of the solution set to avoid the singularities of these systems [Le+23]. Paradoxical behaviours may arise if the space of mobility of the arm has more dimension than one could expect from the equations [Sch21].

- **Computer vision:** Many polynomial systems arising in vision consist of matching problems between snapshots captured by cameras (see Figure 1.2). Usually, thousands of polynomial systems have to be simultaneously solved [Duf+18; Kuk13] so small differences in the computations will be helpful in the final result. As one thinks of the cameras as linear projections, interesting algebraic objects with sparse structure (Chow forms [OT19], distortion varieties [Kil+16]...) arise.

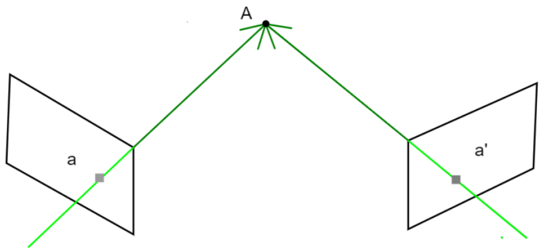


Figure 1.2: A single point A expressed from two different camera positions, with a, a' being the images. The condition of both images corresponding to A is expressed as a polynomial system.

- **Physics:** $\mathcal{A}$-hypergeometric systems arise naturally in the study of Euler integrals in particle physics. The structure of the Newton polytope that we will discuss in this thesis is closely attached to these systems and their sparse structure [MHMT23].

- **Other applications** include multivariate cryptography [CG20; CG23; FVP08], coding theory [JA11; Sop13], optimization [Lau14], topological data analysis [Sch22], game theory [PS22] or algebraic statistics [Sul18].

Finding different representations of the solution set, eliminating variables from the system or solving polynomial systems with a finite number of solutions are computationally hard problems: the complexity of the algorithms will, in general, grow exponentially in the number of variables. Therefore, if we want to find robust and efficient algorithms, we cannot just tell the computer "`solve this`" and expect it to respond in reasonable time: we ought to look at the structure of those polynomials and see if we manage to find better algorithms that take advantage of the properties of each family of systems. On the other hand, understanding the structure of the polynomials from the point of view of commutative algebra or algebraic geometry is an interesting problem on its own. In this thesis, we focus on the sparse (or toric) structure of polynomial systems and propose some novel constructions that intend to describe and exploit such structure.

Newton polytopes and projective toric varieties When we look at a polynomial, only a finite number amongst its coefficients will be nonzero. Thus, using only the monomials with nonzero coefficient provides a shortcut to understand the structure of polynomial equations. Namely, every $F \in \mathbb{C}[x_1, \ldots, x_n]$ is of the form:

$$F = \sum u_b x^b \quad b \in \mathbb{Z}^n \quad x^b = x_1^{b_1} \cdots x_n^{b_n} \quad (b_1, \ldots, b_n) \in \mathbb{Z}^n$$

with a finite number of nonzero coefficients $u_b \in \mathbb{C}$. The exponents b appearing in a polynomial F form a finite subset $\mathcal{A} \subset \mathbb{Z}^n$, which we refer as the *supports* of F. The convex hull of the set of supports in $\mathbb{R}^n$ are named the *Newton polytope* $\Delta = \text{conv}(\mathcal{A})$.

From the point of view of algebraic geometry, we would like to understand the solution set of a polynomial system from the algebra described by these polynomials. Namely, the solution set of a system given by polynomials $F_1, \dots, F_r \in \mathbb{C}[x_1, \dots, x_n]$ over a subset $K \subset \mathbb{C}$ is:

$$V_{K^n}(F_1, \dots, F_r) = \{(\overline{x}_1, \dots, \overline{x}_n) \in K^n \quad F_i(\overline{x}_1, \dots, \overline{x}_n) = 0 \text{ for all } i = 1, \dots, r\}. \tag{1.1}$$

The usual candidates for K in the above applications are the complex numbers $\mathbb{C}$, the nonzero complex numbers $\mathbb{C}^* = \mathbb{C} - \{0\}$, the real numbers $\mathbb{R}$ or the real positive numbers $\mathbb{R}_{>0}$. A possible way of introducing the connection between the structure of the Newton polytopes and the geometry of polynomial systems is stating the famous theorem of Bernstein-Khovanskii-Kushnirenko that counts the number of solutions over $\mathbb{C}^*$ in terms of the mixed volume of the polytopes [Ber75; Kus75].

Theorem. Given a polynomial system

$$F_1 = \cdots = F_n = 0 \tag{1.2}$$

of n equations and n variables, with finitely many zeros and Newton polytopes $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n$, the number of solutions of this system in $\left(\mathbb{C}^*\right)^n$ (counting multiplicity) is bounded above by the mixed volume of the polytopes,

$$\text{MV}(\Delta_1, \dots, \Delta_n)$$

which is the coefficient of $t_1 \cdots t_n$ in the polynomial $\text{Vol}(t_1\Delta_1 + \cdots + t_n\Delta_n)$. This bound is attained for a general choice of the coefficients of $F_1, \dots, F_n$.

The beauty of this theorem expresses the type of results that have motivated many lines of research in the last decades (and, actually, centuries): the geometry of the solution set is related to the properties of the Newton polytopes! Therefore, studying the relations between algebra, geometry and combinatorics underlying to a polynomial system is a very wide and interesting mathematical challenge.

The generic conditions for the number of solutions over $(\mathbb{C}^*)^n$ to be exactly the mixed volume can be found in [HS95; Roj94]. On the other hand, if the previous bound is not attained, we can see that the ”missing” solutions lie in a certain variety, which is defined from the Newton polytopes of the system. For simplicity, consider Δ to be Newton polytope of all the polynomials in the system (1.2) and let $\mathcal{A} = \Delta \cap \mathbb{Z}^n = \{m_1, \dots, m_s\}$ be the lattice points in this polytope. Then, we can look at the following map:

$$\Phi_{\mathcal{A}} : (\mathbb{C}^*)^n \to \mathbb{P}^{s-1} \quad t := (t_1, \dots, t_n) \to (t^{m_1} : \cdots : t^{m_s}) \tag{1.3}$$

and consider the variety X_Δ defined by the Zariski closure of the image of $\Phi_\mathcal{A}$. One can see that the natural action of the torus $(\mathbb{C}^*)^n$ on itself extends to an action on X_Δ, providing the name *projective toric variety* [CLS12]. The theory that motivates the use of this type of relations between polytopes (as well as other combinatorial objects) and the geometry of polynomial systems is known as *toric geometry* and goes back to the seminal works of Danilov, Demazure or Khovanskii [Dan78; Dem70; Kho77].

Revisiting the problem of the "missing" solutions of the polynomial system $F_1 = \cdots = F_n = 0$, we can find a version of the Bernstein-Khovanskii-Kushnirenko [Tel22; GKZ94; Roj96] theorem considering the solutions in X_Δ after homogenizing the system with respect to this toric variety: if the homogeneous system still has a finite number of solutions, this number is exactly the mixed volume. This homogeneization can also be described in terms of the Newton polytopes. The solutions of this new homogeneous polynomial system which do not lie in $(\mathbb{C}^*)^n$ are in the lower dimensional orbits of the action of $(\mathbb{C}^*)^n$ in X_Δ and are often refered as "solutions at infinity".

The construction above demonstrates the suitability of using toric varieties to analyze the geometry of polynomial systems. However, the paradigm of trying to exploit the combinatorial structure underlying to polynomial systems exceeds the definition that we provided. In general, any variety X at which the torus $(\mathbb{C}^*)^n$ is in correspondence with an open dense subset of X and the natural action of torus $(\mathbb{C}^*)^n$ extends to the rest of the variety is called toric (not even necessarily projective). The ideals defining these varieties are prime and binomial [ES96] and appear in many applications. The ubiquity of this type of structures allows us to claim that *the world is toric* [MS21].

The point of view of looking at the solutions set from the perspective of the Newton polytope allows us to classify polynomial systems in the following three categories (see Fig. 1.3):

- **Dense (*or homogeneous*) polynomial systems:** The simplest case of study are dense polynomial systems, in which we allow all the monomials up to a certain degree d to have a nonzero coefficient. In this case, the Newton polytopes are simplices and homogenizing means adding a new variable x_0 and multiplying every monomial by a power of x_0 until all the monomials have degree d. Most of the results presented in this thesis were already well-established for this type of systems. In this case, the underlying projective toric variety is the projective space $\mathbb{P}^n$.
- **Multihomogeneous polynomial systems:** In a slightly more general case, we can group the variables $x_1, \dots, x_n$ into r families for $r \geq 1$ and consider polynomial systems in which the degree of the monomials with respect to each group of variables does not exceed a certain tuple of degrees $(d_1, \dots, d_r)$. In this case, we homogenize by adding one variable for each group of variables and the underlying projective toric variety is the multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

This intermediate case is present in numerous applications. Some of the computational problems that we will adress have already been widely studied for the multihomogeneous case. However, some other studies, specially those dealing with the relation between the multigraded Castelnuovo-Mumford regularity and Gröbner bases, are already quite intricate in this case, and the literature is more scarce.

- **Sparse polynomial systems:** In the most general case, we only fix the Newton polytopes of the polynomial system, and thus allow any of the monomials corresponding to lattice points in the polytope to have a nonzero coefficient. If the Newton polytopes of the system are $\Delta_1, \ldots, \Delta_r$ the toric variety that we consider is associated to the polytope $\Delta = \sum_{i=1}^{r} \Delta_i$, as in (1.3). In the flavour of the Bernstein-Khovanskii-Kushnirenko theorem, the combinatorics of the polytopes takes a very interesting role in the description of the properties of these systems. For instance, the homogenization can described in terms of the Newton polytopes; see (2.21).

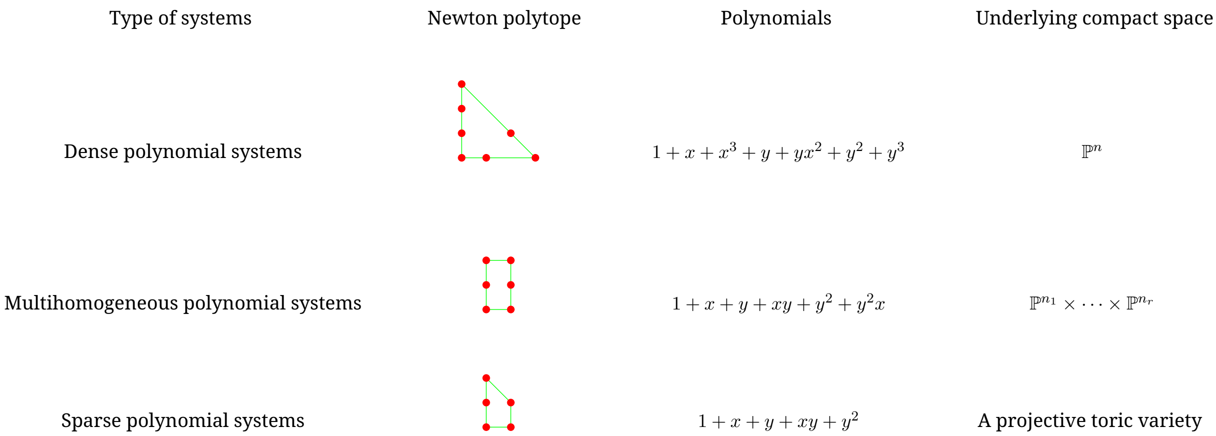


Figure 1.3: Exploiting the Newton polytope structure of polynomial systems is advantageous for computations.

Symbolic methods for computing with polynomial systems The computational tools that we consider in this thesis belong to the category of symbolic methods. These methods comprise algorithms that output the solution as the result of a series of computations that consider the variables and coefficients as symbols. On the other hand, the coefficients of polynomial systems coming from scientific applications are usually represented as floating point numbers and thus, they may come with numerical errors that have to be taken into account. The compacity of projective toric varieties is also advantageous in this case as small perturbations in the coefficients will not derive in big differences in the solution set. The details on numerical algebraic geometry exceed the scope of this thesis and the expertise of the author. For a thorough exposition on the relation of the discussed methods with numerical analysis, we refer to [SW05].

The methods that we consider are *elimination matrices and Gröbner bases*. Of course, the methods of algebraic computation are much broader and interact with each other in many ways. However, in this thesis we stick to the basic idea that having a good understanding of these two methods and how they relate to the geometry of polynomial systems will provide a general knowledge on their structure, specially in the case of systems with a finite number of solutions.

- **Elimination matrices**: A classical approach towards non-linear algebra is linearization, this is, the exploitation of linear algebra methods (gaussian elimination, eigenvalues, eigenvectors...) after transforming a problem given by a polynomial system into a linear one. A possible way to make such transformation is the following: consider a polynomial system $F_1 = \cdots = F_r = 0$ in a ring $S = \mathbb{C}[x_1, \dots, x_n]$ of degrees $d_1, \dots, d_r$ i.e.

$$F_i = \sum_{\deg(x^a) \leq d_i} u_{i,a} x^a \quad i = 1, \dots, r. \tag{1.4}$$

Consider the polynomial ideal $I \subset S$ generated by $F_1, \dots, F_r$ and the polynomial map $\mathbb{M}$ defined as:

$$\mathbb{M} : S^r \to S \quad (G_1, \dots, G_r) \to \sum_{i=0}^{r} G_i F_i. \tag{1.5}$$

This map is a very natural way to present the algebra of the polynomial system: a polynomial $G \in S$ belongs to the ideal I, if and only if, G is in the image of $\mathbb{M}$. Moreover, once we consider a degree $\nu \in \mathbb{Z}_{\geq \max d_i}$, the previous map is transformed into a map between vector spaces:

$$\mathbb{M}_\nu : \oplus_{i=1}^r S_{\leq \nu - d_i} \to S_{\leq \nu} \quad (G_1, \dots, G_r) \to \sum_{i=0}^{n} G_i F_i \tag{1.6}$$

where $S_{\leq \nu}$ is the vector space spanned by all the monomials of degree lower or equal than ν. The map $\mathbb{M}_\nu$ is a map between two vector spaces and thus, by employing the degree ν, we have turned a non-linear structure of polynomials into a linear problem. The matrix $\mathcal{M}_\nu$ associated with the map $\mathbb{M}_\nu$ in monomial bases appeared for the first time in the study of resultants by Sylvester [Syl18] and Macaulay [Mac03]. For this reason, it is often refered as the *Macaulay* (or *Sylvester-Macaulay*) matrix.

Example. Consider the following three polynomials in $\mathbb{C}[x_1, x_2]$:

$$F_0 = 1 - x_1 + x_1^2 \quad F_1 = 2 + x_1 + 2x_1x_2 + x_2^2 - 2x_1^2 \quad F_2 = 1 - x_1 + x_2$$

of degrees $2, 2$ and 1, respectively. If $\nu = 3$, the Macaulay matrix $\mathcal{M}_\nu$ is the following:

$$
\begin{array}{r@{}c}
 & \begin{array}{cccccccccc} 1 & x_1 & x_1^2 & x_1^3 & x_2 & x_1x_2 & x_1^2x_2 & x_2^2 & x_1x_2^2 & x_2^3 \end{array} \\
\begin{array}{r} F_0 \\ x_1F_0 \\ x_2F_0 \\ F_1 \\ x_1F_1 \\ x_2F_1 \\ F_2 \\ x_1F_2 \\ x_2F_2 \\ x_1^2F_2 \\ x_1x_2F_2 \\ x_2^2F_2 \end{array} &
\begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\
2 & 1 & -2 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 2 & 1 & -2 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & -2 & 1 \\
1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{bmatrix}
\end{array}.
$$

The matrix-based constructions offer the advantage of universality in coefficients: once the degrees of the polynomials are fixed (or, more broadly, the Newton polytopes), the coefficients can be specialized to any values, and the matrix construction remains the same. Consequently, extracting properties of the solution set via these matrices relies solely on identifying the degrees ν at which the geometric properties of the system can be retrieved. This versatility enables us to employ these matrices in solving various computational problems effectively.

Motivated by the applications in geometric modelling and the theory of resultants, we can consider the case where there are $n+1$ polynomials in n variables and find degrees ν at which the matrices $\mathcal{M}_\nu$ such that we recover the following two properties: $i)$ their corank is positive when we specialize to systems that have a solution (over the corresponding projective toric variety) and $ii)$ if the specialized system is 0-dimensional (i.e. has a finite number of solutions), then the corank is precisely this number of solutions, counting multiplicities [Bus06; EM99; Tel20]. If the degree is big enough $(\nu \gg 0)$, the matrix $\mathcal{M}_\nu$ always has both properties and thus, the focus usually relies in trying to find the smallest ν with those properties.

Once we have fixed ν, we can use $\mathcal{M}_\nu$ to eliminate variables from a polynomial system or find the solutions of 0-dimensional systems. To describe how to perform those operations, we can turn to conventional methods involving resultants or eigenvalue methods. We outline the description of these methods for the case of dense polynomial systems, but the tecniques we will describe also follow in the multihomogeneous and sparse cases; see [CLO98, Chapter 3, Chapter 8] for a longer and more precise description.

- *Resultant-based methods*: As we mentioned above, the Macaulay matrix appeared in the first place in the study of resultants, which considers the case of $n+1$ polynomials $F_0, \dots, F_n \in \mathbb{C}[x_1, \dots, x_n]$ of degrees $d_0, \dots, d_n$.

For general values of the coefficients, these systems will not have any solution in $\mathbb{C}$. A powerful tool for elimination is to find a polynomial in the coefficients of the system whose zeros provide precisely those systems that have solutions. This polynomial is known as the *resultant* of the system, i.e.

$$\text{Res}(F_0, \ldots, F_n) = 0 \iff F_0 = \cdots = F_n = 0 \text{ has a solution in } \mathbb{C}^n.$$

The resultant depends only on the degrees of $F_0, \ldots, F_n$, so we can denote it as $\text{Res}_{d_0,\ldots,d_n}$. These polynomials can be very useful for solving systems with a finite number of solutions. Namely, if we are given a system $F_1, \ldots, F_n$ with a finite number of solutions, we can introduce an additional linear polynomial $F = u_0 + u_1 x_1 + \cdots + u_n x_n$ and consider the resultant of $F, F_1, \ldots, F_n$. If the system $F_1 = \cdots = F_n = 0$ has no solutions at infinity, the resultant decomposes as:

$$\text{Res}_{1,d_1,\ldots,d_n}(F, F_1, \ldots, F_n) = C \prod_{\overline{x} \in V_{(\mathbb{C}^*)^n}(F_1,\ldots,F_n)} (u_0 + u_1 \overline{x}_1 + \cdots + u_n \overline{x}_n)^{\mu_{\overline{x}}}$$

where C is a nonzero constant and $\mu_{\overline{x}}$ is the algebraic multiplicity of the point $\overline{x}$ in the variety $V_{(\mathbb{C}^*)^n}(F_1, \ldots, F_n)$ [CLO98, Chapter 3].

Another possibility could be to consider $F_1, \ldots, F_n$ as polynomials in the variables $x_1, \ldots, x_{n-1}$ whose coefficients are polynomials in x_n. Then, $\text{Res}^{x_n}_{d_1,\ldots,d_n}(F_1, \ldots, F_n)$ is a polynomial in x_n whose roots are precisely the x_n-components of the roots of the system; see also [CLO98, Chapter 3]. This tecnique makes resultants very useful for eliminating variables from polynomial systems, independently of the dimension of the solution set. However, if the polynomial system $F_1 = \cdots = F_n = 0$ has components of positive dimension, the resultant will vanish identically when evaluated to the coefficients of the system (which depend on $x_1, \ldots, x_{n-1}$) difficulting the recovery of the solutions. This problem can be treated with some techniques [Can90; Pog24; Roj97], but still remains a challenge of the use of resultants.

Resultants are naturally attached to the linearization method above: if $\mathcal{M}_\nu$ is an elimination matrix, the resultant can be computed as the greatest common divisor of the nonzero maximal minors of $\mathcal{M}_\nu$ considered as polynomials in the coefficients, i.e.

$$\text{Res}_{d_0,\ldots,d_n} = \text{gcd}(\text{Nonzero maximal minors of } \mathcal{M}_\nu).$$

In the best case, the matrix $\mathcal{M}_\nu$ is already square and one has a determinantal formula $\text{Res}_{d_0,\ldots,d_n} = \det(\mathcal{M}_\nu)$. For instance, this is the case of the classical Sylvester matrix for the resultant of two polynomials in one variable [Syl18]. However, $\mathcal{M}_\nu$ is usually not square, and computing the greatest common divisor of the maximal minors of the matrix can be quite challenging. Instead, we can just note that $\text{Res}_{d_0,\ldots,d_n}$ divides the determinant of every nonzero maximal minor of $\mathcal{M}_\nu$. Therefore, a standard way to find formulas for the resultant is considering the ratio of two determinants:

$$\mathrm{Res}_{d_0,\dots,d_n} = \frac{\det(\mathcal{H})}{\det(\mathcal{E})}$$

where $\mathcal{H}$ is a maximal submatrix of $\mathcal{M}_\nu$ with nonzero determinant. The matrix $\mathcal{E}$ might correspond to a submatrix of $\mathcal{H}$ [Mac03; CE95] or to another matrix closely related to $\mathcal{M}_\nu$. For instance, it is common to consider the resultant as a determinant of a complex (such as the Koszul complex [GKZ94] or the Weyman complex [WZ92]). The computations for solving 0-dimensional systems or eliminating variables can be performed directly with the matrix $\mathcal{H}$, but one has to take into account the presence of the factor $\det(\mathcal{E})$ whose vanishing at the coefficients of the system is a potential inconvenient.

There are other formulas for computing the resultant in which the entries of the matrix can be other polynomials in the coefficients. Examples of such formulas appear in the very classical works of Bézout [Bez79], Dixon [Dix09] or Morley and Coble [MC27]. Most of these formulas follow from the basic idea of adding inertia forms [Hur13], i.e. polynomials in the saturation of the given ideal. The literature for computing these forms in different degrees includes the works of Hurwitz, Mertens, Van der Waerden and Zariski [Zar37].

All in all, the search for more compact formulas for the resultant, specially in its more sparse versions, is a very extense area of research at which the work of this thesis aims to contribute [Jou97; EM12; Ben+21; CDS97; DE03; EM99; D'A01; GKZ94; Ben+21; SZ94].

Other types of resultants also become interesting if one wants to exploit further structure of polynomial systems with the purpose of algebraic elimination. Here, we can mention the special cases of residual resultants [Bus01; EM01] and subresultants [DJ05; ADTGV09; Sza08].

- *Eigenvalue methods:* In the same line of trying to exploit the linear algebra associated to polynomial computations, one can also recover the solutions of a 0-dimensional polynomial system as the eigenvalues of a matrix. Namely, if A is the (finite) algebra $\mathbb{C}[x_1,\dots,x_n]/(F_1,\dots,F_n)$, then one can consider a polynomial $F \in \mathbb{C}[x_1,\dots,x_n]$ and the multiplication map:

  $$m_F : A \xrightarrow{\cdot F} A.$$

  A classical result, firstly due to Stickelberger [Cox21] but developed in the modern language by Auzinger and Stetter [AS88], states that the eigenvalues of the map m_F correspond to the evaluations of F at the solutions of the polynomial system $F_1 = \cdots = F_n = 0$. Under good numerical conditions, this result can be very advantageous from the point of view fast computations.

In seeking ways to construct the maps m_F, one method involves finding a basis for A and subsequently multiplying this basis by F. However, the challenge arises when determining the representation of this multiplication result in the basis of A. To streamline this process, a more efficient strategy is to consider the maximal minors of the elimination matrices $\mathcal{H}$ associated with the system $F_1, \ldots, F_n, F$ and considering the *Schur complement* construction. This means writing the matrix $\mathcal{H}$ as:

$$\mathcal{H} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

where last rows and columns associated to the polynomial F. Then, the multiplication map can be constructed as:

$$m_F = M_{11} - M_{12} M_{22}^{-1} M_{21}.$$

Another similar construction from exploiting the cokernel of the matrix $\mathcal{M}_\nu$. Namely, if $\mathcal{M}_\nu$ is an elimination matrix, a cokernel matrix $\mathcal{N}_\nu$ has rank equal to the number of solutions. Therefore, the multiplication maps m_F can be recovered as nonzero maximal submatrices of

$$\mathcal{N}_\nu (\mathcal{M}_\nu^F)^T$$

where $\mathcal{M}_\nu^F$ (at degree ν) is the Macaulay matrix formed solely by the polynomial F [Ben22; BT21].

- **Gröbner bases**: Finding generators of an ideal I with good properties is an intrinsic problem to multivariate non-linear algebra. A classical way to introduce the need for good basis of the ideal I is deciding whether a homogeneous polynomial $F \in \mathbb{C}[x_1, \ldots, x_n]$ belongs to the ideal generated by $F_1, \ldots, F_r \in \mathbb{C}[x_1, \ldots, x_n]$.

  In the univariate case, this can be done through the divison algorithm and noting that for any polynomial $G \in \mathbb{C}[x]$, any polynomial F with $\deg(F) \geq \deg(G)$ can be written as:

  $$F = GQ + \overline{R}$$

  where $\deg(\overline{R}) < \deg(G)$. In this case, R is zero, if and only if, F belongs to the ideal generated by G.

  However, in the multivariate case, we could find ways to write the polynomial F as:

  $$F = \sum_{i=1}^{r} G_i F_i + \overline{R}$$

  where $\overline{R} \neq 0$ but F belongs to the ideal generated by $F_1, \ldots, F_r$. Imposing degree conditions in $\overline{R}$ is usually not sufficient for using $\overline{R}$ to decide whether F belongs to the ideal generated by $F_1, \ldots, F_r$ [CLO98, Chapter 1]. As we mentioned in the case of matrices, the problem that we are treating can be decided by checking whether the polynomial F belongs to the image of the Macaulay map.

Example. Consider the ideal generated by the following homogeneous polynomials in $\mathbb{C}[x_1, x_2]$:

$$F_1 = 1 - x_1 + x_2^2 + x_1^2 \quad F_2 = 1 - x_1 + x_2$$

which geometrically consists of two points in $\mathbb{C}^2$. In some applications, it is relevant to check whether a polynomial F (or some power of it) belongs to the ideal generated by F_1 and F_2, as it allows to decide whether these polynomials vanish at these two points.

As an example, we consider $F = 1 + x_1^2 + x_2^2$ and $\nu = 2$, we can consider Macaulay matrix $\mathcal{M}_\nu$:

$$\begin{array}{c} \\ F_1 \\ x_1F_2 \\ x_2F_2 \\ F_2 \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} x_1^2 & x_1x_2 & x_2^2 & x_1 & x_2 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

If F is a combination of the polynomials F_1 and F_2, then the last row of the matrix (in blue) is a combination of the rest of rows and thus, the above matrix cannot be of full rank. An effective way to check the rank of the matrix is to perform Gaussian elimination and transform the matrix to a row echelon form. If we consider the row echelon form of the first rows of the matrix in the previous example (associated to F_1, F_2) , we get:

$$\begin{array}{r} \\ F_1 \\ x_1F_2 + F_1 \\ (x_1 + x_2)F_2 + F_2 \\ F_2 \\ F - F_1 - F_2 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1^2 & x_1x_2 & x_2^2 & x_1 & x_2 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array}.$$

With this form, we can write the row associated to F so that it has zeros in the rows of the pivots associated to the Gaussian elimination (in purple). This implies that F can be written as:

$$F = F_1 + F_2 + R \text{ where } R = x_2 - 1.$$

The condition of imposing that none of the monomials associated to the terms of R divides the monomials associated to the pivots, is the key for guaranteeing that $R \neq 0$ implies that F is not a combination of F_1 and F_2. However, writing the matrix in the previous form depends on the order of the monomials corresponding to the columns.

This order must come from a total order in the monomials. If the monomial order is multiplicative (i.e. $x^\alpha < x^\beta$, implies that $x^\alpha x^\gamma < x^\beta x^\gamma$ for every triple of monomials $x^\alpha, x^\beta, x^\gamma \in S$), then the pivots of the gaussian elimination in the matrices $\mathcal{M}_\nu$ for all $\nu \in \mathbb{Z}$ form a monomial ideal in S.

By the Hilbert basis theorem [CLO98, Chapter 1], this is a finitely generated ideal generated by all the initial terms in the ideal I. Note that once we fixed the order, the process of choosing the pivots of the gaussian elimination consists on being able to choose the *initial terms* (or *leading terms*) for each polynomial F, i.e. finding the highest term of F with respect to the monomial order, denoted as in(F). Thus, the previous ideal corresponds to

$$\text{in}(I) = (\text{in}(F) \quad F \in I)$$

which is known as the *initial ideal.*

Definition. A *Gröbner basis* [Buc65] of I is a set of generators of I such that their initial terms generate the initial ideal in(I). The Gröbner basis is called minimal if the set of initial terms is a minimal set of generators of I.

Employing Gröbner bases is very advantageous in many senses: many of the algebraic operations that one can try to perform with I can be performed first in in(I), and then lifted to I; see [Sch80] for the case of computing syzygies (2.2) of I using the initial ideal.

In the best case, the computations are independent of which monomial order we are choosing, getting *universal* Gröbner bases. However, this is not the general case and choosing of a good monomial order can be important. The following two monomial orders are specially important when using Gröbner bases in algebraic elimination.

Definition. The *degree lexicographical* monomial order takes two different monomials $x^b, x^{b'} \in \mathbb{C}[x_1, \ldots, x_n]$ where $b, b' \in \mathbb{Z}^n$ and orders $x^b <_{\text{lex}} x^{b'}$, if and only if,

$\deg(x^b) < \deg(x^{b'})$ or $\deg(x^b) = \deg(x^{b'})$ and the first entry of $b' - b$ is positive.

The *degree reverse lexicographical* considers $x^b <_{\text{DRL}} x^{b'}$, if and only if,

$\deg(x^b) < \deg(x^{b'})$ or $\deg(x^b) = \deg(x^{b'})$ and the last entry of $b' - b$ is negative.

In the case of the degree lexicographical monomial monomial order eliminate variables in an easy way. Usually, if one wants to eliminate the variables $x_1, \ldots, x_l$ the ideal to consider (*elimination ideal*) is:

$$I_l = I \cap \mathbb{C}[x_{l+1}, \ldots, x_n].$$

Then, if G_{lex} is a degree lexicographical Gröbner basis of I, the set $G_l = G_{\text{lex}} \cap \mathbb{C}[x_{l+1}, \ldots, x_n]$ is a Gröbner basis of I_l. With this, one can also solve polynomial systems with a finite number of solutions by successively finding the zeros of univariate polynomials generating each of the elimination ideals.

Regarding the methods to compute Gröbner bases, the *Buchberger algorithm* [Buc65] is the most used method to find them. This algorithm consists on choosing two generators $F, G \in I$ and consider the least common multiple of their initial forms. With this, one can consider the *S-polynomial*:

$$S(F,G) = \frac{\mathrm{lcm}(\mathrm{in}(F), \mathrm{in}(G))}{\mathrm{in}(G)} G - \frac{\mathrm{lcm}(\mathrm{in}(F), \mathrm{in}(G))}{\mathrm{in}(F)} F.$$

Once this is done, one can consider the residue of dividing the S-polynomial in the given set of generators and add this residue as a new generator. A well-known result in commutative algebra (Hilbert's syzygy theorem) implies that the algorithm consisting of repeatedly adding these elements terminantes and provides a Gröbner basis.

The introduction of Gröbner bases through gaussian elimination in Macaulay matrices is not arbitrary. Some variants of the Buchberger's algorithm [Fau99; Fau02] try to reduce the computational workload through using the Macaulay matrices with the minimal possible unnecessary calculations of S-pairs. Finding the best strategy for choosing the S-pairs is a problem which has aroused a large number of studies [LL91; Gio+91; Tra96].

Furthermore, the elimination algorithm discussed in the preceding sections relies on employing the lexicographical order. However, employing other orders may lead to shorter computation times. Therefore, another interesting problem is to find algorithms for transforming a Gröbner basis with respect to any monomial order into a lexicographical one; see [Fau+93].

The methods of elimination that we described above are rather standard. There is a wide literature on how to exploit the multihomogeneous or sparse structure for polynomial systems both for the case of elimination matrices [Emi14; Stu94; BT21; DJS22] and for Gröbner bases [BFT18; FSS14; FM17]. In this thesis, we will focus in the measure of the degrees involved in the Gröbner bases computations as measure of the complexity of using these methods, specially in the case of multihomoeneous and sparse polynomial systems. For the case of matrices, the main focus of this thesis is to find ν (corresponding to a multi-degree or a Newton polytope) such that $\mathcal{M}_\nu$ is an elimination matrix of the smallest possible size.

For the above description, we choose monomial bases for the vector spaces in (1.5) as it is coherent with the idea of exploiting the the monomial structure given by the Newton polytopes. However, it is not difficult to find computational problems for which this paradigm is not sufficient. For instance, in computer-aided design, the Bernstein basis [MRR05] is commonly employed. Additionally, various other bases have been proposed for different purposes [BPT23; MT14; MTVB18] for other notable cases. The use of methods of computational algebra that do not rely Gröbner bases and depend on other combinatorial constructions has been a topic of discussion [CLM22] in the last years.

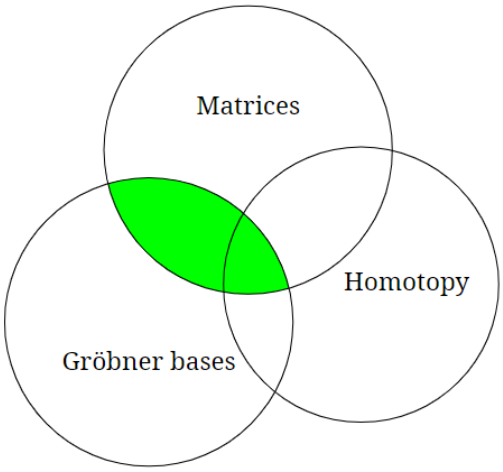


Figure 1.4: Some methods of computation. This thesis deals with elimination matrices and Gröbner bases.

A third family of methods (which we will not discuss in this thesis) related to polynomial computations consists on deforming an initial system with known solutions to the given target system [VVC94; MS87; HS95], providing the *homotopy continuation method*. These methods have many advantages and also exploit the underlying algebraic structure of the solutions; see [Duf+18; Duf+21; BT18]. All in all, we can situate the studies of this thesis in the green region of Figure 1.4.

In the above descriptions, we assumed that the field over which we aim to find the solutions is $\mathbb{C}$, which algebraically closed. However, many of the problems appearing in applications require finding solutions in the field of real numbers $\mathbb{R}$ or other fields of positive characteristic. However, another big family of methods applied to real algebraic geometry also exploit the representations of matrices and Gröbner bases that we will discuss; see [BPR06] for more details on real algebraic geometry.

The Castelnuovo-Mumford regularity and the complexity of computing with Gröbner bases. As we discussed in the description of algebraic elimination methods, the degree involved in computations serves as a important measure of their complexity. Adopting this perspective offers the advantage of exploiting the algebraic structure of the polynomial system, providing an effective communication between computational and commutative algebra. In the case of dense polynomial systems., the Castelnuovo-Mumford regularity [MB66] arises as a bound for the degrees implicated in these computations.

Definition. Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring, let $I \subset S$ be an ideal and let $m \in \mathbb{Z}$. The ideal I is called m-regular if the degrees of its j-th syzygies (Betti numbers) are bounded above by $m+j$. The Castelnuovo-Mumford regularity reg(I) is the minimal m such that I is m-regular; see Chapter 2 for a review on commutative algebra.

From the point of view of commutative algebra, this invariant also reflects many interesting properties of I, through its relation with local cohomology modules [EG84]. As it bounds the degrees of the syzygies, the regularity of in(I) bounds the degrees in a Gröbner basis of I, i.e.

$$\max\{\text{degrees of the minimal generators of } \operatorname{in}(I)\} \leq \operatorname{reg}(\operatorname{in}(I)).$$

Moreover, a relation between the computations made with the initial ideal and the algebraic structure of I arises by noticing that

$$\operatorname{reg}(I) \leq \operatorname{reg}(\operatorname{in}(I)).$$

Due to the difficulty of determining, in general, the degrees at which the Gröbner basis is generated, Bayer and Stillman [BS87a] provided a criterion to determine when computations at degrees higher than reg(I) are redundant, or dependent on the monomial order. Their criterion enlightened the following two ideas:

- Choosing the *degree reverse lexicographical monomial* order [Eis95, † 15.2] is advisable to minimize the size of the degrees involved in the computations. This idea had already appeared in the work of Lazard [Laz83] or Trinks [Tri78].
- Considering a generic linear change of coordinates in the ideal is also recommended. In other words, one should compute with the *generic initial ideal* gin(I) as defined by Galligo in [Gal74], instead of in(I).

Most computer algebra systems incorporate these two concepts. The algebraic explanation for the answer of Bayer and Stillman follows from the fact that, under these two assumptions, the complexity of computing Gröbner bases relies solely on the Castelnuovo-Mumford regularity of I, and not of its initial ideal. Namely, under the assumption of using the degree reverse lexicographical monomial order, they proved the equalities

$$\max\{\text{degrees of the minimal generators of } \operatorname{gin}(I)\} = \operatorname{reg}(\operatorname{gin}(I)) = \operatorname{reg}(I). \quad (1.7)$$

As a consequence, the Castelnuovo-Mumford regularity describes tightly the complexity of computing with Gröbner bases.

Unfortunately, the bounds for the regularity appearing in the works of Giusti [Giu84] and Galligo [Gal79] are doubly exponential.

Theorem. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal, which is generated by polynomials of degree $\leq d$.

$$\operatorname{reg}(I) \leq (2d)^{2^{n-2}}$$

These bounds cannot be imporoved, as shown by a famous example due to Mayr and Meyer [MM82]. For this reason, performing computations by Gröbner-based methods is discarded by many different problems of non-linear algebra. However, under general assumptions in the polynomials, the bounds for the regularity can be a better than this case. For instance, if one takes generic polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$, they form a regular sequence [Par10]. Namely, for each $r' \in \{2, \ldots, r\}$, $f_{r'}$ is a nonzero divisor in $\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_{r'-1})$. For these sequences, the minimal free resolution is given by the Koszul complex providing the following value for the Castelnuovo-Mumford regularity.

Theorem. (Macaulay bound) If $I = (f_1, \ldots, f_r)$ is a regular sequence, then

$$\text{reg}(I) = d_1 + \cdots + d_r - r + 1.$$

This bound also explains the relation between the Castelnuovo-Mumford regularity and elimination matrices. For $r = n+1$, $\nu = d_0 + \cdots + d_n - n$ is the first degree at which $\mathcal{M}_\nu$ is an elimination matrix. Moreover, this is precisely the degree at which the classical resultant formula of Macaulay [Mac03] can be built.

In terms of solving 0-dimensional polynomial systems through using the cokernel for extrating the solutions as eigenvalues of a matrix, it is also possible to show that the first degree at which one can do that is the Castelnuovo-Mumford regularity [TMVB17]. Overall, the Castelnuovo-Mumford regularity, extensively studied in commutative algebra, governs the complexity of computations using the methods outlined above in the dense case. A key objective of the research described in this thesis is to establish invariants that clarify and characterize the complexity of computations in the multihomogeneous and sparse cases.

Contributions The contributions of this thesis are based in the three following papers, which will appear in journals and have been presented in international conferences.

- **The Canny-Emiris formula** [CE22; CE23]: The choice of the minor of $\mathcal{M}_\nu$ providing the resultant formulas in the sparse case comes from a combinatorial rule given by Canny and Emiris [CE93] which resembles the classical formula of Macaulay for the dense multivariate resultant [Mac03]. In this formula, the rows of the minor correspond to lattice points in a translation of the polytope $\Delta = \sum_{i=0}^{n} \Delta_i$

  $$\mathcal{B} = \mathbb{Z}^n \cap (\Delta + \delta) \tag{1.8}$$

  where δ is a generic translation vector. Providing a mixed subdivsion on Δ corresponds to matching the rows and some of the columns of $\mathcal{M}_\nu$ and giving a maximal minor of this matrix. The mixed subdivision also indicates the rows and columns that form the matrix appearing in the denominator of the formula.

A proof of this formula was given by D'Andrea, Sombra and Jerónimo [DJS22] under certain hypotheses. In [CE22], we gave a family of subdivisions satisfying these hypotheses. Moreover, different possible algorithms for dealing with the lattice points may provide smaller matrices.

We considered using the greedy algorithm proposed by Canny and Pedersen [CP93] for the previous family of subdivisions under suitable hypotheses on the Newton polytopes (n-zonotopes and multihomogeneous systems) and characterized combinatorially the lattice points of $\mathcal{B}$ labeling the rows and columns of these matrices. We also provide a `JULIA` implementation of the Canny-Emiris formula based in the above combinatorial characterization.

- **Toric sylvester forms** [BC22]: For dense polynomial systems, it is possible to reduce the size of the elimination matrices $\mathcal{M}_\nu$ to the cost of introducing forms in the saturation $I^{\mathrm{sat}} = (I : \mathfrak{m}^\infty)$, where I is the ideal generated by the homogeneous polynomials and $\mathfrak{m}$ is the irrelevant ideal of $\mathbb{P}^n$ [Jou97]. The construction of these forms consists on noticing that under suitable hypotheses on ν, the module $(I^{\mathrm{sat}}/I)_\nu$ is free and explicitly finding a basis in terms of some elements of I^{sat} known as Sylvester forms. With these, we can transform the matrices of (1.5) to:

$$\mathbb{M}_\nu : \left(\oplus_{i=0}^n S(-\alpha_i) \oplus S^{\mathcal{I}} \to S \right)_\nu \quad (G_0, \dots, G_n, l_\mu) \to \sum_{i=0}^n G_i F_i + \sum_{\mu \in \mathcal{I}} l_\mu \,\mathrm{sylv}_\mu \tag{1.9}$$

  for some set of indices $\mathcal{I}$ labeling the basis of $(I^{\mathrm{sat}}/I)_\nu$. This construction was extended first to the multiprojective case in [BCN22] and we reproduced it for any smooth projective toric variety satisfying a certain hypothesis [BC22]. As a consequence of this construction, we also found novel formulas for computing sparse resultants and toric residues.

- **Multigraded Castelnuovo-Mumford regularity and Gröbner bases** [Ben+24]: From the perspective of commutative algebra, it is rather well-established that the generalization of the Castelnuovo-Mumford regularity that preserves many of the good geometric properties of the dense case was provided by Maclagan and Smith [MS04]. An important part of my research was devoted to studying the relation between Gröbner bases and multigraded Castelnuovo-Mumford regularity in the multihomogeneous case, i.e. $X_\Sigma = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. In this case, the generic change of coordinates must preserve the grading, proving the multi-generic initial ideal. Unlike in the classical case, the relative order of the variables of different degree plays a very relevant role.

  In this context, we established bounds on the degrees involving the degree reverse lexicographical Gröbner basis for an multihomogeneous ideal and compared our results with other type of descriptions appearing in the literature [ACDN00; Röm01]. In our work, the central object in the relation between regularity and initial ideals is provided by a partial regularity region described in terms of local cohomology.

The results that appear in the rest of chapters comprise most of the results and ideas appearing in [CE22; CE23; BC22]. However, there are a few results which are not contained in any of these papers. These are Theorem 2.12, which presents a slight modification of the existence of the bigeneric ideal; Corollary 4.1, which gives a family of toric varieties which satisfy the σ-positive property and Theorem 5.10, regarding the bound on the cohomological dimension. Moreover, in the section of computations, the package of Sylvester forms had not appeared in any of the articles. The manuscript is structured as follows:

- In Chapter 2, a list of preliminaries from commutative algebra, toric geometry, sparse resultants and multigraded regularity are presented.
- In Chapter 3, we present the results regarding the Canny-Emiris formula, the greedy algorithm and the study of the case of n-zonotopes and multihomogeneous systems. We also added a chapter on how to refine mixed subdivisions using tropical geometry.
- In Chapter 4, we have listed all the results regarding toric Sylvester forms, hybrid elimination matrices and the applications in the computation of sparse resultants and toric residues.
- In Chapter 5, we listed the results regarding multigraded Castelnuovo-Mumford regularity and its relation with Gröbner bases, specifically with multigeneric initial ideals. In this chapter, we give a special focus to the partial regularity region and its properties.
- In Chapter 6, we aplpy some of our constructions to problems in geometric modelling and computer vision, such as finding the implicit equation of a rational surface or the 5-point problem. Moreover, we describe some of the `JULIA` code that we developed as implementation of the results in the chapters 3 and 4.
- In Chapter 7, we list a long series of open problems or interesting questions that were discovered during the research.

Chapter 2

Preliminaries

In this chapter, we review all the results that are required for the presentation of our contributions. Some of the results of this section are quite standard in commutative algebra and algebraic geometry and can be found in the books of Eisenbud [Eis95] and Hartshorne [Har77], to which every student in these areas is familiar. On the other hand, some knowledge on toric geometry is also required for the statements and proofs in the next sections. Our main reference for that topic is the book of Cox, Little and Schenck [CLS12]. In the introduction, we already provided a quick review on Gröbner bases, resultants and regularity. However, in this section we also provide the concrete definitions of the sparse resultants [CLO98] and generic initial ideals [Gre98], which are required in the contributions. The results of Bayer and Stillman [BS87a] which we aimed to generalize to the multigraded setting are also reviewed. In the last section, we also discuss the topic of the multigraded Castelnuovo-Mumford regularity, which is very relevant to the results in Chapter 5.

1. Aspects of commutative algebra and algebraic geometry

In this first preliminaries section, we review aspects of commutative algebra and algebraic geometry that are relevant to the further developments of the thesis.

Polynomial ideals and affine varieties. Let $\mathbf{k}$ be a field and let $R = \mathbf{k}[x_1, \dots, x_n]$ be a polynomial ring.

Definition 2.1. A *(left) R-module* M is an abelian group $(M, +)$ with an action $(\cdot) : R \times M \to M$ satisfying that for every $x, y \in R$ and $m, n \in M$:

- $x \cdot (f + g) = x \cdot f + x \cdot g$
- $(x + y) \cdot f = x \cdot f + y \cdot f$

- $x \cdot (y \cdot f) = xy \cdot f$
- $1 \cdot f = f$

A *(left) R-submodule* $N \subset M$ is a subgroup of M which is closed under the action of R. A *morphism* of R-modules is a map $\delta : M \to N$ preserving all the above properties. An R-module M is *free* of dimension n if it is isomorphic to R^n.

In particular, when encoding the structure of polynomial systems, we are interested in a precise type of modules called ideals.

Definition 2.2. A *polynomial ideal* $I \subset R$ is a submodule of R, considered as an R-module with the multiplication as an action over itself.

The Hilbert basis Theorem [Eis95, Chapter 1, Theorem 1.2] implies that polynomial rings over a field are noetherian, i.e. every ascending chain of ideals stabilizes. As a consequence, every polynomial ideal $I \subset R$ has a finite number of generators. The following constructions are quite standard when one works with polynomial ideals.

Definition 2.3. Let $I, J \subset R$ be two polynomial ideals.

- The *sum* $I + J$ is the ideal generated by the sums $f + g$ where $f \in I$ and $g \in J$.
- The *intersection* $I \cap J$ is the ideal generated by the polynomials that belong to I and J.
- The *product* IJ of two ideals is the ideal generated by fg where $f \in I$ and $g \in J$.
- For $k \in \mathbb{Z}_{\geq 1}$, the *power ideal* I^k is recursively defined as $I^k = (I^{k-1})I$, where $I^0 = R$.
- The *radical* $\sqrt{I}$ the ideal generated by the polynomials $f \in R$ such that $f^k \in I$ for some $k \in \mathbb{Z}_{\geq 1}$.
- The *colon ideal* $I : J$ is the ideal generated by the polynomials $f \in R$ such that $f \cdot J \subset I$.
- The *saturation* of I with respect to I, $I : J^\infty$, is the ideal generated by the polynomials $f \in R$ such that $f \cdot J^k \subset I$ for some $k \in \mathbb{Z}_{\geq 1}$.
- An ideal I is *prime* if for every pair of polynomials $f, g \in R$ such that $fg \in I$, then either $f \in I$ or $g \in I$.
- An ideal I is *primary* if for every pair of polynomials $f, g \in R$ such that $fg \in I$, then either $f \in I$ or $g^k \in I$ for some $k \in \mathbb{Z}_{\geq 1}$.

Theorem 2.1. [Eis95, Theorem 3.10] Let $I \subset R$ be a polynomial ideal. Then, there are primary ideals $Q_1, \ldots, Q_r \subset R$ such that:

$$I = Q_1 \cap \cdots \cap Q_r$$

such that for every $i \in \{1, \ldots, r\}$, $Q_i \not\supset \cap_{j \neq i} Q_j$ and for every $i, j \in \{1, \ldots, r\}$ such that $i \neq j$, $\sqrt{Q_i} \neq \sqrt{Q_j}$. This decomposition is not unique but the prime ideals $\sqrt{Q_1}, \ldots, \sqrt{Q_r}$ are always the same.

Definition 2.4. The set $Q_1, \ldots, Q_r$ is a minimal *primary decomposition* of I. The prime ideals $\sqrt{Q_i}$ for $i \in \{1, \ldots, r\}$ are called the *associated primes* of I. The minimal ideals in $\sqrt{Q_i}$ with respect to inclusion are called the *minimal primes* of I and the others are called the *embedded primes* of I.

Let $\mathbb{A}^n_{\mathbf{k}}$ be the affine space of dimension n over $\mathbf{k}$. We can consider the polynomials $f \in R$ as functions $f : \mathbb{A}^n_{\mathbf{k}} \to \mathbf{k}$.

Definition 2.5. Let $I \subset R$ be an ideal. The *affine variety* $\mathbb{V}(I)$ is the subset of $\mathbb{A}^n_{\mathbf{k}}$ defined as:

$$\mathbb{V}(I) = \{p \in \mathbb{A}^n_{\mathbf{k}} \quad f(p) = 0 \quad \forall f \in I\}.$$

The sets of the form $\mathbb{V}(I)$ are closed under intersections and finite unions. Therefore, they constitute the closed subsets of a topology in $\mathbb{A}^n_{\mathbf{k}}$, which is known as the *Zariski* topology.

Definition 2.6. Let $V \subset \mathbb{A}^n_{\mathbf{k}}$ be a a subset. The *ideal of* V is the ideal of polynomials that vanish in all the points of V, i.e.

$$\mathbb{I}(V) = \{f \in R \quad f(p) = 0 \quad \forall p \in V\}.$$

The *Zariski closure* of an affine subset $V \subset \mathbb{A}^n_{\mathbf{k}}$ is the smallest closed subset in the Zariski topology containing V, i.e. $\overline{V} := \mathbb{V}(\mathbb{I}(V))$.

Theorem 2.2. [Eis95, Theorem 1.6] (Hilbert's Nullstellensatz) Given an ideal $I \subset R$,

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

In other words, if a polynomial $f \in R$ vanishes at all the points of the variety $\mathbb{V}(I)$, then there is $k \in \mathbb{Z}_{\geq 1}$ such that $f^k \in I$. In particular, if $\mathbb{V}(I) = \emptyset$ then $1 \in I$.

All in all, there is a correspondence between affine varieties and ideals which can be summarized in the following features. Let $I, J \subset R$ be ideals and V, W be affine varieties.

- Radical ideals are in correspondence with affine varieties.
- Inclusion of ideals $I \subset J$ corresponds to the reverse inclusion between varieties $\mathbb{V}(I) \supset \mathbb{V}(J)$, and viceversa.

- The sum of two ideals $I+J$ is sent to intersection of varieties $\mathbb{V}(I)\cap\mathbb{V}(J)$. The intersection of two varieties $V\cap W$ is sent to the radical of the sum $\sqrt{\mathbb{I}(V)+\mathbb{I}(W)}$.

- The product of ideals IJ corrersponds to the union of affine varieties $\mathbb{V}(I)\cup\mathbb{V}(J)$. Conversely, $V\cup W$ corresponds to the intersection of ideals $\mathbb{I}(V)\cap\mathbb{I}(W)$.

- Considering the colon ideal $I:J$ corresponds to considering the Zariski closure of the set-theoretical difference of varieties $\overline{\mathbb{V}(I)-\mathbb{V}(J)}$. Conversely, the Zariski closure of the set-theoretical difference of two affine varieties $\overline{V-W}$ corrresponds to the ideal $\mathbb{I}(V):\mathbb{I}(W)$.

- If an ideal is prime, then $\mathbb{V}(I)$ is irreducible, i.e there is no pair of affine varieties V_1, V_2 such that $\mathbb{V}(I) = V_1\cup V_2$ and $\mathbb{V}(I)\neq V_1, V_2$. Conversely, if V is irreducible, then $\mathbb{I}(V)$ is prime.

Definition 2.7. Let $I\subset R$ be an ideal. The *quotient ring* R/I is the ring defined by the classes of equivalence of polynomials through the relation $f\simeq_I g$, if and only if, $f-g\in I$.

The same correspondence between ideals and affine varieties as above can be described by considering the set of prime ideals of the ring R/I (Spec(R/I)), which is known as the *affine scheme* of R/I. The idea of dimension, to which we might have a geometric intuition, can also be described in terms of this correspondence between varieties and ideals.

Definition 2.8. The dimension of a variety V is the supremum of all integers n such that there is a chain of distinct irreducibe subvarieties $W_0\subset\cdots\subset W_n = V$.

Definition 2.9. The *height* of a prime ideal P is the supremum of all integers n such that there is a chain of distinct prime ideals $P_0\subset\cdots\subset P_n = P$. The *Krull dimension* of a ring R is the maximum of all the heights of its prime ideals $P\subset R$. The Krull dimension of the ring R/I coincides with the dimension of the variety $\mathbb{V}(I)$.

We end this section with the famous theorem of Noether normalization [Eis95, Section 8.2.1] which describes an interesting way to look at the dimension of an ideal.

Theorem 2.3. A set of elements $y_1,\ldots,y_m\in R$ is algebraically independent elements if there is no nonzero polynomial $P\in\mathbf{k}[y_1,\ldots,y_m]$ such that $P(y_1,\ldots,y_m)=0$. Let d be the dimension of R/I. For every chain of prime ideals $P_0\subset\cdots\subset P_d$ of maximal height in R/I, there are algebraically independent elements $y_1,\ldots,y_d\in R$ such that R/I such that R/I is a finite module over $\mathbf{k}[y_1,\ldots,y_d]$ and $P_i\cap\mathbf{k}[y_1,\ldots,y_d] = (y_1,\ldots,y_i)$ for $i=0,\ldots,d$.

In the case that R/I is 0-dimensional, this theorem implies that R/I is $\mathbf{k}$-vector space. The dimension of the vector space corresponds (up to multiplicity) to the number of points in $\mathbb{V}(I)$ [Eis95, Corollary 2.15].

Graded ideals and projective varieties. In the introduction, we motivated the need of working with homogeneous polynomial systems and projective varieties. Next, we provide the basic notions of graded modules and homogeneous polynomial ideals.

Definition 2.10. A ring R is $\mathbb{Z}^r$-graded if it can be written as a direct sum $R = \oplus_{d \in \mathbb{Z}^r} R_d$ where R_d is a finite $\mathbf{k}$-vector space, such that $R_0 = \mathbf{k}$ and $R_d R_{d'} \subset R_{d+d'}$. An R-module M is $\mathbb{Z}^r$-graded if it can be decomposed as as $M = \oplus_{d \in \mathbb{Z}^r} M_d$ where M_d is a $\mathbf{k}$-vector space, such that $R_d M_{d'} \subset M_{d+d'}$. For any $\mathbb{Z}^r$-graded module and $d, d' \in \mathbb{Z}^r$, the *twisted module* $M(-d)$ whose graded piece of degree d' is $M_{d'-d}$.

If M, N be two $\mathbb{Z}^r$ graded modules, a homomorphism of modules $\delta : M \to N$ is $\mathbb{Z}^n$-graded of degree $d \in \mathbb{Z}^r$ if for every $d' \in \mathbb{Z}^r$, $\delta(M_{d'}) \subset N_{d+d'}$.

From now on, R is a polynomial ring over a field with $n+1$ variables, i.e. $R = \mathbf{k}[x_0, \dots, x_n]$. The grading in $R = \mathbf{k}[x_0, \dots, x_n]$ is defined by a map deg $: R \to \mathbb{Z}^r$ such that deg$(1) = 0$ and for two polynomials $f, g \in \mathbf{k}[x_0, \dots, x_n]$, one has deg$(fg) =$ deg$(f) +$ deg(g). Thus, the grading is defined solely by the degrees of the variables $x_0, \dots, x_n$. The vector space R_m is spanned by the monomials of degree m.

The grading provided by deg is *standard* if deg(x_i) is an element of the canonical basis of $\mathbb{Z}^n$ for $i \in \{1, \dots, r\}$. For instance, the only standard $\mathbb{Z}$-grading considers all the variables as elements of degree 1.

Definition 2.11. A polynomial $f \in R$ is homogeneous if it belongs to R_d for some $d \in \mathbb{Z}^n$. An ideal $I \subset R$ is homogeneous if it is generated by homogeneous polynomials.

In the rest of this section, we restrict to the case of the standard $\mathbb{Z}$-grading in $R = \mathbf{k}[x_0, \dots, x_n]$. However, once we have defined the setting of toric varieties, we will show that some of the constructions that appear in the next pages can be reproduced in that case.

Definition 2.12. The *projective space* $\mathbb{P}^n$ is the quotient $\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts in $\mathbb{C}^{n+1} - \{0\}$ by homotheties $\lambda(a_0, \dots, a_n) = (\lambda a_0, \dots, \lambda a_n)$ for $\lambda \in \mathbb{C}^*$.

Polynomial functions are not well-defined over the projective space as they do not necessarily behave well under the action that defines it. However, the zeros of homogeneous polynomials are well defined in the projective space. Namely, for $d \in \mathbb{Z}_{\geq 0}$ and $f \in R_d$, we have

$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n) \quad (a_0, \dots, a_n) \in \mathbb{C}^{n+1} - \{0\} \quad \lambda \in \mathbb{C}^*$$

and so

$$f(\lambda a_0, \dots, \lambda a_n) = 0 \iff f(a_0, \dots, a_n) = 0.$$

This allows us to give the following definition.

Definition 2.13. Let $I \subset R$ be a homogeneous ideal. The *projective variety* $\mathbb{V}_{\mathbb{P}^n}(I)$ is defined as:

$$\mathbb{V}_{\mathbb{P}^n}(I) = \{p \in \mathbb{P}^n \quad f(p) = 0 \quad \forall f \in I\}.$$

Similarly as before, this defines the Zariski topology in $\mathbb{P}^n$.

The polynomials in the ideal $\mathfrak{m} = (x_0, \dots, x_n)$ only vanish simultaneously in $\mathbb{A}^{n+1}_{\mathbf{k}}$ at the point 0. This point was removed in the definition of the projective space, implying that $\mathbb{V}_{\mathbb{P}^n}(\mathfrak{m}) = \emptyset$. This justifies that the ideal $\mathfrak{m}$ is known as the *irrelevant ideal*. Thus, if $I \subset R$ is any homogeneous ideal, considering the colon ideal with respect to any power of $\mathfrak{m}$ does not change the variety $\mathbb{V}_{\mathbb{P}^n}(I)$. In other words, the saturation with respect to $\mathfrak{m}$, i.e.

$$I^{\mathsf{sat}} = (I : \mathfrak{m}^\infty)$$

will satisfy $\mathbb{V}_{\mathbb{P}^n}(I) = \mathbb{V}_{\mathbb{P}^n}(I^{\mathsf{sat}})$. In fact, I^{sat} is the largest ideal in the class of all ideals giving the same projective variety as I [BS87b]. This justifies the use of saturations to understand the set of projective points that vanish in an ideal.

Definition 2.14. Let $V \subset \mathbb{P}^n$ be a subset. The homogeneous ideal $\mathbb{I}_{\mathbb{P}^n}(V)$ is defined as:

$$\mathbb{I}_{\mathbb{P}^n}(V) = \{f \in R \quad f(p) = 0 \quad \forall p \in V \quad f \text{ homogeneous}\}.$$

Theorem 2.4. [Eis95, Theorem 1.6] (Projective Hilbert's Nullstellensatz) Given a homogeneous ideal $I \subset R$,

$$\mathbb{I}_{\mathbb{P}^n}(\mathbb{V}_{\mathbb{P}^n}(I)) = \sqrt{I}.$$

In particular, if $\mathbb{V}_{\mathbb{P}^n}(I) = \emptyset$, then for every $i = 0, \dots, n$, there is $k \in \mathbb{Z}_{>0}$ such that $x_i^k \in I$.

The definition of the dimension of a variety can be reproduced also in this case as maximal chain of distinct irreducible varieties. However, if we consider the Krull dimension of of the quotient ring R/I, we will find the affine dimension of that quotient ring, which involves one more variable. Therefore, the geometric dimension will coincide with the algebraic Krull dimension minus one, i.e.

$$\dim(\mathbb{V}_{\mathbb{P}^n}(I)) = \dim(R/I) - 1. \tag{2.1}$$

On the other hand, the projective varierty $\mathbb{V}_{\mathbb{P}^n}(I)$ can be described in terms of the homogeneous prime ideals in R/I that do not contain the irrelevant ideal, getting the *projective scheme* $\mathrm{Proj}(R/I)$.

$$\mathrm{Proj}(R/I) = \{P \text{ homogeneous prime ideal of } R/I \text{ such that } P \not\supset \mathfrak{m}\}.$$

Using the graded structure, we can provide another object that allows us to understand the dimension of the variety $\mathbb{V}_{\mathbb{P}^n}(I)$.

Definition 2.15. The *Hilbert function* of R/I is the function:

$$\mathrm{HF}_{R/I} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \quad d \to \dim_{\mathbf{k}}(R/I)_d.$$

It is standard in commutative algebra [Eis95] that there is a univariate polynomial $\mathrm{HP}_{R/I}$ such that:

$$\mathrm{HP}_{R/I}(t) = \mathrm{HF}_{R/I}(t) \quad t \gg 0.$$

If this polynomial is of degree D, then the dimension of $\mathbb{V}_{\mathbb{P}^n}(I)$ is D. Moreover, the Hilbert polynomial can be written as:

$$\mathrm{HP}_{R/I}(t) = \frac{e}{(\dim R/I - 1)!} t^D + \{\textit{terms of lower degree in } t\}$$

for some $e \in \mathbb{Z}_{\geq 0}$; see [Eis95, Section 1.9]. Geometrically, it can be shown that e corresponds to the number of intersection points between with D general hyperplanes $H_i = \{l_i = 0\}$ for $i = 1, \dots, D$ where $l_1, \dots, l_D$ are general linear forms, i.e. considering the variety $\mathbb{V}_{\mathbb{P}^n}(I, l_1, \dots, l_D)$. This provides a geometric definition of the *degree* of the variety $\mathbb{V}_{\mathbb{P}^n}(I)$.

Homological constructions. Some of the tools of commutative algebra that we will use are based on homological algebra, which studies of properties of modules (in particular, of polynomial ideals and quotient rings) in terms of the homology of chain complexes.

Definition 2.16. Let R be a ring. A *chain complex* $(M_i, \delta_i)_{i \in \mathbb{Z}}$ is a sequence of R-modules $(M_i)_{i \in \mathbb{Z}}$ together with a sequence of morphisms $\delta_i : M_i \to M_{i-1}$, i.e.

$$M_\bullet = (\dots \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \xrightarrow{\delta_{i-1}} \dots)$$

satisfying $\delta_{i-1} \circ \delta_i = 0$ for $i \in \mathbb{Z}$, in other words, $\mathrm{im}(\delta_i) \subset \ker(\delta_{i-1})$. A chain complex is bounded if there are $a, b \in \mathbb{Z}$ with $a < b$ such that $M_i = 0$ for all i such that $i < a$ or $i > b$. The i-th homology of a complex is the module:

$$H_i = \frac{\ker(\delta_{i-1})}{\mathrm{im}(\delta_i)}.$$

A chain complex is *exact* is $H_i = 0$ for all $i \in \mathbb{Z}$.

If the ring R, the modules $(M_i)_{i \in \mathbb{Z}}$ and the morphisms $\delta_i : M_i \to M_{i-1}$ are $\mathbb{Z}$-graded, then the complex $M_\bullet$ inherits the graded structure, and so do its homology modules. Considering the graded pieces of degree d of a chain complex, as a complex of $\mathbf{k}$-vector spaces is known as considering a *strand* of a complex and denoted as $(M_\bullet)_d$.

A possible way to comprise the algebraic structure of an module M is precisely by attaching to it a chain complex, called *free resolution*. Initially, the algebra of M is described by a minimal set of generators, $f_1, \dots, f_r$, where every other element in I can be written as a combination of them and none of them can be removed.

However, the algebraic relations between these generators are also part of the description of M. For this purpose, it is natural to consider the *syzygies* of $f_1, \ldots, f_r$, i.e.

$$\text{Syz}(f_1, \ldots, f_r) = \{(g_1, \ldots, g_r) \in \oplus_{i=1}^r R \quad \sum_{i=1}^r g_i f_i = 0\} \tag{2.2}$$

With this, we get an exact chain complex (short exact sequence) of the form:

$$0 \to \text{Syz}(f_1, \ldots, f_r) \to \oplus_{i=1}^r R(-d_i) \xrightarrow{\phi} M \to 0.$$

However, the algebra of I is also comprised by the generators of Syz$(f_1, \ldots, f_r)$ as an R-module. Thus, we can find a set of generators $f_1', \ldots, f_{r'}'$ of this R-module of degrees $d_1', \ldots, d_{r'}'$. Finding these generators is equivalent to finding a surjective map from $\oplus_{i=1}^{r'} R(-d_i')$ to ker(ϕ), getting a new exact chain complex of the form:

$$0 \to \text{Syz}(f_1', \ldots, f_r') \to \oplus_{i=1}^{r'} R(-d_i') \to \oplus_{i=1}^r R(-d_i) \xrightarrow{\phi} M \to 0.$$

If we repeat this process, we will get a chain complex formed by free R-modules. Hilbert's syzygy theorem [Eis95, Theorem 1.13] implies that this process will finish after a finite number of steps, which is bounded by the number of variables in R.

Definition 2.17. A *finite free resolution* of a graded R-module M is a bounded chain complex of the form

$$0 \to F_s \xrightarrow{\delta_s} \ldots \to F_1 \xrightarrow{\delta_1} F_0 \to 0$$

satisfying:

- The 0-th homology equals M, i.e. $H_0 = M$.
- For $i > 0$, the i-th homology vanishes, i.e. $H_i = 0$.
- Each of the modules F_i is free.

Free resolutions need not be unique. For instance, one can consider the trivial exact chain complex induced by the identity map on R, i.e.

$$0 \to R \to R \to 0.$$

If we are given a free resolution $F_\bullet$ of M, we can modify its i-th map by considering the map:

$$\overline{\delta}_i : F_i \oplus R \to F_{i-1} \oplus R \quad (m, r) \to (\delta_i(m), r)$$

Then, the new chain complex after this modiffication is still a free resolution; see [Eis95, p. 20.1]. In order to avoid the presence of these trivial complexes, we can consider minimal free resolutions.

Definition 2.18. A *minimal free resolution* of a graded R-module M is a free resolution such that:

$$\delta_i(F_i) \subset \mathfrak{m} F_{i-1} \quad \forall i \geq 1.$$

Equivalently, F_i maps to a minimal set of generators of coker(δ_i); see [Eis95, Lemma 19.4].

Minimal free resolutions exist and they are unique in the graded setting [Eis95, p. 1.1.6], thus they provide a method to comprise the algebraic structure of M. The minimal free resolution is described in terms of the Betti numbers. Namely, as the F_i are free modules, one can write them as:

$$F_i = \oplus_d S(-d)^{\beta_{i,d}(M)}.$$

Here, $\beta_{i,d}(M)$ are the Betti numbers of M and only a finite number of them can be nonzero.

Definition 2.19. Given an ideal $I \subset R$ generated by r polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r \in \mathbb{Z}$, the *Koszul complex* is

$$\mathcal{K}_\bullet(I) : K_r(I) \xrightarrow{\delta_r} \cdots \to K_1(I) \xrightarrow{\delta_1} K_0(I)$$

where the terms are the free modules $\mathcal{K}_i(I) = \bigoplus_{1 \le j_1 < \dots j_i \le k} R(-\sum_{k \in \{j_1, \dots, j_i\}} d_k)$. The differentials $\delta_i : \mathcal{K}_i(I) \to \mathcal{K}_{i-1}(I)$ are defined as the direct sum of the maps $\delta_i = \bigoplus_{1 \le j_1 < \dots < j_i \le k} \delta^{j_1, \dots, j_i}$ defined as:

$$\delta^{j_1,\dots,j_i} : R(-\sum_{k \in \{j_1,\dots,j_i\}} d_k) \to \bigoplus_{j \in \{j_1,\dots,j_i\}} R(-\sum_{k \in \{j_1,\dots,j_r\} - \{j\}} d_k)$$
$$\delta^{j_1,\dots,j_i}(g) = \sum_{k \in \{j_1,\dots,j_i\}} (-1)^{\tau(k)} f_k g \quad (2.3)$$

where $\tau(k)$ is such that $j_{\tau(k)} \le k \le j_{\tau(k)+1}$.

If we are given homogeneous polynomials $f_1, \dots, f_r$ with degrees $d_1, \dots, d_r$ and general coefficients, the Koszul complex provides the minimal free resolution [Eis95, Corollary 19.3]. This result is based on the fact that general polynomials of degrees $d_1, \dots, d_r$ form a regular sequence, i.e.

$$(f_1, \dots, f_{i-1} : f_i) = (f_1, \dots, f_{i-1}).$$

As we will often consider this general case for the coefficients of the system, the Koszul complex will be a very useful tool in many of our construcitons.

Finally, there is another homological algebra construction which we can use to summarize the properties of R-modules.

Definition 2.20. Let M be an R-module and let $f_1, \dots, f_r \in R$ be a sequence of elements generating an ideal $J \subset R$. The *Čech complex* is formed by the modules $\mathcal{C}^i_{f_1,\dots,f_r}(M)$ where:

$$\mathcal{C}^i_{f_1,\dots,f_r}(M) = \bigoplus_{1 \le j_1 \le \dots \le j_i \le r} M_{(f_j)_{j \in \{j_1,\dots,j_i\}}}$$
$$\text{where } M_{(f_j)_{j \in \{j_1,\dots,j_i\}}} = \langle \{ \frac{g}{f_j} \quad g \in M \quad j \in \{j_1, \dots, j_i\} \} \rangle \quad (2.4)$$

and the differentials $\Delta^i : \mathcal{C}^i_{f_1,\dots,f_r}(M) \to \mathcal{C}^{i+1}_{f_1,\dots,f_r}(M)$ defined as $\Delta^i = \oplus_{1\leq j_1<\dots j_i\leq r}\delta^{j_1,\dots,j_i}$ where:

$$\delta^{j_1,\dots,j_r}(m) = \sum_{k\notin\{j_1,\dots,j_i\}} (-1)^{\tau(k)}\phi_k(m_{j_1,\dots,j_i})$$

where $\tau(k)$ is such that $j_{\tau(k)} \leq k \leq j_{\tau(k)+1}$ and $\phi_k : M_{(f_j)_{j\in\{j_1,\dots,j_i\}}} \to M_{(f_j)_{j\in\{j_1,\dots,j_i,k\}}}$ is defined by inclusion.

Definition 2.21. The homology of the Čech complex is independent of the choice of the set of generators of the ideal J and known as *local cohomology*. We denote it as $H^i_J(M)$.

Definition 2.22. The *cohomological dimension* of M with respect to J is:

$$\mathrm{cd}_J(M) = \max(\{0\} \cup \{i \in \mathbb{Z}_{>0} \text{ s.t. } H^i_J(I) \neq 0\}). \tag{2.5}$$

For our case of interest of polynomial systems, we study local cohomology modules of the ideal I and the quotient ring R/I with respect to the irrelevant ideal $\mathfrak{m}$. In this cases, local cohomology exhibits very interesting properties. For instance, the 0-th cohomology module with respect to $\mathfrak{m}$ corresponds to the quotient I^{sat}/I [Bus06, Section 1], i.e.

$$H^0_{\mathfrak{m}}(R/I) = H^1_{\mathfrak{m}}(I) = I^{\mathrm{sat}}/I. \tag{2.6}$$

Moreover, as we will base many of the computations of local cohomology modules on minimal free resolutions, an important object to understand is the local cohomology of the ring R itself, which is well understood.

Theorem 2.5. [Bus06, Section 1.3.4] For $R = \mathbf{k}[x_0,\dots,x_n]$ and $\mathfrak{m} = \langle x_0,\dots,x_n\rangle$, we have:

$$H^i_{\mathfrak{m}}(R) = \begin{cases} 0 & i \neq n+1 \\ \frac{1}{x_0\cdots x_n}\mathbf{k}[x_0^{-1},\dots,x_n^{-1}] & i = n+1 \end{cases}.$$

In particular, $H^i_{\mathfrak{m}}(R)_d = 0$ unless $i = n+1$ and $d \leq -(n+1)$.

In order to derive the non-vanishing graded pieces of the local cohomology modules $H^i_{\mathfrak{m}}(I)$ for an ideal I, we will often use the the study of the spectral sequences associated to a Čech-resolution (or Čech-Koszul) double complex $\mathcal{C}^\bullet_{\mathfrak{m}}(F_\bullet)$. These two spectral sequences [Wei94, Section 5.6] appear after taking homologies in the two different possible directions. If one considers the homologies with respect to $F_\bullet$ first, then in the second page of the spectral sequence, we will obtain $H^\bullet_{\mathfrak{m}}(I)$. On the other hand, if we start taking homologies with respect to $\mathcal{C}^\bullet$, the first page will be formed by modules of the form $H^\bullet_{\mathfrak{m}}(F_\bullet)$, whose supports can be computed using the description in Theorem 2.5. The two spectral sequences must converge to the same limit, providing a method to understand $H^\bullet_{\mathfrak{m}}(I)$.

In order to connect the importance of local cohomology with the geometric ideas that we introduced at the beginning of this section, we can state a formula, usually named after Grothendieck and Serre, which relates local cohomology with Hilbert functions and polynomials.

Theorem 2.6. [BH98, Theorem 4.3.5] For any homogeneous ideal $I \subset R$, and $d \in \mathbb{Z}$ we have:

$$\mathrm{HF}_{R/I}(d) = \mathrm{HP}_{R/I}(d) + \sum_{i=0}^{n} \dim_{\mathbf{k}} H^i_{\mathfrak{m}}(R/I)_d.$$

The Castelnuovo-Mumford regularity. As a way to end the review on commutative algebra, we state the classical definition of the Castelnuovo-Mumford regularity [MB66].

Definition 2.23. Let $I \subset R$ be a homogeneous ideal. The ideal is called m-regular for $m \in \mathbb{Z}$, if it satisfies any of the following three equivalent conditions:

- $H^i_{\mathfrak{m}}(I)_d$ for $i \geq 0$ and $d > m - i$
- $\beta_{i,d} = 0$ for $i \geq 0$ and $d > m + i$.
- The truncated ideal $I_{\geq m} = \oplus_{d \geq m} I_d$ has a linear resolution.

The Castelnuovo-Mumford regularity is the set of m-regular degrees, i.e.

$$\mathrm{reg}(I) = \{m \in \mathbb{Z} \quad I \text{ is } m\text{-regular}\}$$

There is a wide literature on understanding the insights of this invariant; see [Cha07] for a summary. In this literature, the Castelnuovo-Mumford regularity is often seen as the minimal degree in $\mathrm{reg}(I)$. However, for the insights that we will describe Chapter 5, it is interesting to see the regularity as a subset of degrees in $\mathbb{Z}$. The equivalence between the three definitions was provided by Eisenbud and Goto in [EG84]. An elegant proof can be found in [Eis05, Proposition 4.16].

Remark 2.1. Using Theorem 2.6, we can see that for $d \in \mathrm{reg}(I)$, then:

$$\mathrm{HF}_{S/I}(d) = \mathrm{HP}_{S/I}(d).$$

2. Newton polytopes and toric varieties

In this section, we discuss the theory of projective toric varieties, which forms the foundation for much of the progress made in this thesis. It is important to have in mind that the starting point of our description are the Newton polytopes. The theory of toric geometry has many more insights (see [CLS12]), which exceed the results that we required in our work.

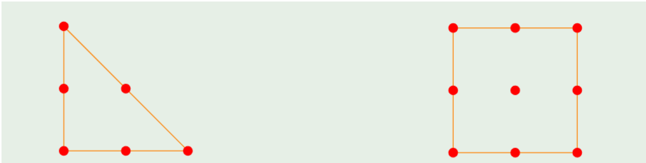


Figure 2.1: Two lattice polytopes in $\mathbb{R}^2$, defined as the convex hull of the sets $S = \{(0,0),(1,0),(2,0),(0,1),(1,1),(0,2)\}$ and $S = \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)\}$, respectively.

Polytopes and normal fans. In order to develop the theory of toric varieties for polynomial systems, one can work over the lattice $\mathbb{Z}^n$. However, one can also consider that the polynomials lie in some other lattice (for instance, $\frac{1}{D}\mathbb{Z}^n$ for some $D \in \mathbb{Z}_{>1}$), providing the same theory for generalized polynomials with rational exponents. Moreover, in most of our developments in Chapter 4, we will require that $\mathbf{k}$ is the field of complex numbers $\mathbb{C}$. The theory of toric varieties can be developed for any other field, even of positive characteristic.

Notation 2.1. We denote by $N = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z})$ the dual of $\mathbb{Z}^n$. Let $(\mathbb{C}^*)^n$ be the complex torus of dimension n. We also set $N_{\mathbb{R}}$ the dual vector space to $\mathbb{R}^n$. Denote as $\langle \cdot, \cdot \rangle$ the natural pairing between $\mathbb{Z}^n$ and N.

Definition 2.24. Let $S \subset \mathbb{Z}^n$ be a finite subset. The *convex hull* of S is the subset of $\mathbb{R}^n$ given as:

$$\mathrm{conv}(S) = \Big\{ \sum_{u \in S} \lambda_u u \quad \lambda_u \in \mathbb{R}_{\geq 0} \quad \sum_{u \in S} \lambda_u = 1 \Big\} \subset \mathbb{R}^n$$

A *lattice polytope* Δ is a subset of $\mathbb{R}^n$ of the above form. The *dimension* of a polytope is the minimal d such that there is an affine linear subspace $V \subset \mathbb{R}^n$ of dimension d such that $\Delta \subset V$.

The affine linear subspaces defining the dimension can be described in terms of affine hyperplanes:

$$H_{u,a} = \{ m \in \mathbb{R}^n \quad \langle u, m \rangle = a \}$$

for some $u \in N$ quad $a \in \mathbb{Z}$. Similarly, one can describe the *closed half-subspaces* associated to to u, a as:

$$H^+_{u,a} = \{ m \in \mathbb{R}^n \quad \langle u, m \rangle \geq a \}.$$

A face Δ' of a polytope Δ is a polytope, denoted as $\Delta' \lesssim \Delta$, such that there are $u_1, \ldots, u_r \in N$ and $a_1, \ldots, a_r \in \mathbb{Z}$ such that:

$$\Delta' = \Delta \cap H_{u_1,a_1} \cap \cdots \cap H_{u_r,a_r} \quad \Delta \subset H^+_{u_1,a_1} \cap \cdots \cap H^+_{u_r,a_r}.$$

The faces of a polytope can be classified in terms of their dimension. In particular, we can discuss *facets* (of dimension $\dim \Delta - 1$), *edges* (of dimension 1) or *vertices* (of dimension 0).

Remark 2.2. Most of the properties that we will discuss are invariant under considering translations of the polytopes, this is:

$$\Delta + t = \{m \in \mathbb{R}^n \quad m = m' + t \quad m' \in \Delta\}.$$

for some $t \in \mathbb{Z}^n$.

Using the notation above, any lattice polytope Δ can be presented as an intersection of closed half-spaces, i.e.

$$\Delta = H^+_{u_1,a_1} \cap \cdots \cap H^+_{u_r,a_r}.$$

If there are two indices $i, j \in \{1, \dots, r\}$ such that $u_i = -u_j$ and $a_i = a_j$, then Δ is not n-dimensional. In that case, up to a translation of the polytope, we can study its properties in the sub-lattice of $\mathbb{Z}^n$ defined by the equation $\langle m, u_i\rangle = 0$. Thus, for the next discussions, *we can assume that Δ is n-dimensional.* In that case, Δ can be presented as the intersection of the inequalities defining its facets, i.e.

$$\Delta = \{m \in \mathbb{R}^n \quad \langle u_F, m\rangle \geq -a_F \quad F \text{ facet}\}$$

where the facet $F \lesssim \Delta$ is defined by as $\Delta \cap H_{u_F,-a_F}$ for some $u_F \in N$ and $a_F \in \mathbb{Z}$.

If the polytope Δ is very ample (see [CLS12, Definition 2.2.17]), then a toric variety X_Δ associated with this polytope can be constructed by using the Zarizki closure of the map $\Phi_{\mathcal{A}}$, i.e.

$$\Phi_{\mathcal{A}} : (\mathbb{C}^*)^n \to \mathbb{P}^{s-1}_{\mathbb{C}} \quad t := (t_1, \dots, t_n) \to (t^{m_1} : \cdots : t^{m_s}), \tag{2.7}$$

where $\mathcal{A} = \Delta \cap M = \{m_1, \dots, m_s\}$. If we are given any n-dimensional lattice polytope Δ (not necessarily very ample) the toric variety to consider is the one associated to $l\Delta$ for $l \gg 0$, which has the same normal fan and is very ample; see [CLS12, Definition 2.3.14].

Example 2.1. Consider the simplex Δ_n in $\mathbb{R}^n$, which is the convex hull of the lattice points $\mathcal{A} = \{0, e_1, \dots, e_n\}$ where $(e_i)_{i=1,\dots,n}$ is a canonical basis of $\mathbb{Z}^n$, then the closure of the image of $\Phi_{\mathcal{A}}$ is $\mathbb{P}^n$.

Definition 2.25. Let $S \subset N$ be a finite subset. The *conic hull* is the subset of $N_{\mathbb{R}}$ given as:

$$\text{Cone}(S) = \Big\{ \sum_{u \in S} \lambda_u u \quad \lambda_u \in \mathbb{R}_{\geq 0} \Big\} \subset N_{\mathbb{R}}.$$

A *rational cone* σ is a subset of $N_{\mathbb{R}}$ of the above form. A cone σ is *strongly convex* if $\sigma \cap (-\sigma) = 0$. The *dual* cone of σ^* in $\mathbb{R}^n$ is the subset:

$$\sigma^* = \{m \in \mathbb{R}^n \quad \langle u, m\rangle \geq 0 \quad \forall u \in \sigma\}.$$

A *face* of a cone is a subcone $\sigma' \lesssim \sigma$ for the form: $\sigma' = \sigma \cap H_{m,0}$ for some $m \in \sigma^*$. The *dimension* of a cone σ is the smallest dimension of the linear subspace $V \subset N_{\mathbb{R}}$ such that $\sigma \subset V$.

Definition 2.26. A *fan* Σ is union of strongly convex cones of $\mathbb{R}^n$ such that:

- If $\sigma \in \Sigma$, every face of σ is also in Σ.
- The intersection $\sigma_1 \cap \sigma_2$ of two cones $\sigma_1, \sigma_2 \in \Sigma$ is a face of each.

A fan Σ' refines another fan Σ, if every cone $\sigma' \in \Sigma'$ is contained in a cone $\sigma \in \Sigma$, i.e. $\sigma' \subset \sigma$. We denote as $\Sigma(d)$ the set of d-dimensional cones of a fan. The 1-dimensional cones are called *rays*.

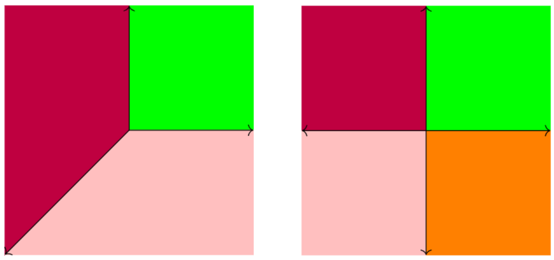


Figure 2.2: The normal fans of the polytopes in Figure 2.1.

If Δ is a polytope and $\Delta' \lesssim \Delta$ is a face of it, defined by the hyperplanes $H_{u_1,b_1}, \ldots, H_{u_r,b_r}$, one can consider the cones of the form:

$$\sigma_{\Delta'} = \text{Cone}(u_i \quad i \in \{1, \ldots, r\}) \subset N_{\mathbb{R}}.$$

These cones are independent of $b_1, \ldots, b_r$ and thus, invariant under translations of the polytope.

Definition 2.27. The *normal fan* Σ of a polytope Δ is the union of the cones $\sigma_{\Delta'}$ for all the faces $\Delta' \lesssim \Delta$.

From the dual cones of the normal fan Σ, one can recover the affine varieties defining Δ. Namely, one can consider the *semigroup* $S_\sigma = \sigma^* \cap \mathbb{Z}^n$ and study the *affine toric* variety

$$\text{Spec}(\mathbf{k}[S_\sigma]) \quad \mathbf{k}[S_\sigma] = \mathbf{k}[x^m \quad m \in \sigma^* \cap \mathbb{Z}^n].$$

where $\mathbf{k}[S_\sigma]$ is the *semigroup algebra* generated by S_σ. The generators of S_σ as a semigroup provide the generators of $\mathbf{k}[S_\sigma]$ as an algebra. Gluing the affine varieties defined by the semigroup algebras for every cone $\sigma \in \Sigma$, following the face structure of the fan, one can recover the variety X_Δ; see [CLS12, Chapter 3, 3.1].

Example 2.2. The normal fan Σ of the first polytope in Example 5.1 is the fan with rays $\{e_1, \ldots, e_n, -\sum_{i=1}^n e_i\}$; see Figure 2.2. If we consider the cone σ generated by $e_1, \ldots, e_n$, then the semigroup algebra is generated by the lattice points in $\mathbb{Z}^n_{\geq 0}$ and the affine toric variety is $\mathbb{A}^n_{\mathbf{k}}$; see Figure 2.3. Thus, gluing the semigroup algebras defined by Σ corresponds to the gluing of affine spaces that provides the projective space $\mathbb{P}^n$.

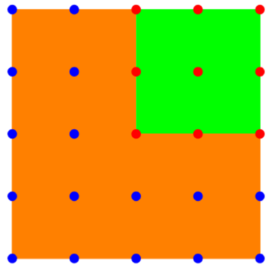


Figure 2.3: In green, the dual of the cone generated by $(1,0)$ and $(0,1)$. In red, the lattice points in this dual cone. In green and orange, the dual to the cone generated by $(0,0)$. In red and blue, the lattice points in this dual cone.

The Cox ring and the quotient construction Finally, we can provide a definition of a toric variety which relies directly on the fan Σ and does not require to glue affine pieces. Recall that the starting point of our discussion is a polytope Δ. As we assumed that Δ is n-dimensional, its normal fan is formed by strongly convex cones and it is *complete*, i.e.

$$\cup_{\sigma\in\Sigma}\sigma = N_{\mathbb{R}}.$$

The normal fan of a polytope must have $n+r$ rays for $r \geq 0$. Moreover, we can assume that the generators of the rays $u_\rho \in N$ for $\rho \in \Sigma(1)$ are primitive and span the vector space $N_{\mathbb{R}}$. By [CLS12, Corollary 3.3.10], this condition is equivalent to the toric variety X_Σ having no torus factors. Thus, we denote as $u_1, \ldots, u_{n+r}$ the generators of the rays in some order. In addition, if we denote the class group of X_Σ by $\mathrm{Cl}(X_\Sigma)$, there is a short exact sequence:

$$0 \to \mathbb{Z}^n \xrightarrow{\mathbf{F}} \mathbb{Z}^{n+r} \xrightarrow{\pi} \mathrm{Cl}(X_\Sigma) \to 0, \tag{2.8}$$

where $\mathbf{F}$ is an $(n+r)\times n$ matrix whose rows are the generators of the rays in $\Sigma(1)$ and π is chosen accordingly to be a cokernel matrix; see [CLS12, Theorem 4.1.3].

Definition 2.28. The *Cox ring* is defined as the ring $R = \mathbf{k}[x_1, \ldots, x_{n+r}]$. This ring is $\mathrm{Cl}(X_\Sigma)$-graded through the map π. Namely, a monomial $\prod_{i=1}^{n+r} x_i^{a_i}$ has degree $\pi((a_i)_{i=1,\ldots,n+r})$.

This short exact sequence induces a transposed short sequence of groups by considering the functor $\mathrm{Hom}(-,\mathbb{C}^*)$.

$$0 \to G \xrightarrow{\pi^T} (\mathbb{C}^*)^{n+r} \xrightarrow{\mathbf{F}^T} (\mathbb{C}^*)^n \to 0, \tag{2.9}$$

where $G = \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$. In particular, one can show that G is the subgroup of $(\mathbb{C}^*)^n$ defined by the equations:

$$G = \{(t_\rho)_{\rho\in\Sigma} \in (\mathbb{C}^*)^{n+r} \quad \prod_{\rho\in\Sigma(1)} t_\rho^{\langle u_\rho, m\rangle} = 1 \quad \forall m \in M\}.$$

Thus, the natural action of $(\mathbb{C}^*)^{n+r}$ in $\mathbb{C}^{n+r}$ induces an action of G in $\mathbb{C}^{n+r}$.

Definition 2.29. The *irrelevant ideal* is:

$$\mathfrak{b} = (\tilde{x}^\tau \text{ such that } \tau \in \Sigma(n)), \text{ where } \tilde{x}^\tau = \prod_{\rho \notin \tau(1)} x_\rho. \tag{2.10}$$

The *irrelevant subset* is defined as $V(\mathfrak{b}) = \{x \in \mathbb{C}^{n+r} \quad f(x) = 0 \quad \forall f \in \mathfrak{b}\}$.

The primary decomposition of $\mathfrak{b}$ can also be defined in terms of the combinatorics of the fan Σ.

Definition 2.30. A subset $C \subset \Sigma(1)$ is a *primitive collection* if:

- There is no cone $\sigma \in \Sigma$ such that $C = \sigma(1)$.
- For every proper subset $C' \subsetneq C$, there is a cone $\sigma \in \Sigma$ such that $C' = \sigma(1)$.

The irrelevant ideal satisfies $\mathfrak{b} = \cap_C (x_\rho \quad \rho \in C)$ where the intersection runs over all the primitive collections in Σ; see [CLS12, Definition 5.1.6]. Thus, the irrelevant subset can be decomposed as:

$$V(\mathfrak{b}) = \cup_C V(x_\rho \quad \rho \in C). \tag{2.11}$$

We can pay particular attention to the toric varieties for which the primitive collections are disjoint.

Definition 2.31. A fan Σ *splits* if the primitive collections are disjoint. A toric variety X_Σ has a *splitting fan* if the fan Σ splits.

Each of the irreducible components in (2.11) is an orbit of the action of $(\mathbb{C}^*)^{n+r}$ in $\mathbb{C}^{n+r}$. Thus, G also induces an action in $\mathbb{C}^{n+r} - V(\mathfrak{b})$.

Definition 2.32. The toric variety X_Σ defined from the normal fan of Δ is defined as:

$$X_\Sigma := \mathbb{C}^{n+r} - V(\mathfrak{b})//G$$

This definition is equivalent to the two definitions above; see [CLS12, Proposition 5.1.9].

Example 2.3. The irrelevant ideal of the fan giving the projective space is precisely $\mathfrak{m} = (x_0, \ldots, x_n)$ and $G = \{(\lambda, \ldots, \lambda) \in (\mathbb{C}^*)^{n+1} \quad \lambda \in \mathbb{C}^*\}$. With this, one recovers Definition 2.12.

The action of $(\mathbb{C}^*)^{n+r}$ in $\mathbb{C}^{n+r} - V(\mathfrak{b})$ induces an action of $(\mathbb{C}^*)^n$ in X_Σ after considering the quotient by G. There are other possible definitions of a toric variety which do not involve starting with a polytope or a fan. All of them involve the idea

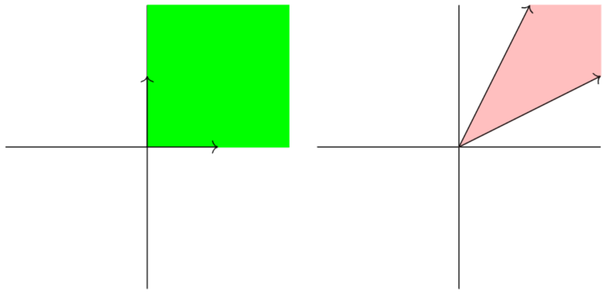


Figure 2.4: In the first drawing, the cone generated by $(1,0)$ and $(0,1)$ is smooth as its generators are a basis of $\mathbb{Z}^2$. In the second case, the cone generated by $(1,2)$ and $(2,1)$ is not smooth.

of having a variety X where the torus $(\mathbb{C}^*)^n$ is an open and dense subset with respect to the Zariski topology and such that the action of $(\mathbb{C}^*)^n$ in this subset extends to the rest of the toric variety.

Further algebraic structure can be analyzed on toric varieties, by noting that their defining ideals are *prime ideals generated by binomials* [ES96]. With the combinatorial information of polytopes and fans, one can easily this structure in the ideals defining these varieties; see [CLS12, Proposition 1.1.9]. In some areas of computational algebraic geometry, one can find ideals defining toric varieties whose polytopes and fans can be more difficult to grasp; see [MS21, Section 8.3].

Defining toric varieties form the combiatorics of polytopes and fans gives us the advantage of connecting the geometric properties of X_Σ with the properties of the fan Σ.

Definition 2.33. A rational cone σ is *simplicial* if its generators are linearly independent over $\mathbb{Z}^n$. The fan Σ is simplicial if every cone $\sigma \in \Sigma$ is simplicial. A rational cone σ is *smooth* if its generators are part of a basis of $\mathbb{Z}^n$. The fan is *smooth* Σ if every cone $\sigma \in \Sigma$ is smooth.

Theorem 2.7. The variety X_Σ is simplicial, if and only if, Σ has, at most, finite quotient singularities. The variety X_Σ is smooth, if and only if, Σ is smooth. Moreover, the variety X_Σ is compact (in the classical topology), if and only if, Σ is complete; see [CLS12, Theorem 3.1.19].

Polytopes and divisors By looking a bit more closely at the short exact sequence in (2.9), one can see that the elements of $\mathbb{Z}^{n+r}$ correspond to Weil divisors in the toric variety X_Σ, which are invariant under the action of $(\mathbb{C}^*)^n$. Namely these divisors are of the form:

$$D_\nu = \sum_{j=1}^{n+r} \nu_j D_j$$

where D_j is the Weil divisor defined by the equation $\{(x_i)_{i=1,\dots,n+r} \in X_\Sigma \quad x_j = 0\}$; see [CLS12, § 4.1]. In particular, by (2.9), the principal Weil divisors are of the form:

$$\mathbf{F}(m) = \sum_{j=1}^{n+r} \langle u_j, m\rangle D_j \tag{2.12}$$

for some $m \in \mathbb{Z}^n$. On the other hand, to each Weil divisor $\sum_{j=0}^{n+r} \nu_j D_j$, one can associate the following polytope:

$$\Delta_\nu = \{m \in \mathbb{R}^n \, : \, \langle u_j, m\rangle \geq -\nu_j, \, j = 1, \dots, n+r\}. \tag{2.13}$$

Properties of divisors, such as being *Cartier, nef* or *ample*, can be understood from the combinatorics of the polytope Δ_ν; see [CLS12, Theorem 4.2.8, Proposition 6.1.1, Theorem 6.3.12, Proposition 7.2.3] for proofs.

Theorem 2.8. Let $D_\nu = \sum_{j=0}^{n+r} \nu_j D_j$ be a Weil divisor in X_Σ and let Δ_ν be the associated polytope.

- The Weil divisor D_ν is Cartier, if and only if,

$$\forall \tau \in \Sigma(n), \text{ there is } m_\tau \in M \text{ such that if } \rho_j \in \tau(1) \; \langle u_j, m_\tau\rangle = -\nu_j. \tag{2.14}$$

- The Cartier divisor D_ν is nef, if and only if,

$$\forall \tau \in \Sigma(n), \text{ there is } m_\tau \in \Delta_\nu \cap M \text{ such that if } \rho_j \in \tau(1) \; \langle u_j, m_\tau\rangle = -\nu_j. \tag{2.15}$$

  Moreover, the previous conditions are equivalent to D_ν being basepoint free.

- The divisor D_ν is ample, if and only if, the normal fan of Δ_ν is Σ.

Assumption 2.1. Our goal is to construct homogeneous polynomial systems that only depend on a polytope Δ and its normal fan Σ. Under this assumption, if D_ν is a Weil divisor in X_Σ, then the normal fan of Δ_ν is refined by Σ; see [CLS12, Proposition 6.2.5].

In some of our results, we will require that X_Σ is a smooth projective toric variety. We can make this assumtion by considering a resolution of singularities of X_Σ which is constructed through a fan Σ' which refines Σ; see [CLS12, Theorem 10.1.10]. After changing X_Σ by $X_{\Sigma'}$, D_ν remains a nef divisor in $X_{\Sigma'}$ and the normal fan of Δ_ν is refined by Σ'. Moreover, if the lattice polytope Δ_ν is fixed, we can also assume that D_ν is nef and Cartier divisor, by choosing $(\nu_j)_{j=1,\dots,n+r}$ to be of the form:

$$\nu_j = -\min_{m \in \Delta} \langle u_j, m\rangle.$$

minimum is attained at a vertex $m \in \Delta_\nu$; see [CLS12, Proposition 6.2.5]. All in all, our constructions will be based on a correspondence between nef Cartier divisors in X_Σ and lattice polytopes, i.e.

$$\left\{ \text{Nef Cartier divisors } D_\nu \text{ in } X_\Sigma \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Lattice polytopes } \Delta_\nu \text{ whose} \\ \text{normal fan refines } \Sigma \end{array} \right\}. \tag{2.16}$$

The assumption that X_Σ is smooth also implies that every Weil divisor is Cartier. In other words, the class group $\mathrm{Cl}(X_\Sigma)$ coincides with the Picard group $\mathrm{Pic}(X_\Sigma)$, which is a free abelian group isomorphic to $\mathbb{Z}^r$; see [CLS12, Proposition 4.2.5, Proposition 4.2.6].

On the other hand, if two polytopes $\Delta_\nu, \Delta_{\nu'}$ correspond to Weil divisors in the same class in $\mathrm{Cl}(X_\Sigma)$, then by (2.9) they are translations of each other, i.e. there is $m' \in \mathbb{Z}^n$ such that:

$$\nu_j - \nu'_j = \langle u_j, m' \rangle \quad j = 1, \ldots, n+r$$

which implies that $\Delta_\nu + m' = \Delta_{\nu'}$; see [CLS12, §4.2, §4.3].

Assumption 2.2. Some of the constructions that we will cosnider (for instance, sparse resultants) are invariant under translations of the polytope Δ_ν. Therefore, for each $\sum_{j=1}^{n+r} \nu_j D_j$, we will choose a representative of its class in $\mathrm{Cl}(X_\Sigma)$ in the following way: each maximal cone $\sigma \in \Sigma(n)$ corresponds to a vertex in Δ_ν. In particular, under Assumption 2.1, we can fix a smooth n-dimensional cone $\sigma \in \Sigma(n)$ and we can translate Δ_ν so that the vertex associated to σ is $0 \in \mathbb{Z}^n$. This choice of the representative of the class of Δ_ν has the following implications:

- Choosing $\sigma \in \Sigma(n)$ allows us to order and label the variables in the Cox ring as $x_1, \ldots, x_n$ for those variables associated to rays $\rho \in \sigma(1)$ and $z_1, \ldots, z_r$ for the rest of variables.
- The matrix π which is a cokernel for $\mathbf{F}$ in the short exact sequence (2.9) can be written as:
$$\pi = \begin{pmatrix} \mathcal{P} & \mathrm{Id}_r \end{pmatrix}, \tag{2.17}$$
where $\mathcal{P}$ is a block matrix $(\mathcal{P}_{j,k})_{1 \le j \le r, 1 \le k \le n}$ with entries in $\mathbb{Z}$. The first n columns of π correspond to the rays $\rho \in \sigma(1)$. The rows of π correspond to the relations between u_{n+j} and the basis given by $u_1, \ldots, u_n$ for $j = 1, \ldots, r$, i.e. relations of the form:
$$u_{n+j} + \sum_{k=1}^{n} \mathcal{P}_{j,k} u_k = 0 \quad j = 1, \ldots, r. \tag{2.18}$$
- If $0 \in \Delta_\nu$ is the vertex associated to σ, we imply that $\nu_j = 0$ for all $j = 1, \ldots, n$. Hence, we are choosing a representative of the class of polytopes of Δ_ν that only depends on $\nu_{n+1}, \ldots, \nu_{n+r}$.
- Under Assumptions 2.1, the Cox ring is $\mathbb{Z}^r$-graded by the map π. In particular, the way we wrote the map π in (4.2) implies that every monomial $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} z_1^{\mu_{n+1}} \cdots z_r^{\mu_{n+r}}$ of degree ν is mapped via π to $(\nu_{n+j})_{j=1,\ldots,r} \in \mathbb{Z}^r$ and thus satisfies the relations:
$$\nu_{n+j} = \mu_{n+j} + \sum_{k=1}^{n} \mathcal{P}_{j,k} \mu_k \text{ for all } j = 1, \ldots, r. \tag{2.19}$$

- The lattice points in Δ_ν correspond to the monomials in R of degree $\nu \in \mathrm{Pic}(X_\Sigma)$. We will use this idea to homogenize and dehomogenize polynomials in the toric setting.

Example 2.4. Let $X_\Sigma = \mathbb{P}^n$ and σ is the cone generated by the canonical basis of $\mathbb{Z}^n$, the polytopes Δ_ν only depend on a positive integer $a \in \mathbb{Z}_{>0}$ and one recovers the Newton polytopes of the polynomials of degree a.

Remark 2.3. Writing the polytopes in the presentation (2.13) and imposing that $0 \in \Delta$ also implies that for any $\nu \in \mathrm{Cl}(X_\Sigma)$, we have $\nu_{n+j} \geq 0$ for $j = 1, \dots, r$. Otherwise, $0 = \langle u_{n+j}, 0\rangle \geq -\nu_{n+j} > 0$. In particular, if $\nu_{n+j} < 0$ for some $j \in \{1, \dots, r\}$, then there are no lattice points in Δ.

Generic polynomial systems and homogenization Going back to the polynomial systems, we will consider polynomials with supports in a subset $\mathcal{A} \subset \mathbb{Z}^n$, i.e.

$$\tilde{F} = \sum_{m \in \mathcal{A}} c_m x^m \in \tilde{R} = \mathbf{k}[x_1, \dots, x_n] \quad c_m \in \mathbf{k} \tag{2.20}$$

where $x^m = x_1^{m_1} \cdots x_n^{m_n}$ is a monomial that can be identified with a character $x^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$. This is the general form of a polynomial whose Newton polytope is $\Delta = \mathrm{conv}(\mathcal{A}) \subset \mathbb{R}^n$. However, the polynomial $\tilde{F}$ is not homogeneous in the setting of the Cox ring that we previously defined.

Using Assumptions 2.2, we can assume that $0 \in \Delta$ and the cone σ associated to 0 is smooth. Moreover, we write the variables in the Cox ring as $x_1, \dots, x_n$ for the variables of the rays in σ and $z_1, \dots, z_r$ for the rest of variables. If Δ is the polytope associated to a nef Cartier divisor $\sum_{j=1}^{n+r} \nu_j D_j$ in a toric variety X_Σ, then we can homogenize $\tilde{F}$ to be a homogeneous polynomial in the Cox ring by considering:

$$\tilde{F} \to F = \sum_{m \in \mathcal{A}} c_m x^{\mathbf{F}m+\nu} \in \mathbf{k}[x_1, \dots, x_n, z_1, \dots, z_r] \tag{2.21}$$

where $\mathbf{F}$ is the matrix in (2.9). We note that we can chose a monomial basis of the graded piece of R of degree ν (denoted as R_ν) corresponding to x^μ where $\mu = \mathbf{F}m + \nu$ for each $m \in \mathcal{A}$. On the other hand, if we are given a homogeneous polynomial, given as a sum of the monomials in R_ν, i.e.

$$F = \sum_{x^\mu \in R_\nu} c_\mu x^\mu \in R_\nu \quad c_\mu \in \mathbf{k}$$

then, we can dehomogenize by first changing $z_1 = \cdots = z_r = 1$ getting an affine polynomial of the form:

$$F \to \tilde{F} = \sum_{m \in \mathcal{A}} c_\mu x^{\overline{\mathbf{F}}m} \in \mathbf{k}[x_1, \dots, x_n, z_1, \dots, z_r]. \tag{2.22}$$

where $\overline{\mathbf{F}}$ is the submatrix of $\mathbf{F}$ considering the rows. This matrix is invertible over $\mathbb{Z}$ as we assumed that σ is a smooth cone and the generators of the rays $\rho \in \sigma(1)$

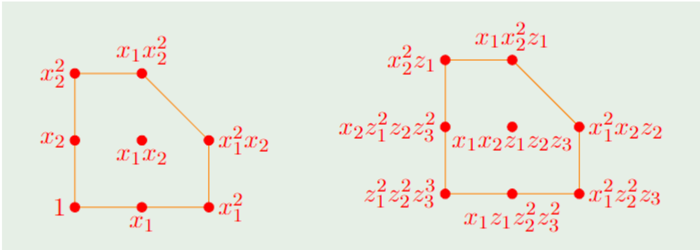


Figure 2.5: Affine and homogeneous monomials of a polynomial system.

form a basis of the lattice $\mathbb{Z}^n$, implying that the matrix has determinant ± 1. As the elimination constructions we will describe are invariant under a change of bases of the lattice, we can replace $x^{\bar{\mathbf{F}}m}$ by x^m, recovering a polynomial with supports in $\mathcal{A}$.

Example 2.5. The polynomial in Figure 2.5 has a Newton polytope whose normal fan has rays:

$$\rho_1 = (1,0)\ \rho_2 = (0,1), \rho_3 = (-1,0)\ \rho_4 = (-1,-1)\ \rho_5 = (0,-1).$$

Therefore, the matrices $\mathbf{F}$ and π are of the form:

$$\mathbf{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \pi = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the polytope Δ_ν in the Figure can be defined as in (2.13) with $\nu = (0,0,2,3,2)$, providing the monomials and their homogenizaiton in Figure 2.5.

We refer the reader to [BT22, Section 2.2] for more details about homogenization and dehomogenization of sparse polynomial systems.

Homological constructions over the irrelevant ideal In the case of projective toric varieties, the irrelevant ideal $\mathfrak{b}$ assumes the role previously held by $\mathfrak{m}$, as discussed in Section 1.. Consequently, local cohomology modules over $\mathfrak{b}$ gain significance in the when we try to distinguish the geometry of X_Σ from the algebra of homogeneous ideals in the Cox ring. However, if $X_\Sigma \neq \mathbb{P}^n$, then the irrelevant ideal is (in general) not a prime ideal and the structure of $H^i_{\mathfrak{b}}(R)$ cannot be described as in Theorem 2.5. Nonetheless, alternative techniques exist for its characterization, such as exploring its relationship with sheaf cohomology modules.

Let S be a finitely generated Cl(X_Σ)-graded R-module with associated coherent sheaf $\mathcal{S}$ in X_Σ and $\alpha \in$ Cl(X_Σ). If $p \geq 2$, then

$$H^p_{\mathfrak{b}}(S)_\alpha \simeq H^{p-1}(X_\Sigma, \mathcal{S}(\alpha)), \tag{2.23}$$

where $H^p_{\mathfrak{b}}(S)_\alpha$ is the graded piece of $H^p_{\mathfrak{b}}(S)$ of degree α and $\mathcal{S}(\alpha)$ is the sheaf defined by $\mathcal{S} \otimes \mathcal{O}_\Sigma(D)$, for a divisor D with $[D] = \alpha$ and $\mathcal{O}_\Sigma$ the structure sheaf of X_Σ; see [CLS12, Theorem 9.5.7] for proofs. Furthermore, the following exact sequence holds:

$$0 \to H^0_{\mathfrak{b}}(S)_\alpha \to S_\alpha \to H^0(X_\Sigma, \mathcal{S}(\alpha)) \to H^1_{\mathfrak{b}}(S)_\alpha \to 0.$$

If $S = R$, then $R_\alpha = H^0(X_\Sigma, \mathcal{O}_\Sigma(\alpha))$ and therefore

$$H^0_{\mathfrak{b}}(R) = H^1_{\mathfrak{b}}(R) = 0. \tag{2.24}$$

Notation 2.2. For the sake of simplicity in the notation, for any Cartier divisor D and any integer $p \geq 0$, we will write $H^p(X_\Sigma, \alpha)$ in place of $H^p(X_\Sigma, \mathcal{O}_\Sigma(D))$, where $\alpha = [D] \in \mathrm{Cl}(X_\Sigma)$.

The following theorems, that are originally due to Demazure and Batyrev-Borisov, will be our main tools to analyze the vanishing of sheaf cohomology modules over toric varieties; see [CLS12, Theorem 9.2.3, Theorem 9.2.7] for proofs.

Theorem 2.9 (Demazure)**.** Let X_Σ be a toric variety such that Σ is complete and D be a nef Cartier divisor, then $H^p(X_\Sigma, \alpha) \simeq 0$ for all $p > 0$ and $\alpha = [D]$.

Theorem 2.10 (Batyrev-Borisov)**.** Let X_Σ be a complete toric variety and D be a nef Cartier divisor, then

$$H^p(X_\Sigma, -\alpha) \simeq \begin{cases} 0 & \text{if } p \neq \dim \Delta_\alpha \\ \bigoplus_{m \in \mathrm{Relint}(\Delta_\alpha) \cap M} \mathbf{k}\chi^{-m} & \text{if } p = \dim \Delta_\alpha \end{cases}$$

where $\alpha = [D] \in \mathrm{Cl}(X_\Sigma)$ and $\mathrm{Relint}(\Delta_\alpha)$ denotes the relative interior of the polytope Δ_α associated with α.

Remark 2.4. We notice that the two above theorems are proved in more generality in [CLS12], we stated them with assumptions that are sufficient in our context.

Another important result we will use is the toric version of Serre duality (see [CLS12, Theorem 9.2.10] for a proof): for any Cartier divisor D and any integer $p \geq 0$,

$$H^p(X_\Sigma, \alpha) \cong H^{n-p}(X_\Sigma, -K_X - \alpha)^\vee, \tag{2.25}$$

where K_X is the anticanonical class in $\mathrm{Cl}(X_\Sigma)$ and $\alpha = [D] \in \mathrm{Cl}(X_\Sigma)$.

Let X_Σ be a projective toric variety and let R be its Cox ring. The Hilbert function of a finitely generated graded R-module S is defined by

$$\mathrm{HF}(S, -) : \mathrm{Cl}(X_\Sigma) \to \mathbb{Z}_{\geq 0} \quad \alpha \mapsto \mathrm{HF}(S, \alpha) := \dim_{\mathbf{k}}(S_\alpha). \tag{2.26}$$

Assuming that X_Σ is a *smooth* toric variety, then for $\alpha \gg 0$ (component-wise), this function becomes a (multivariate) polynomial called the Hilbert polynomial and is denoted by $\mathrm{HP}(S, \alpha)$; see [MS03, Lemma 2.8].

Remark 2.5. If $S = R/J$ with J homogeneous ideal of R defining a 0-dimensional subscheme in X_Σ, then the Hilbert polynomial of S is a constant which is equal to the number of points (over an algebraic closure of $\mathbf{k}$) in this subscheme, counted with multiplicity.

The Grothendieck-Serre formula appearing in Theorem 2.6 also generalizes to the case of the irrelevant ideal, namely, for any $\alpha \in \mathrm{Cl}(X_\Sigma)$,

$$\mathrm{HF}(S, \alpha) = \mathrm{HP}(S, \alpha) + \sum_{i=0}^{n} (-1)^i \dim_{\mathbf{k}} H^i_{\mathfrak{b}}(S)_\alpha. \tag{2.27}$$

see [MS03, Proposition 2.14] for a proof.

3. The sparse resultant

In the introduction, we motivated resultants as central tools in elimination theory, refering to several a wide literature on various methods to compute them; see for instance [GKZ94; DD00; WZ92; DJS22; Ben+21]. However, when we move towards the sparse (or toric) setting, their definition can be intricate. Classically, sparse resultants are studied in the situation where the family of exponents of the given monomials is essential, that is, when the sparse resultant depends on the coefficients of all the polynomials and, in addition, the affine span of these families of exponents coincides with the ambient lattice; see [Stu94]. In this section, we give the definition provided in [DJS22], which is more general than the one provided for an essential family and recall some of the properties of these objects.

Moreover, it is important that we are able to manage an object that eliminates variables also in the case of homogeneous polynomials in the Cox ring of a toric variety. This object was also defined in the book *Discriminants, resultants and multidimensional determinants* by Gelfand, Kapranov and Zelevisnky [GKZ94]. Under some assumptions on the supports of the polynomials, these two objects coincide, thus the methods for computing the resultant can be used both in the affine and homogeneous settings.

The affine case. The setting for the resultant is that of $n+1$ sets of supports $\mathcal{A}_0, \dots, \mathcal{A}_n \subset \mathbb{Z}^n$, providing a *universal* system of polynomials:

$$\tilde{F}_i = \sum_{m \in \mathcal{A}_i} c_{i,m} x^m \quad i = 0, \dots, n \tag{2.28}$$

Let $\Delta_i = \mathrm{conv}(\mathcal{A}_i)$ for $i = 0, \dots, n$ be the Newton polytopes of $F_0, \dots, F_n$.

Definition 2.34. The family of polytopes $\Delta_0, \dots, \Delta_n$ is *essential* if $\dim(\sum_{i=0}^n \Delta_i) = n$ and for every $J \subset \{0, \dots, n\}$, we have:

$$\dim(\sum_{j\in J} \Delta_j) \geq |J|. \tag{2.29}$$

The *fundamental subfamily* is the unique family $(\Delta_i)_{i\in I}$ for a subset $I \subset \{0, \dots, n\}$ satisfying $\dim(\sum_{i\in I} \Delta_i) = |I| - 1$ and $\dim(\sum_{j\in J} \Delta_j) \geq |J|$ for every $J \subset I$.

From this definition, we derive that if the family $(\Delta_0, \dots, \Delta_n)$ is essential, then it is the fundameltal subfamily of that system of polytopes. On the other hand, if the family is not essential, there is strictly smaller fundamental subfamily $(\Delta_i)_{i\in I}$ for some proper subset $I \subset \{0, \dots, n\}$. In this setting, it makes sense to consider that the lattice spanned by this subfamily, i.e.

$$L_I = \big\{ \sum_{i\in I} \lambda_i m_i \quad \lambda_i \in \mathbb{Z} \quad m_i \in \mathcal{A}_i \, i = 0, \dots, n \big\}. \tag{2.30}$$

This lattice might not be saturated, i.e. there might be lattice points in the real vector space it spans that do not belong to L_I. For this reason, we can consider the following lattice.

$$L_I^{\text{sat}} = (L_I \otimes \mathbb{R}) \cap \mathbb{Z}^n. \tag{2.31}$$

The space of coefficients of the $\tilde{F}_i$'s has a natural structure of multi-projective space $\prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i}$, as the zeros of $\tilde{F}_0 = \cdots = \tilde{F}_n = 0$ are not modified after multiplication by a nonzero scalar. Consider the *incidence variety*

$$Z(\tilde{F}) = \{x \times (\dots, c_{i,m}, \dots) \in (\mathbb{C}^*)^n \times \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i} \quad \tilde{F}_0(x) = \cdots = \tilde{F}_n(x) = 0\}$$

and let π be the canonical projection onto the second factor

$$\pi : (\mathbb{C}^*)^n \times \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i} \to \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i}.$$

Definition 2.35. The *sparse resultant*, denoted as $\text{Res}_{\mathcal{A}}$, is a primitive polynomial in in $\mathbb{Z}[c_{i,m}]$ defining the direct image $\pi_*(Z(\tilde{F}))$. This polynomial is a power of the *sparse eliminant*, denoted as $\text{Elim}_{\mathcal{A}}$, which is the irreducible polynomial defining the closure of the image of $Z(\tilde{F})$, i.e. $\overline{\pi(Z(\tilde{F}))}$, if this is a hypersurface, and 1 otherwise. In other words,

$$\text{Res}_{\mathcal{A}} = \pm \, \text{Elim}_{\mathcal{A}}^{d_{\mathcal{A}}} \tag{2.32}$$

for some $d_{\mathcal{A}} \in \mathbb{Z}_{\geq 0}$.

Note that as we are considering the closure of $\pi(Z)$, the preimages of the zeroes of $\text{Res}_{\mathcal{A}}$ are of the form $x \times (\dots, c_{i,m}, \dots)$ where x lies in some compactification of $(\mathbb{C}^*)^n$.

From [Stu94, Corollary 1.1], we can derive that if $(\mathcal{A}_i)_{i\in I}$ is the fundamental subfamily of $\mathcal{A}_0, \ldots, \mathcal{A}_n$, then the sparse resultant (and eliminant) only depend on the coefficients of the polynomials in this family. In fact, we can recude the general theory to working over L_I^{sat}. In [DJS22, Proposition 3.6], the exponent $d_{\mathcal{A}}$ is computed.

Proposition 2.1. Let $(\mathcal{A}_i)_{i\in I}$ be the fundamental subfamily of $\mathcal{A}_0, \ldots, \mathcal{A}_n$ and suppose that $I \neq \emptyset$. Then, $\dim(L_I) = |I| - 1$, and the exponent in (2.32) can be written as:

$$d_{\mathcal{A}} = [L_I^{\text{sat}} : L_I]\, \mathrm{MV}_{\mathbb{Z}^n/L_I^{\text{sat}}}(\pi(\Delta_j)_{j\notin J})$$

where π is the projection $\pi : \mathbb{Z}^n \to \mathbb{Z}^n/L_I^{\text{sat}}$.

Imposing that the Δ_i's are n-dimensional and the $\mathcal{A}_i$ span the lattice $\mathbb{Z}^n$ is sufficient for ensuring that the fundamental subfamily is $\mathcal{A}_0, \ldots, \mathcal{A}_n$ and that the sparse resultant is an irreducible polynomial, i.e. $d_{\mathcal{A}} = 1$; see [GKZ94, Chapter 8].

Lemma 2.1. [DJS22, Proposition 3.2] Let $\phi : M \to M'$ be an injective morphism of lattices of rank n. Then, $\mathrm{Res}_{\phi(\mathcal{A})} = \mathrm{Res}_{\mathcal{A}}^{[M':\phi(M)]}$.

Remark 2.6. Moreover, the sparse resultant is invariant under translations. Therefore, we can always assume $0 \in \mathcal{A}_i$ for all $i = 0, \ldots, n$.

The degree of the resultant with respect to the coefficients of each equation must coincide with the maximal number of generic solutions of the system, which following Theorem 1, is the mixed volume.

Definition 2.36. The mixed volume of n polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$, denoted as

$$\mathrm{MV}_M(P_1, \ldots, P_n)$$

is the coefficient of $\prod_{i=1}^n \lambda_i$ in $\mathrm{Vol}_n(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ which is a polynomial in $\lambda_1, \ldots, \lambda_n$; see [CLO98, Theorem 6.7].

Proposition 2.2. [Stu94, Lemma 1.2] For $i = 0, \ldots, n$, the degree of the sparse resultant with respect to the coefficients of the i-th polynomial can be computed as:

$$\deg_{\mathcal{A}_i}(\mathrm{Res}_{\mathcal{A}}) = \sum_{J\subset\{0,\ldots,i-1,i+1,\ldots,n\}} (-1)^{|J|}\, \mathrm{Vol}(\sum_{j\in J} \Delta_j) = \mathrm{MV}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n).$$

The homogeneous case. Let $F_0, \ldots, F_n$ be homogeneous polynomials of degrees $\alpha_0, \ldots, \alpha_n \in \mathrm{Cl}(X_\Sigma)$ corresponding to the Newton polytopes $\Delta_0, \ldots, \Delta_n$. Using (2.21), these polynomials are of the form:

$$F_i = \sum_{m\in\mathcal{A}_i} c_{i,m} x^{\mathbf{F}m+\nu} \in C \quad i = 0, \ldots, n. \tag{2.33}$$

for $\mathcal{A}_i = \Delta_i \cap \mathbb{Z}^n$, i.e. these polynomials are general elements in the graded pieces of the Cox ring of degree α_i for $i = 0, \ldots, n$, i.e. S_{α_i}. By [Cox95, Proposition 1.1], these are global sections of the line bundles $\mathcal{O}_{X_\Sigma}(D_i)$ where D_i are nef Cartier divisors in the class of α_i for $i = 0, \dots, n$.

Using Assumption 2.2 and the notation of the previous section, we write:

$$\Delta_i = \{m \in \mathbb{R}^n \,:\, \langle u_j, m\rangle \geq -a_{i,j},\, j = 1, \dots, n+r\} \tag{2.34}$$

where $a_{i,j} = 0$ for $j = 1, \dots, n$ under Assumptions 2.2. To begin with, we can restrict to the case where $\mathcal{O}_{X_\Sigma}(D_i)$ are very ample line bundles, which means that the normal fan of Δ_i is Σ for $i = 0, \dots, n$. With this, we can define the *homogeneous incidence variety* locus of

$$Z(F) = \{(x, c_{i,m}) \in X_\Sigma \times \prod_{i=0}^{n} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_i)) \quad F_i(x) = 0 \quad i = 0, \dots, n\}$$

From [GKZ94, Chapter 3, Proposition 1.3], we can derive that the image of $Z(F)$ after the projection $p : X_\Sigma \times \prod_{i=0}^{n} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_i)) \to \prod_{i=0}^{n} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_i))$ is an irreducible hypersurface.

Definition 2.37. The *homogeneous sparse resultant* of $F_0, \dots, F_n$, which we denote as Res_D, is the unique irreducible polynomial defining the closure image of $Z(F)$ after p.

Notation 2.3. Note that the study of the toric variety associated with the resultant relates to using the Cox ring $C = A[x_1, \dots, x_n, z_1, \dots, z_r]$ where:

$$A = \mathbf{k}[c_{i,m} \quad m \in \mathcal{A}_i \quad i = 0, \dots, n].$$

In other words, we are studying a projection from the toric variety $X_\Sigma \times_{\mathbf{k}} \prod_{i=0}^{n} \mathbb{P}^{\mathcal{A}_i}$.

However, the assumption that the polytopes Δ_i correspond to very ample divisors can be to restrictive. The assumptions to overpass this restriction in [GKZ94, Chapter 8] are the that *the polytopes* Δ_i *span the vector space* $\mathbb{R}^n$.

Under these conditions, these polytopes correspond to nef divisors D_i and, as in (2.7), there is a map:

$$\Phi_{\mathcal{A}_i} : (\mathbb{C}^*)^n \to \mathbb{P}^{\mathcal{A}_i} \quad i = 0, \dots, n.$$

which, altogether, define a product map:

$$\Phi_{\mathcal{A}_0} \times \cdots \times \Phi_{\mathcal{A}_n} : (\mathbb{C}^*)^n \to \mathbb{P}^{\mathcal{A}_0} \times \cdots \times \mathbb{P}^{\mathcal{A}_0}$$

whose image defines a toric variety $X_{\Delta_{0,\dots,\Delta_n}}$, which is isomorphic to X_Δ for $\Delta = \Delta_0 + \cdots + \Delta_n$; see [GKZ94, Chapter 8, Proposition 1.4].

Now, the map $\Phi_{\mathcal{A}_0} \times \cdots \times \Phi_{\mathcal{A}_n}$ provides the following injective map:

$$H^0(\mathbb{P}^{\mathcal{A}_i}, \mathcal{O}(1)) \to H^0(X_\Delta, \mathcal{O}_{X_\Sigma}(D_i)) \tag{2.35}$$

which need not be surjective. It will be surjective if the variaety is *linearly normal*, which means that X_{Δ_i} is not contained in a hyperplane and it can not be represented as the projection from a higher dimensional projective space; see [GKZ94, Chapter 1, Definition 4.6].

Using the above maps, we can see that if the starting points are the polytopes whose span is $\mathbb{R}^n$, then the affine and homogeneous resultant will coincide.

Proposition 2.3. [GKZ94, Chapter 8, Proposition 1.5] Under the map in (2.35) and the assumptions above, the affine and homogeneous resultants coincide

Once this *affine versus homogeneous* situation that one encounters in the definition of resultants is clear, we can shift our attention to the methods of computation, which will occupy a big part of our time during this text. A classical method for computing the sparse resultant is to consider the determinant of the Koszul complex $K_\bullet(F)$ of the sequence of homogeneous polynomials $F_0, \dots, F_n$. Under these assumptions, we can compute $\mathrm{Res}_{\mathcal{A}}$ as the determinant of some graded pieces of the complex

$$K_\bullet(F) : K_{n+1} = C(-\sum \alpha_i) \xrightarrow{\partial_{n+1}} \dots \xrightarrow{\partial_3} K_2 = \oplus_{k,k'} C(-\alpha_k - \alpha_{k'}) \\ \xrightarrow{\partial_2} K_1 = \oplus_k C(-\alpha_k) \xrightarrow{\partial_1} C. \quad (2.36)$$

As a consequence, we get that the resultant can be computed as the determinant of some strands of this complex; see [GKZ94, Chapter 3, Theorem 4.2].

Theorem 2.11. There is a nonempty subset of $\mathrm{Pic}(X_\Sigma)$ such that for $\alpha \in \Gamma_{\mathrm{Res}} \subset \mathrm{Pic}(X_\Sigma)$, the strand $K_\bullet(F)_\alpha$ is an acyclic complex of free A-modules and $H^0(K_\bullet(F)_\alpha) = B_\alpha$. Moreover, if we also consider α such that $(I^{\mathrm{sat}}/I)_\alpha = 0$, then $\det(K_\bullet(F)_\alpha)$ equals the sparse resultant $\mathrm{Res}_{\mathcal{A}}$ up to multiplication by a nonzero scalar.

4. Generic initial ideals and the regularity criterion

In the introduction, we gave the definition of a Gröbner bases as useful tools in algebraic elimination; see the definition of Gröbner bases in Section 1. However, for the developments of the results in Chapter 5, we do not deal with directly Gröbner bases as much as with generic initial ideals. As we explained in the introduction, these are the Gröbner basis that appear after performing a generic change of coordinates that preserves the grading of an ideal.

Generic initial ideals For general toric varieties, it is not clear that these objects are well-defined. In fact, as shown by Maclagan and Smith in [MS04, Example 4.11], the grading might be too restrictive to allow changes of coordinates that alter the initial ideal. However, for standard gradings, these initial ideals exist, following the work of Galligo in [Gal74]. For the standard $\mathbb{Z}$-graded case, these ideals are very well-studied; see [Gre98].

The generic inital ideal exists for standard $\mathbb{Z}^n$-gradings, where the underlying toric variety is $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. The ideal appearing in this case is known as the *multigeneric initial ideal*. For the sake of simplicity, we stick to the bigraded case, i.e. for $\mathbb{P}^n \times \mathbb{P}^m$, and $R = \mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ and thus, to the the *bigeneric initial ideal*, denoted as $\mathrm{bigin}(I)$

The proof of the existence of $\mathrm{bigin}(I)$ of follows from the same lines in [Eis95, Proposition 15.12]. We aim to reproduce this proof in the coming pages. However, there are results in Chapter 5, in which using the generic coordinates is only required with respect to one group of variables. In the next theorem, we show that it is possible to consider the generic initial ideal $\mathrm{gin}_x(I)$ after performing only a change of coordinates with respect to the x block of variables and the results that are shown in the previous sections are preserved. Our proof follows the same lines as the proof of the existence of the generic initial ideal; see [Eis95, Proposition 15.18].

Notation 2.4. Let $u \in \mathrm{GL}(n+1) \times \mathrm{GL}(m+1)$ be block-diagonal matrix with nonzero determinant and two blocks u^x and u^y. This matrix defines a linear change of coordinates in S as:

$$u = (u^x, u^y) : R \to R \quad x_i \to u^x_{i0} x_0 + \cdots + u^x_{in} x_n \quad y_i \to u^y_{i0} y_0 + \cdots + u^y_{im} y_m. \tag{2.37}$$

For each homogeneous polynomial $f \in S$, we define the polynomial $u \circ f$ as $f(u(x, y))$, which has the same bi-degree as f. For each bihomogeneous ideal $I \subset S$, u defines the ideal $u \circ I = \langle u \circ f \mid f \in I \rangle$. However, in the next theorem the change of variables with respect to the second group of variables is fixed, and thus we consider $u = g \times \mathrm{id}$ for $g \in \mathrm{GL}(n+1)$.

Theorem 2.12. There is an open set $U \subset \mathrm{GL}(n+1)$ such that for $g \in U$, the ideal $\mathrm{in}((g \times \mathrm{id}) \circ I)$ is constant. Moreover, this ideal is preserved by linear changes of coordinates in $\mathrm{GL}(n+1)$.

Proof. Namely, consider $(g_{ij})_{i,j=0,\ldots,n} \in \mathrm{GL}(n+1)$ as a general transformation. Consider $f_1, \ldots, f_r$ to be the generators of I in degree (a, b). Consider $g \circ f := (g \times \mathrm{id}) \circ f_1 \wedge \cdots \wedge (g \times \mathrm{id}) \circ f_r$ as an element of $\wedge^r S_{(a,b)}$. Assume that $m_1 \wedge \cdots \wedge m_r$ be the highest monomial in $g \circ f$ and let $p(g_{ij})$ be the coefficient of this monomial. Consider $U_{(a,b)} \subset \mathrm{GL}(n+1)$ to be the open set given by $p(g_{ij}) \neq 0$. Consider $J_{(a,b)}$ to be the ideal generated by $m_1, \ldots, m_r$.

Next, we show that $J = \oplus_{(a,b) \in \mathbb{Z}^2} J_{(a,b)}$ is an ideal. As $U_{(a,b)}$ and $U_{(a+1,b)}$ are open and dense, we can find $g \in U_{(a,b)} \cap U_{(a+1,b)}$ such that $\mathrm{in}((g \times \mathrm{id}) \circ I)_{(a,b)} = J_{(a,b)}$ and $\mathrm{in}((g \times \mathrm{id}) \circ I)_{(a+1,b)} = J_{(a+1,b)}$. Thus, $S_{(1,0)} J_{(a,b)} \subset J_{(a+1,b)}$ (similarly, $S_{(0,1)} J_{(a,b)} \subset J_{(a,b+1)}$).

Finally, we show that $U = \cap_{(a,b)\in\mathbb{Z}^2} U_{(a,b)}$ is open and dense. Namely, we show U is a finite intersection. Let $\mathcal{J}$ be the set of bi-degrees (a,b) such that there is some generator of J of degree $\geq (a,b)$, this set is finite as J is an ideal. Consider $g_0 \in \cap_{(a,b)\in\mathcal{J}} U_{(a,b)}$. We know that $\text{in}((g_0 \times \text{id}) \circ I)_{(a,b)} = J_{(a,b)}$ for $(a,b) \in \mathcal{J}$. Therefore, $J \subset \text{in}(g \circ I)$. Moreover,

$$\dim_{\mathbf{k}} J_{(a,b)} = \dim_{\mathbf{k}} I_d = \dim_{\mathbf{k}} (g_0 I)_{(a,b)} \quad (a,b) \in \mathbb{Z}^2_{\gtrsim 0}$$

implying that $\text{in}((g_0 \times \text{id}) \circ I) = J$.

Claim 1: The strategy for showing that J is preserved by changes of coordinates in $\text{GL}(n+1)$ is noting that if $J_{(a,b)}$ is a vector space of dimension t, then the vector space $\wedge^t J_{(a,b)}$ of dimension 1 is spanned by the greatest monomial appearing in $\wedge^t \text{in}((g \times \text{id}) \circ I)_{(a,b)}$ for all $g \in \text{GL}(n+1)$.

The above claim is proved by showing that if g is lower triangular with one nonzero entry g_{ij} for $i < j$ then:

$$\wedge^t J_{(a,b)} = \wedge^t \text{in}((g \times \text{id}) \circ I)_{(a,b)}.$$

If $f_1, \ldots, f_t$ are generators of $J_{(a,b)}$, we consider $m_i = \text{in}(f_i)$ for $i = 1, \ldots, t$ assuming that $m_1 > \cdots > m_t$. Assume that g is strictly upper triangluar, implying that if $n \in S_{(a,b)}$, we write $n = x_i^{\omega} m$ for m not divided by x_i. The monomials appearing in $g \circ f$ are of the form $x_i^{\omega-s} x_j^s m$ for $0 \leq s \leq \omega$ implying that $\text{in}((g \times \text{id}) \circ f) = \text{in}(f)$ and so $\text{in}((g \times \text{id}) \circ f_1 \wedge \cdots \wedge (g \times \text{id}) \circ f_t) = \text{in}(f_1 \wedge \cdots \wedge f_t)$.

Finally, we show that the ideal J is preserved by lower triangular matrices. For simplicity, we assume that $J = \text{in}(I)$. In particular, we will show that if g is a lower triangular matrix γ with one nonzero entry $g_{ij} \neq 0$ for $i > j$, we have:

$$(1+\gamma) \circ \text{in}(I)_{(a,b)} = \text{in}(I)_{(a,b)}. \tag{2.38}$$

We prove this by considering a basis $f_1, \ldots, f_t$ of $S_{(a,b)}$ and consider $f = f_1 \wedge \cdots \wedge f_t$. If we assume that (2.38) does not happen, then we have $(1+\gamma) \circ \text{in}(f) \neq \text{in}(f)$. As γ is lower triangular, all the terms of $(1+\gamma) \circ \text{in}(f)$ are strictly bigger than $\text{in}(f)$. Therefore, we can get a contradiction with Claim 1 if we consider one of these monomials m and find a diagonal matrix δ such that m appears with nonzero coefficient in $(1+\gamma) \circ \delta \circ f$.

Consider the weight of a monomial $n = n_1 \wedge \cdots \wedge n_t \in \wedge^t S_{(a,b)}$ to be the monomial $n_1 \cdots n_t \in S$. The polynomial f can be summed as $f = \sum_{\omega} f_{\omega}$ where f_{ω} is the sum of the terms of weight $\omega \in S$. Each of these terms can be a sum of different monomials except for the term f_{ω_0} associated to $\text{in}(f)$. Consider the action of a diagonal matrix δ. Then, the result in:

$$(1+\gamma) \circ \delta \circ f = \omega_0(\delta_1, \ldots, \delta_n)(1+\gamma) \circ \text{in}(f) + \sum_{\omega \neq \omega_0} \omega(\delta_1, \ldots, \delta_n)(1+\gamma) \circ f_{\omega}$$

The coefficient of m in $(1+\gamma)\circ\delta\circ\text{in}(f)$ is:

$$a_0\omega_0(\delta_1,\dots,\delta_n)+\sum_{\omega\neq\omega_0}a_\omega\omega(\delta_1,\dots,\delta_n).$$

This is a nonzero polynomial, so as the field is infinite, we can find a diagonal matrix δ such that the above polynomial is nonzero. $\square$

The same proof as above but applying the change of coordinates to two groups of variables, provides the bigeneric initial ideal, denoted as $\text{bigin}(I)$. In the case that we only perform the generic change of coordinates with respect to the x block of variables (resp the y block), we denote the corresponding monomial ideal as $\text{gin}_x(I)$ (resp. $\text{gin}_y(I)$). In the next example, we can show that $\text{gin}_x(I)$ and $\text{bigin}(I)$ need not coincide.

Example 2.6. The ideal $\text{gin}_x(I)$ in the previous Theorem need not coincide with the generic initial ideal $\text{gin}(I)$. The example in $\mathbb{P}^1\times\mathbb{P}^1$ (i.e., in the ring $\mathbb{C}[x_0,x_1,y_0,y_1]$) is $I=(x_0y_1-x_1y_0, x_0y_0y_1+x_1y_1^2)$. The classical bi-generic initial ideal is

$$\text{bigin}(I)=(x_1y_1,x_1y_0^2,x_0y_1^3)$$

while the generic initial ideal $\text{gin}_x(I)$ is $(x_1y_1,x_1y_0^2,x_0y_0y_1^2)$.

Bi-generic initial ideals preserve many of the interesting properties of the classical generic initial ideals; see [Gre98]. In particular, the property that we will use in the coming sections is the following; see [BS87a, Proposition 2.7] for a proof.

Lemma 2.2. Let **k** be a field of characteristic 0. Then, $\text{bigin}(I)$ has the following two properties:

- If $x_ix^\alpha y^\beta\in\text{bigin}(I)$, then $x_jx^\alpha y^\beta\in\text{bigin}(I)$ for all $j\in\{i,\dots,n\}$.
- If $y_ix^\alpha y^\beta\in\text{bigin}(I)$, then $y_jx^\alpha y^\beta\in\text{bigin}(I)$ for all $j\in\{i,\dots,m\}$.

Using $\text{gin}_x(I)$ or $\text{gin}_y(I)$, one can recover the properties of the above lemma, only with respect to each group of variables. The property is also known as ***bi-Borel fixed*** and can also be relevant from the point of view of monomial ideals [BGC13].

The Bayer-Stillman criterion To end this section, we give a bit more of detail on the proof of the results by Bayer and Stillman [BS87a], which in Chapter 5 we will try to extend to the bigraded setting. Assume that I is a homogeneous ideal in a $\mathbb{Z}$-graded ring $S=\mathbb{C}[x_0,\dots,x_n]$. To highlight the problem, we recall that Bayer and Stillman showed that using the degree reverse lexicographical monomial order and generic coordinates, the following equaility holds:

$$\max\{\text{degrees of the minimal generators of } \text{gin}(I)\}=\text{reg}(\text{gin}(I))=\text{reg}(I). \quad (2.39)$$

where $\text{reg}(I)$ is the Castelnuovo-Mumford regularity as defined in 2.23. We can remark the importance of using generic coordinates with the following example.

Example 2.7. Consider the ideal $I = \langle x_2^2 - x_0^2, x_2x_1 + x_0^2\rangle$ in the polynomial ring $\mathbb{C}[x_0, x_1, x_2]$. If compute a Gröbner basis of I, using the degree reverse lexicographic monomial order with $x_0 < x_1 < x_2$, then in(I) is generated by $\langle x_2^2, x_1x_2, x_1^2x_0^2\rangle$. However, if we first perform a generic change of coordinates, then gin(I) is generated by $\langle x_2^2, x_2x_1, x_1^3\rangle$. Therefore, we see in Figure 2.6, the degree of the computations is reduced to only depend on reg(I).

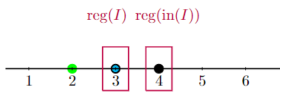


Figure 2.6: The Gröbner basis of I is generated in degree 4 which, in this case, coincides with the regularity of in(I). After the change of coordinates, Bayer and Stillman's result implies that gin(I) is generated in degree 3, which coincides with the reguilarity of I.

Remark 2.7. Recall that reg(in(I)) is only a bound for the degrees of the generators of the Gröbner bases. It is possible to find ideals for which these generators have degree lower than reg(I). However, this analysis requires understanding more structure of the ideal than only its Castelnuovo-Mumford regularity and the degrees of its generators. The study of Bayer and Stillman is, since their prominent work in the eighties, the most general answer that has been provided to this question.

In order to show the equalities in (2.39), they showed that the Castelnuovo-Mumford regularity in Definition 2.23 can be characterized using the following theorem; see [BS87a, Theorem 1.10].

Theorem 2.13. Let d be the dimension of $V_{\mathbb{P}^n}(I)$. Then, the following are equivalent:

- $m \in$ reg(I).
- For d general linear forms $h_1, \ldots, h_d \in S_1$, we have:

$$(I, h_1, \ldots, h_{j-1} : h_j)_{m'} = (I, h_1, \ldots, h_{j-1})_{m'} \quad j = 1, \ldots, d \quad m' \geq m$$

  and

$$(I, h_1, \ldots, h_d)_{m'} = S_{m'} \quad m' \geq m$$

The idea of the proof of this criterion begins by noting that, in the case that $V_{\mathbb{P}^n}(I) = \emptyset$, the regularity is only determined by the degree of saturation, i.e.

$$\mathrm{HF}_{S/I}(m) = 0 \iff \dim_{\mathbb{C}}(I^{\mathsf{sat}}/I)_m = 0 \iff m \in \mathrm{reg}(I).$$

Moreover, a general linear form in S is not a zero divisor in the quotient ring S/I^{sat}. Thus, one can deduce that these forms $h \in S_1$ will satisfy:

$$(I : h)_{m'} = I_{m'} \iff I^{\mathsf{sat}}_{m'} = I_{m'} \quad \forall m' \geq m$$

In particular, if I is m-regular, then one can use this to compare the regularity of I with the regularity of (I,h) and proceed by induction with the proof of the above criterion; see [BS87a, §1].

In particular, after a generic change of coordinates, one can assume that the general linear forms are the coordinate variables $x_0,\dots,x_n$, implying that:

$$m \in \operatorname{reg}(I) \iff (g\circ I, x_0,\dots,x_{j-1} : x_j)_{m'} = (g\circ I, x_0,\dots,x_{j-1})_{m'} \quad j=1,\dots,d \quad m' \geq m$$

for a general linear change of coordinates $g \in \mathrm{GL}(n+1)$. Here is where the DRL order (see the Definition in Chapter 1) plays a relevant role. The main property of the degree reverse lexicographical monomial order that we need to use is that if x_0 divides a monomial x^α, then it also divides every monomial $x^{\alpha'} < x^\alpha$. In particular,

$$\text{if } x_0 \text{ divides } \operatorname{in}(f) \to x_0 \text{ divides } f. \tag{2.40}$$

This very simple property of the DRL monomial order, allowed Bayer and Stillman to also prove that the colon property above, behaves well under considering initial ideals, i.e.

$$(\operatorname{gin}(I), x_0,\dots,x_{j-1} : x_j)_{m'} = (\operatorname{gin}(I), x_0,\dots,x_{j-1})_{m'} \quad j=1,\dots,d \quad m' \geq m$$

$$\iff (g\circ I, x_0,\dots,x_{j-1} : x_j)_{m'} = (g\circ I, x_0,\dots,x_{j-1})_{m'} \quad j=1,\dots,d \quad m' \geq m$$

for a general linear change of coordinates $g \in \mathrm{GL}(n+1)$. Finally, using the Borel-fixed property of gin(I), one can show that its generators must have degree $\leq m$, deriving that this degree is precisely reg(I). More knowledge on the structure of generic initial ideals can be derived by using similar properties; see [BG06; Has12].

5. Multigraded Castelnuovo-Mumford regularity

The goal of the developments of Chapter 5 is to relate the generators of the multigraded generic initial ideals with the multigraded version of the Castelnuovo-Mumford regularity. As a last part of this preliminary section, we review the main aspects of the study of the multigraded Castelnuovo-Mumford regularity, the supports of local cohomology and the multigraded Betti numbers that we will use in Chapter 5. For the sake of simplicity, our results will be proved in the (standard) bigraded case, even though they also hold in the (standard) multigraded setting.

Notation 2.5. Let $\mathbf{k}$ be a field of characteristic 0. Let $S = \mathbf{k}[x_0,\dots,x_n,y_0,\dots,y_m]$ be a ring with a (standard) $\mathbb{Z}^2$-grading, such that $\deg(x_i) = (1,0)$ and $\deg(y_j) = (0,1)$. We will write the monomials in S as $x^\alpha y^\beta = x_0^{\alpha_0}\cdots x_n^{\alpha_n} y_0^{\beta_0}\cdots y_m^{\beta_m}$ for a vector $(\alpha,\beta) \in \mathbb{Z}^{n+m+2}$. A monomial $x^\alpha y^\beta$ has degree (a,b) if $\sum_{i=0}^n \alpha_i = a$ and $\sum_{j=0}^m \beta_j = b$. Let $\mathfrak{m}_x$ (resp. $\mathfrak{m}_y$) be the ideal generated by the x (resp. y) variables. The ambient biprojective space is $\mathbb{P}^n \times \mathbb{P}^m$ and the irrelevant ideal is $\mathfrak{b} = \mathfrak{m}_x\mathfrak{m}_y$.

In Chapter 4, where the main tool to compute the supports of local cohomology modules are the theorems of Demazure (Theorem 2.9), Batyrev-Borisov (Theorem 2.10) and Serre duality (Theorem 2.25). In the case of Chapter 5, the main idea is to reduce the computations of local cohomology with respect to $\mathfrak{b}$ to the local cohomology modules over $\mathfrak{m}_x$ and $\mathfrak{m}_y$. To state these results, we need first to introduce the following notation.

Definition 2.38. Let $E \subset \mathbb{Z}^2$ be a subset. The subset $E^\star$ is defined as:

$$E^\star = \{(a,b) \in \mathbb{Z}^2 \quad \exists (a',b') \geq (a,b) \quad (a',b') \in E\}$$

where $(a',b') \geq (a,b)$ denotes (component-wise) $a' \geq a$ and $b' \geq b$.

In particular, we are interested in the case where E are the supports of the local cohomology modules with respect to a homogeneous ideal J (for instance, with respect to $\mathfrak{b}, \mathfrak{m}_x$ or $\mathfrak{m}_y$).

Definition 2.39. Let $J \subset R$ be a homogeneous ideal. The supports of the local cohomology modules with respect to J are the bi-degrees $(a,b) \in \mathbb{Z}^2$ such that there is $i \geq 1$ with $H^i_J(I)_{(a,b)} \neq 0$, i.e.

$$\operatorname{Supp}_{\mathbb{Z}^2}(H^\bullet_J(I)) = \{(a,b) \in \mathbb{Z}^2 \quad \exists i \geq 1 \quad H^i_J(I)_{(a,b)} \neq 0\}. \tag{2.41}$$

The following theorem relates the supports of the local cohomology modules with respect to $\mathfrak{b}$ with the supports of the local cohomology modules with respect to $\mathfrak{m}_x$ and $\mathfrak{m}_y$.

Theorem 2.14. [CH22, Theorem 3.11] Let $I \subset R$ be a bihomogeneous ideal, then:

$$\operatorname{Supp}_{\mathbb{Z}^2}(H^\bullet_{\mathfrak{b}}(I))^\star = \operatorname{Supp}_{\mathbb{Z}^2}(H^\bullet_{\mathfrak{m}_x}(I))^\star \cup \operatorname{Supp}_{\mathbb{Z}^2}(H^\bullet_{\mathfrak{m}_y}(I))^\star.$$

Remark 2.8. If $I = R$, then it is easier to obtain the supports of the local cohomology with respect to $\mathfrak{m}_x$, that is:

$$\operatorname{Supp}_{\mathbb{Z}^2}(H^\bullet_{\mathfrak{m}_x}(R)) = (-n-1, 0) + (-\mathbb{N} \times \mathbb{N}) \tag{2.42}$$

and, similarly, for the supports of local cohomology with respect to $\mathfrak{m}_y$; see [CH22, Example 2.3].

Another aspect which was analyzed by Chardin and Holanda is the relation of local cohomology with truncated ideals.

Definition 2.40. For any bihomogeneous ideal $I \subset R$ and $(a,b) \in \mathbb{Z}^2$, consider the truncated ideal $I_{\geq(a,b)} = \oplus_{(a',b')\geq(a,b)} I_{(a',b')}$.

The next lemma relates the local cohomology with respect to $\mathfrak{m}_x$ of the truncated modules $I_{\geq(a,b)}$ with the local cohomology of I.

Lemma 2.3. [CH22, Proposition 4.4] Let $I \subset R$ be a bihomogeneous ideal and $(a,b) \in \mathbb{Z}^2$, then:

i) For all $i \geq 0$, if $(a',b') \geq (a,b)$, then:
$$H^i_{\mathfrak{m}_x}(I_{\geq(a,b)})_{(a',b')} = H^i_{\mathfrak{m}_x}(I)_{(a',b')}.$$

ii) For all $i \geq 2$, then:
$$H^i_{\mathfrak{m}_x}(I_{\geq(a,b)}) = H^i_{\mathfrak{m}_x}(I).$$

Once these tools are established, we can give the following definition of the multigraded Castelnuovo-Mumford regularity, which was established by Maclagan and Smith; [MS04, Definition 1.1].

Definition 2.41. Consider a bihomogeneous ideal $I \subset S$. The bigraded Castelnuovo-Mumford regularity reg(I) is the subset of $\mathbb{Z}^2$ containing bi-degrees (a,b) such that, for all $i \geq 1$ and for all $(a',b') \geq (a-\lambda_x, b-\lambda_y)$, it holds
$$H^i_{\mathfrak{b}}(I)_{(a',b')} = 0,$$
where $\lambda_x + \lambda_y = i-1$, with $\lambda_x, \lambda_y \in \mathbb{Z}_{\geq 0}$.

This definition of regularity preserves some of the classical properties of te Castel-nuovo-Mumford regularity: it bounds the degrees of the equations that cut out the variety defined by I. Moreover, Bruce, Cranton-Heller and Sayrafi proved that $(a,b) \in$ reg(I), if and only if, the truncated ideal $I_{\geq(a,b)}$ has a *quasi-linear* resolution; [BHS21, Theorem A]. Another important feature of the bigraded Castelnuovo-Mumford regularity is its relation with the generators of I.

Theorem 2.15 ([MS04, Theorem 1.3]). Let $I \subset S$ be a multihomogeneous ideal. If $(a,b) \in$ reg(I) then, for all $(a',b') \gneq (a,b)$ there are no minimal generators of I of degree (a',b').

Remark 2.9. Note that Theorem 2.15 implies that reg(bigin(I)) provides an upper bound for the bidegrees generators of bigin(I). Moreover, along the same lines as in [MS04, Proposition 3.16], we can prove that the following inclusion holds:
$$\text{reg}(\text{bigin}(I)) \subset \text{reg}(I). \tag{2.43}$$
However, as we will see in Example 5.4, these two regions will, in general, differ.

Another classical definition of the Castelnuovo-Mumford regularity (in the single-graded case) comes from the Betti numbers.

Definition 2.42. The minimal free resolution of I is of the form
$$0 \to \bigoplus_{(a,b)\in\mathbb{Z}^2} S(-a,-b)^{\beta_{r,(a,b)}(I)} \to \cdots \bigoplus_{(a,b)\in\mathbb{Z}^2} S(-a,-b)^{\beta_{0,(a,b)}(I)} \to I \to 0.$$

Here, $S(-a,-b)$ denotes a shift in the grading, namely $S(-(a,b))_{(a',b')} = S_{(a'-a,b'-b)}$ for $(a',b'),(a,b) \in \mathbb{Z}^2$. We also denote $\beta_i(I) = \{(a,b) \in \mathbb{Z}^2 \quad \beta_{i,(a,b)} \neq 0\}$.

There have been several attempts to describe the bi-degrees appearing in Definition 2.41 in terms of the Betti numbers, with some relevant relations between the two descriptions; see [BC17; BHS21; CH22]. In the later discussions, we will use the following relation, established by Chardin and Holanda.

Theorem 2.16. [CH22, Theorem 1.2] Let I be a bihomogeneous ideal, then,

$$\cup_i \beta_i(I)^\star \subset (n+1, m+1) + \big(\operatorname{Supp}_{\mathbb{Z}^2}(H^\bullet_{\mathfrak{m}_x}(I))^\star \cap \operatorname{Supp}_{\mathbb{Z}^2}(H^\bullet_{\mathfrak{m}_x}(I))^\star\big).$$

Chapter 3

The Canny-Emiris formula

In this chapter, we describe the Canny-Emiris formula [CE93], as one of the main sources for formulas for the sparse resultant. The formula is based in the use of mixed subdivisions of the Minkowski sum of the Newton polytopes. A proof of this formula was provided by D'Andrea, Jerónimo and Sombra in [DJS22] under some conditions in the mixed subdivision, which we will also explain. Therefore, our main goal with this work was to provide mixed subdivisions that $i)$ satisfy these conditions and $ii)$ the size of these matrices can be reduced using the greedy algorithm of Canny and Pedersen [CP93] for the case of n-zonotopes and multihomogeneous systems. We end the chapter with a conjecture on the existence of resultant formulas of Canny-Emiris type.

1. Mixed subdivisions and the Canny-Emiris formula

Definition 3.1. Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope. A mixed subdivision of Δ is a decomposition of into a union of polyhedral cells $\Delta = \cup D$ such that:

i) the intersection of two cells is either a cell or empty,

ii) every face of a cell is also a cell of the subdivision and,

iii) every cell D has a component structure $D = D_0 + \cdots + D_n$ where D_i is a cell of the subdivision in Δ_i.

The usual way to construct mixed subdivisions is by considering piecewise affine convex lifting functions $\rho_i : \Delta_i \to \mathbb{R}$ as explained in [GKZ94]. A global lifting function $\rho : \Delta \to \mathbb{R}$ is obtained after taking the inf-convolution of the previous functions, as explained in [DJS22, Section 2]. The graph of this function can be projected to $\mathbb{R}^n$, providing a mixed subdivision, which we denote as $S(\rho)$; see Figure 3.1.

Definition 3.2. A mixed subdivision of Δ is *tight* if, for every n-cell D, its components satisfy:

$$\sum_{i=0}^{n} \dim D_i = n.$$

In the case of $n+1$ polynomials and n variables, this property guarantees that every n-cell has a component that is 0-dimensional. The cells that have a single 0-dimensional component are called *mixed* (i-mixed if it is the i-th component). The rest of the cells are called *non-mixed*.

Let δ be a generic vector such that the lattice points in the interior of $\Delta + \delta$ lie in n-cells. Then, consider:

$$\mathcal{B} = (\Delta + \delta) \cap \mathbb{Z}^n.$$

Each element $b \in \mathcal{B}$ lies in one of these translated cells $D + \delta$ and let D_i be the components of this cell. As the subdivision is tight, there is at least one i such that $\dim D_i = 0$.

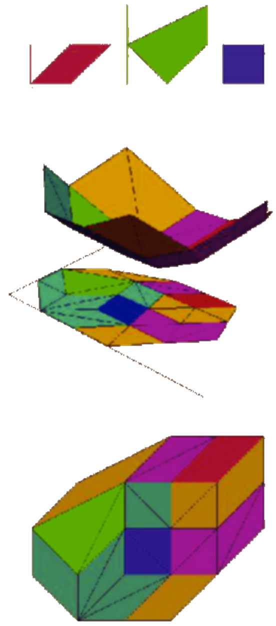


Figure 3.1: The usual way to construct mixed subdivisions is considering piecewise affine convex lifting functions $\rho_i : \Delta_i \to \mathbb{R}$. Then, take the Minkowski sum of their graphs of these functions as polytopes in $\mathbb{R}^{n+1}$. The image source is the book [DE05].

Following the language of [MC00], we call $t_b = (t_{b,0}, \ldots, t_{b,n})$ the *type vector* associated with b, defined as $t_{b,i} = \dim D_i$ for $b \in D + \delta$.

Definition 3.3. The row content is a function

$$\mathrm{rc} : \mathcal{B} \to \cup_{i=0}^{n} \{i\} \times \mathcal{A}_i$$

where, for $b \in \mathcal{B}$ lying in an n-cell D, $\mathrm{rc}(b)$ is a pair $(i(b), a(b))$ with

$$i(b) = \max\{i \in \{0, \ldots, n\} \mid t_{b,i} = 0\} \quad a(b) = D_{i(b)}.$$

This provides a partition of $\mathcal{B}$ into subsets:

$$\mathcal{B}_i = \{b \in \mathcal{B} \quad | \quad i(b) = i\}.$$

Finally, we construct the *Canny-Emiris matrices* $\mathcal{H}_{\mathcal{A},\rho}$ whose rows correspond to the coefficients of the polynomials $\chi^{b-a(b)} F_{i(b)}$ for each of the $b \in \mathcal{B}$. In particular, the entry corresponding to a pair $b, b' \in \mathcal{B}$ is:

$$\mathcal{H}_{\mathcal{A},\rho}[b, b'] = \begin{cases} u_{i(b), b'-b+a(b)} & b' - b + a(b) \in \mathcal{A}_i \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.1. Each entry contains, at most, a single coefficient $u_{i,a}$. In particular, the row content allows us to choose a maximal submatrix of $\mathcal{H}_{\mathcal{A},\rho}$ from the matrix of the map sending a tuple of polynomials $(G_0, \ldots, G_n)$ to $G_0F_0 + \cdots + G_nF_n$ as in (1.5). This class of matrices are called *Sylvester-type* matrices.

Let $\mathcal{C} \subset \mathcal{B}$ be a subset of the supports in translated cells. The matrix $\mathcal{H}_{\mathcal{A},\rho,\mathcal{C}}$ is defined by considering the submatrix of the corresponding rows and columns associated with elements in $\mathcal{C}$. In particular, we look at the set of lattice points lying in translated non-mixed cells and consider:

$$\mathcal{B}^\circ = \{b \in \mathcal{B} \mid b \text{ lies in a translated non-mixed cell}\}.$$

With this, we form the principal submatrix:

$$\mathcal{E}_{\mathcal{A},\rho} = \mathcal{H}_{\mathcal{A},\rho,\mathcal{B}^\circ}.$$

The Canny-Emiris formula computes the sparse resultant is the quotient of the determinants of these two matrices:

$$\operatorname{Res}_{\mathcal{A}} = \frac{\det(\mathcal{H}_{\mathcal{A},\rho})}{\det(\mathcal{E}_{\mathcal{A},\rho})}.$$

This result was conjectured by Canny and Emiris [CE95] and proved by D'Andrea, Jerónimo and Sombra [DJS22] under the restriction that the mixed subdivision $S(\rho)$ given by the lifting ρ satisfies a certain condition, given on a chain of mixed subdivisions.

Definition 3.4. Let $S(\phi), S(\psi)$ be two mixed subdivisions of $\Delta = \sum_{n=0}^{n} \Delta_i$. We say that $S(\psi)$ refines $S(\phi)$ and write $S(\phi) \preceq S(\psi)$ if for every cell $C \in S(\psi)$ there is a cell $D \in S(\phi)$ such that $C \subset D$. An *incremental chain* of mixed subdivisions $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ is a chain of mixed subdivisions of Δ refining each other.

Remark 3.2. In [DJS22, Definition 2.4], a common lifting function $\omega \in \prod_{i=0}^{n} \mathbb{R}^{\mathcal{A}_i}$ is considered and the $S(\theta_i)$ are given by the lifting functions $\omega^{<i} = (\omega_0, \ldots, \omega_{i-1}, 0)$ as long as $S(\theta_i) \preceq S(\theta_{i+1})$. The last zero represents the lifting on $(\Delta_i, \ldots, \Delta_n)$. The resulting mixed subdivision is the same as if we considered the zero lifting in $\sum_{j=i}^{n} \Delta_j$.

Given a tight mixed subdivision $S(\rho)$, we can compute the mixed volume of $\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n$ by considering the volume of the i-mixed cells.

Proposition 3.1. [ER94, Theorem 3.4] Let $S(\rho)$ be a tight mixed subdivision of $\Delta = (\Delta_0, \ldots, \Delta_n)$. For $i = 0, \ldots, n$, the mixed volume of all the polytopes except Δ_i equals the volume of the i-mixed cells.

$$\operatorname{MV}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) = \sum_{D \text{ } i\text{-mixed}} \operatorname{Vol}_n D$$

In particular, $\operatorname{MV}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n)$ equals the degree of the sparse resultant in the coefficients of F_i; see [CLO98, Chapter 7, Theorem 6.3]. Each of the rows of $\mathcal{H}_{\mathcal{A},\rho}$ will correspond to a lattice point b and each entry on that row will have degree 1 with respect to the coefficients of $F_{i(b)}$ (more concretely, it will be one of its coefficients) and zero with respect to the coefficients of the rest of polynomials. Therefore, if we add the lattice points in i-mixed cells, the degree of $\mathcal{H}_{\mathcal{A},\rho}$ with respect to the coefficients of F_i will be at least the degree of the resultant with respect to the same coefficients.

Remark 3.3. Using Proposition 3.1, we can see that if the fundamental subfamily is empty, then the resultant is equal to 1 while if the fundamental subfamily is $\{i\}$ then $\mathcal{A}_i$ is given by a single point $\{a\}$ and the resultant is $u_{i,a}^{m_i}$ for $m_i = \mathrm{MV}(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)$. The Canny-Emiris formula holds [DJS22, Proposition 4.26] in both cases.

Definition 3.5. An incremental chain $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ is *admissible* if for each $i = 0, \dots, n$, each n-cell D of the subdivision $S(\theta_i)$ satisfies either of the following two conditions

i) the fundamental subfamily of $\mathcal{A}_D$ contains at most one support or

ii) $\mathcal{B}_{D,i}$ is contained in the union of the translated i-mixed cells of $S(\rho_D)$. Here, $\mathcal{B}_{D,i}$ is formed by the lattice points in D with row content i.

A mixed subdivision $S(\rho)$ is called *admissible* if it admits an admissible incremental chain $S(\theta_0) \preceq \cdots \preceq S(\theta_n) \preceq S(\rho)$ refining it.

With all these properties, together with the use of the product formulas, one can reproduce the proof of the Canny-Emiris formula given in [DJS22, Theorem 4.27] under the conditions of admissibility in $S(\rho)$; see also [DS15; Stu94].

2. The greedy algorithm

If the set $\mathcal{B}$ only contains the lattice points that lie in mixed cells, then one can recover an exact determinantal formula for $\mathrm{Res}_{\mathcal{A}}$. However, as we explained in the introduction, this is not usually the case. On the other hand, different algorithms for the construction of the Canny-Emiris matrices $\mathcal{H}_{\mathcal{A},\rho}$ can be employed, providing more compact representations.

The first of these algorithms, usually called *incremental* algorithm, was proposed by Canny and Emiris in [CE95]. In this case, they tried to add the lattice points that appeared in one direction, until the degree of the resultant was achieved. Their algorithm was able to recover some of the existing determinantal formulas for multihomogeneous supports. However, their algorithm did not take into account the mixed subdivision in the cosntruction, so it is difficult to make sure that the conditions on the proof of the Canny-Emiris formula were satisfied. Similarly, Canny and Pedersen [CP93] proposed a *greedy* algorithm for the construction of the matrix, which we can describe as follows. Let $b \in \mathcal{B}$ be a lattice point in a translated cell. The first step of the algorithm is to add the row of the matrix corresponding to b, and then continue by considering the lattice points corresponding to the columns that have a nonzero entry in this row. These lattice points are:

$$b - a(b) + \mathcal{A}_{i(b)}.$$

All these lattice points will have to be added as rows of the matrix. If we add the lattice point b' at some point of the algorithm after having added another lattice point b, we say that we *reach* b' from b. The algorithm terminates when there are no more lattice points to add and it might give a square matrix $\mathcal{H}_{\mathcal{G}}$ which has less rows and columns than $\mathcal{H}_{\mathcal{A},\rho}$, which was constructed using all the lattice points in $\mathcal{B}$. The rows and columns associated to lattice points in non-mixed cells also provide a minor $\mathcal{E}_{\mathcal{G}}$ of $\mathcal{H}_{\mathcal{G}}$.

It was not proved by Canny and Pedersen whether this approach would always include all the lattice points in mixed cells as rows of the matrix, independently of the starting point. As these points are necessary to achieve the degree of the resultant, we consider them to be the starting lattice points of the algorithm.

Remark 3.4. We know that the entry corresponding to the diagonal of the matrix $\mathcal{H}_{\mathcal{A},\rho,\mathcal{C}}$ will be $\prod_{b\in\mathcal{C}} u_{i(b),a(b)}$ for any subset $\mathcal{C} \subset \mathcal{B}$. This term can be used in order to deduce that these matrices have non-zero determinant; see [DJS22, Proposition 4.13].

Theorem 3.1. If the Canny-Emiris formula holds for a mixed subdivision $S(\rho)$ and the greedy algorithm provides matrices $\mathcal{H}_{\mathcal{G}}$ and $\mathcal{E}_{\mathcal{G}}$ by starting at the lattice points in mixed cells, then:

$$\mathrm{Res}_{\mathcal{A}} = \frac{\det(\mathcal{H}_{\mathcal{G}})}{\det(\mathcal{E}_{\mathcal{G}})}.$$

Proof. In general, there is a subset $\mathcal{G} \subset \mathcal{B}$ corresponding to the rows and columns of $\mathcal{H}_{\mathcal{G}}$. We are assuming that $\mathcal{G}$ contains all the lattice points in translated mixed cells. Let $\mathcal{H}_{\mathcal{A},\rho}$ be the matrix containing all lattice points in translated cells of $S(\rho)$. Without loss of generality, we can assume that the matrix takes the following form:

$$\mathcal{H}_{\mathcal{A},\rho} = \begin{pmatrix} \mathcal{H}_{\mathcal{G}} & 0 \\ \bullet & \mathcal{H}_{\mathcal{B}-\mathcal{G}} \end{pmatrix}$$

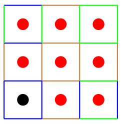
where $\mathcal{H}_{\mathcal{G}}$ is the minor corresponding to the lattice points in $\mathcal{G}$ and $\mathcal{H}_{\mathcal{B}-\mathcal{G}}$ is the minor corresponding to the lattice points not in $\mathcal{G}$. The zeros appear due to the fact that there is no pair $b \notin \mathcal{G}, b' \in \mathcal{G}$ such that $b \in b' - a(b') + \mathcal{A}_{i(b)}$. The same block-triangular structure also appears in the principal submatrix $\mathcal{E}_{\mathcal{A},\rho}$ and all the lattice points that are not in $\mathcal{G}$ must be non-mixed, implying that $\mathcal{E}_{\mathcal{B}-\mathcal{G}} = \mathcal{H}_{\mathcal{B}-\mathcal{G}}$.

Finally, using the fact that the determinant of a block-triangular matrix is the product of the determinants of the diagonal blocks, we can prove the resultant formula:

$$\mathrm{Res}_{\mathcal{A}} = \frac{\det(\mathcal{H}_{\mathcal{A},\rho})}{\det(\mathcal{E}_{\mathcal{A},\rho})} = \frac{\det(\mathcal{H}_{\mathcal{G}})\cdot\det(\mathcal{H}_{\mathcal{B}-\mathcal{G}})}{\det(\mathcal{E}_{\mathcal{G}})\cdot\det(\mathcal{H}_{\mathcal{B}-\mathcal{G}})} = \frac{\det(\mathcal{H}_{\mathcal{G}})}{\det(\mathcal{E}_{\mathcal{G}})}.$$

$\square$

Example 3.1. Let f_0, f_1, f_2 be three bilinear equations corresponding to the supports $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$. A possible mixed subdivision $S(\rho)$ is the following:



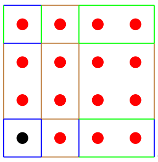
where the dots indicate the lattice points in translated mixed cells. The number of lattice points in translated cells is 9. However, if we construct the matrix greedily starting from the lattice points in translated mixed cells, we have an 8×8 matrix.

Example 3.2. Let f_0, f_1, f_2 be three bihomogeneous equations with supports

$$\mathcal{A}_0 = \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1)\},$$

$$\mathcal{A}_1 = \{(0,0),(1,0),(0,1),(1,1),(0,2),(1,2)\},\ \mathcal{A}_2 = \{(0,0),(1,0),(0,1),(0,1)\}.$$

The expected number of supports lying in translated cells is 16. Let $\rho_0 = (0,3,6,3,6,9)$, $\rho_1 = (0,2,2,4,4,6)$ and $\rho_2 = (0,1,1,2)$ be the lifting functions and $\delta = (-\frac{1}{2}, \frac{1}{2})$ give the following mixed subdivision:

.

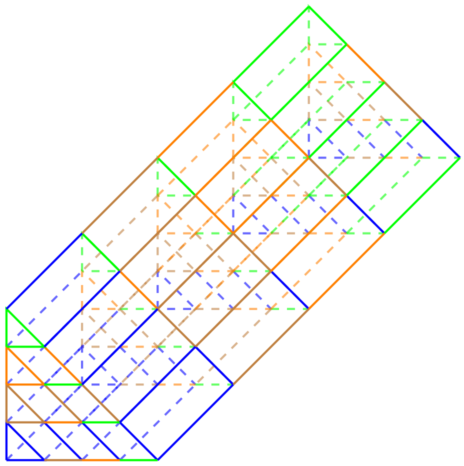
However, if we use the greedy approach, we have an 15×15 matrix, corresponding to the lattice points marked in red.

Example 3.3. Let f_0, f_1, f_2, f_3 be four polynomials with

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1),(0,1,1)\}$$

and $\rho_0 = (0,3,6,3,6,9)$, $\rho_1 = (0,2,4,2,4,6)$, $\rho_2 = (0,1,2,1,2,3)$ and $\rho_3 = (0,0,0,0,0,0)$

gives the mixed subdivision:



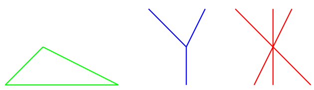
If we take the translation $\delta = (-2/3, -2/3, -1/2)$ the number of points in traslated mixed cells is 24, but the degree of the resultant is $3+3+3+3=12$. If we start at the point $(0,0,0)$ and use the greedy algorithm, we achieve a matrix of size 20×20.

3. A family of mixed subdivisions

In this section, we give a family of lifting functions associated to the polytopes $\Delta_0, \dots, \Delta_n$ and we prove that the Canny-Emiris formula holds for the corresponding mixed subdivisions.

Definition 3.6. We can define a hyperplane arrangement $\mathcal{H} \subset N_{\mathbb{R}}$ by considering the span of the $(n-1)$-dimensional cones of the normal fan of Δ; see [Zie95] for more on polytopes and hyperplane arrangements.

Example 3.4. A polytope Δ (green), together with its normal fan (blue) and the hyperplane arrangement $\mathbb{H}_{\Delta}$ (red).



Definition 3.7. Let $\mathcal{H}$ be the hyperplane arrangement associated to Δ and take a vector $v \in N_{\mathbb{R}}$ which does not lie in $\mathcal{H}$. We consider lifting functions $\omega_i : \mathcal{A}_i \to \mathbb{R}$ defined as:

$$\omega_i(x) = \lambda_i \langle v, x \rangle \quad i = 0, \dots, n \quad x \in \Delta_i$$

for $\lambda_0, \dots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_0 > \dots > \lambda_n \geq 0$ and small enough. Let $\rho = (\omega_0, \dots, \omega_n)$ be a lifting giving a mixed subdivision $S(\rho)$.

Remark 3.5. This choice of the lifting function can also be seen as a case of the approach of [D'A01], in a first proof of the rational formula for generalized unmixed systems. In particular, it is possible to think of the choice of the row content $a(b)$ associated to each lattice point as trying to solve the simplex method with the lifting function as objective, which implies that $S(\rho)$ is tight. This family guarantees that we are always choosing this point in the same direction; see Figure 4.3 for a description of this process.

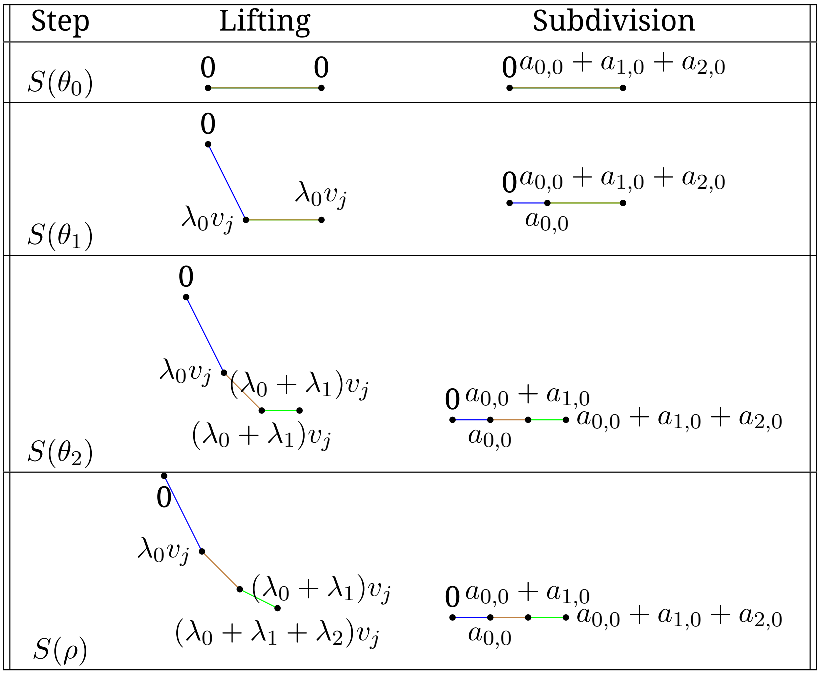


Figure 3.2: This table explains how the process of passing from the proposed lifting on Δ_0, Δ_1, Δ_2 to the mixed subdivision works in the j-th coordinate for $v_j < 0$ for any of the two components of Example 3.1. One clearly sees that, for instance, $\overline{0a_{0,0}e_0} \subset D_0$, if and only if, $x_0 \leq a_{0,0}$ for $x \in D$. The product of two subdivisions of this form gives the mixed subdivision in the figure of Example 3.1.

Theorem 3.2. $S(\rho)$ is an admissible mixed subdivision.

Proof. Let $S(\theta_i)$ be the mixed subdivision obtained from $\theta_i = (\omega_0, \ldots, \omega_{i-1}, 0, \ldots, 0)$. Using [DJS22, Proposition 2.11], for each $i = 0, \ldots, n$, there is an open neighboorhood of $0 \in U \subset \mathbb{R}^{\mathcal{A}_i}$ such that for $\omega_i \in U$ we have $S(\theta_i) \preceq S(\theta_{i+1})$. For $\lambda_i > 0$ small enough, ω_i lies in U. Therefore, the $S(\theta_i)$ form an incremental chain.

All the lattice points with row content 0 are 0-mixed. Therefore, $S(\theta_0)$ satisfies $ii)$ in Definition 3.5. Let D be an n-cell of $S(\theta_i)$. If dim $D_i = 0$, then the fundamental subfamily of $\mathcal{A}_D$ is at most $\{i\}$ as shown in Remark 3.3. We show that, for our choice of the lifting function, the rest of cells D satisfy $ii)$ in Definition 3.5.

Let $D \in S(\theta_i)$ such that dim $D_i > 0$. Suppose that this cell contains a lattice point $b \in \mathcal{B}$ that has row content i but is not i-mixed. Therefore, this lattice point b will be in a cell of $S(\rho)$ with a 0-dimensional j-th component for some $j < i$. Take $C \supset D$ in $S(\theta_j)$ containing the previous lattice point b. If dim $C_j > 0$, then the lifting function

$\omega_j = \lambda_j \langle v, x \rangle$ takes the same value in all the points of C_j. Therefore, the vector v is normal to a face of C_j and has to be contained in the hyperplane arrangement associated to Δ. As this is not the case, $\dim C_j = 0$ and consequently $\dim D_j = 0$, contradicting the initial hypothesis. $\square$

Proving that $S(\rho)$ is an admissible mixed subdivision consists on both proving that it has an incremental chain satisfying $S(\theta_0) \preceq \cdots \preceq S(\theta_n) \preceq S(\rho)$ and that this incremental chain satisfies the conditions in Definition 3.5.

In this proof, we considered the easiest way to prove the chain condition, which is using [DJS22, Proposition 2.11]. In this case, for λ_{i+1} small enough satisfying $\lambda_i > \lambda_{i+1} > 0$, ω_i lies in U. Therefore, the $S(\theta_i)$ form an incremental chain. However, we can drop the restriction that λ_{i+1} is small enough by proving a more general result. In Section 5., we explore this new proof in the more general context of tropical geometry.

4. n-zonotopes and multihomogeneous systems

In this section, we study the previous family of mixed subdivisions in the particular cases of n-zonotopes and multihomogeneous syztems. For simplicity, we suppose that our lattice is $\mathbb{Z}^n$.

Definition 3.8. A zonotope is a polytope given as a sum of line segments. An n-zonotope is generated by n line segments, which span a lattice of dimension n.

Consider linearly independent $v_1, \ldots, v_n \in \mathbb{Z}^n$ and the line segments

$$\overline{0v_1}, \ldots, \overline{0v_n} \subset \mathbb{R}^n$$

forming an n-zonotope $Z = \overline{0v_1} + \cdots + \overline{0v_n} \subset \mathbb{R}^n$. If the Newton polytopes are n-zonotopes whose defining line segments are integer multiples of the $\overline{0v_j}$, we can write the supports of the system as:

$$\mathcal{A}'_i = \Big\{ \sum_{j=1}^{n} \lambda_j v_j \in \mathbb{Z}^n \mid \quad \lambda_j \in \mathbb{Z}, \quad 0 \leq \lambda_j \leq a_{ij} \Big\} \tag{3.1}$$

for some $a_{i,j} \in \mathbb{Z}_{>0}$. Let V be the nonsingular matrix whose columns are the v_j for $j = 1, \ldots, n$ and consider it as a monomorphism of lattices $V : \mathbb{Z}^n \to \mathbb{Z}^n$ of rank n. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{Z}^n$.

Corollary 3.1. Let $\mathcal{A}'_0, \ldots, \mathcal{A}'_n$ be the previous family of supports, then $\mathrm{Res}_{\mathcal{A}'} = \mathrm{Res}_{\mathcal{A}}^{|\det(V)|}$, where:

$$\mathcal{A}_i = \big\{ (b_j)_{j=1,\ldots,n} \in \mathbb{Z}^n \quad \mid \quad 0 \leq b_j \leq a_{ij} \big\} \quad i = 0, \ldots, n \tag{3.2}$$

Proof. Using Lemma 2.1, we can view the map $V : \mathbb{Z}^n \to \mathbb{Z}^n$ as a monomorphism of lattices sending the canonical basis e_i to v_i for $i = 1, \ldots, n$. The determinant $\det(V)$ is the index of the image. □

Remark 3.6. The results that follow in this section could be proved without using Corollary 3.1, after changing b_j by $\langle b, \eta_j \rangle$ for $j = 1, \ldots, n$, where $(\eta_j)_{j=1,\ldots,n}$ are the normal vectors to the zonotope. However, this result extends to the matrices $\mathcal{H}_{\mathcal{G}}$ and gives a major simplification when $\det(V) > 1$.

In order to prove our results, we assume that the a_{ij} are ordered, meaning that

$$0 < a_{0j} \leq a_{1j} \leq \cdots \leq a_{n-1,j} \quad j = 1, \ldots, n \tag{3.3}$$

where we exclude $\mathcal{A}_n$ from this assumption. Consider a translation $\delta \in \mathbb{R}^n$ such that it is negative in each component. Then, the lattice points in translated cells of a mixed subdivision of the previous system are:

$$\mathcal{B} = \big\{(b_j)_{j=1,\ldots,n} \in \mathbb{Z}^n \quad | \quad 0 \leq b_j < \sum_{i=0}^{n} a_{ij} \big\}.$$

Let $v \notin \cup_{i=0}^{n} \{x_j = 0\}$ define the mixed subdivision $S(\rho)$ as in the previous section. We assume $v_j < 0$ and get the following result.

Proposition 3.2. Let $b \in \mathcal{B}$ and $i \in \{0, \ldots, n\}$. Then:

$$t_{b,i} = \big|\{j \in \{1, \ldots, n\} \quad | \quad \sum_{k=0}^{i-1} a_{kj} \leq b_j < \sum_{k=0}^{i} a_{kj} \}\big|$$

and the row content $i(b)$ is the maximum index in $\{0, \ldots, n\}$ such that:

$$\nexists j \in \{1, \ldots, n\} : \quad \sum_{k=0}^{i(b)-1} a_{kj} \leq b_j < \sum_{k=0}^{i(b)} a_{kj}$$

with the support $a(b) \in \mathcal{A}_{i(b)}$ satisfying:

$$a(b)_j = \begin{cases} 0 & b_j < \sum_{k=0}^{i(b)-1} a_{k,j}, \\ a_{i(b),j} & b_j \geq \sum_{k=0}^{i(b)} a_{k,j}. \end{cases}$$

Proof. We analyze the structure of the mixed subdivision $S(\theta_1)$ for $v_j < 0$. The lifting function assign a higher value to the face $\{x_j = 0\} \subset \Delta_0$ with respect to the face $\{x_j = a_{0,j}\} \subset \Delta_0$. When applied to the Minkowski sum Δ, this implies that the hyperplane $\{x_j = a_{0,j}\}$ will divide the cells $D_0 \in S(\theta_1)$ between those such that $\overline{0a_{j,0}e_j} \subset D_0$ $\big(x_j < a_{0,j}\big)$ and those with $\overline{0a_{j,0}e_j} \not\subset D_0$ $\big(x_j > a_{0,j}\big)$.

In terms of the lattice points, those $b \in \mathcal{B}$ lying in a cell $D \in S(\theta_1)$ that contains the line segment $\overline{0a_{j,0}e_j} \subset D_0$ satisfy that $0 \leq b_j < a_{0j}$ and the rest satisfy $b_j \geq a_{0j}$.

As a consequence, $t_{b,0}$ will be the number of $j \in \{1, \dots, n\}$ such that $0 \le b_j < a_{0j}$.

We can reproduce this argument for $i > 1$. As $S(\theta_i)$ refines $S(\theta_{i-1})$, the available interval for D_i is $[\sum_{k=0}^{i-1} a_{k,j}, \sum_{k=0}^{n} a_{k,j})$, so the inequality corresponding to $\overline{0a_{i,j}e_j} \subset D_i$ will be $\sum_{k=0}^{i-1} a_{k,j} \le b_j < \sum_{k=0}^{i} a_{k,j}$. The second claim follows from the definition of row content with respect to the type vector t_b.

Let $b \in \mathcal{B}$ and let $i(b)$ be its row content. For $j = 1, \dots, n$, we either have $b_j < \sum_{k=0}^{i(b)-1} a_{kj}$ or $b_j \ge \sum_{k=0}^{i(b)} a_{kj}$. In the first case, the vertex associated to the row content, will be in the face of $\Delta_{i(b)}$ defined by the equality $\{x_j = 0\}$ and in the second case, the one defined by the equality $\{x_j = a_{i(b),j}\}$. □

Remark 3.7. If $v_j > 0$, we would change the inequalities by $\sum_{k=i}^{n} a_{kj} \le b_j < \sum_{k=i-1}^{i} a_{kj}$, but the results that follow would not change. All the other mixed subdivisions of the system can also be formed this way.

Definition 3.9. The type function $\varphi_b : \{1, \dots, n\} \to \{0, \dots, n\}$ associated to each lattice point $b \in \mathcal{B}$ is defined as the vector of indices satisfying:

$$\sum_{k=0}^{\varphi_b(j)-1} a_{k,j} \le b_j < \sum_{k=0}^{\varphi_b(j)} a_{k,j}.$$

Following Proposition 3.2, it satisfies that $t_{b,i} = |\varphi_b^{-1}(i)|$.

From the components of $a(b)$ in Proposition 3.2, we deduce that the range of values for $(b - a(b) + \mathcal{A}_{i(b)})_j$ is:

$$\begin{cases} [b_j, b_j + a_{i(b),j}] & b_j < \sum_{k=0}^{i(b)-1} a_{kj} \\ [b_j - a_{i(b),j}, b_j] & b_j \ge \sum_{k=0}^{i(b)} a_{kj} \end{cases}$$

Corollary 3.2. The range of possible type functions for $b' \in b - a(b) + \mathcal{A}_{i(b)}$ are:

$$\varphi_{b'}(j) \in \begin{cases} \{\varphi_b(j) - 1, \varphi_b(j)\} & i(b) < \varphi_b(j) \\ \{\varphi_b(j), \dots, i(b)\} & i(b) > \varphi_b(j) \end{cases}$$

Proof. Take I to be the index such that $\sum_{k=0}^{I-1} a_{k,j} \le b_j < \sum_{k=0}^{I} a_{k,j}$. Then, we can derive the inequalities:

$$\begin{cases} b_j - a_{i(b),j} \ge \sum_{k=0}^{I-1} a_{k,j} - a_{i(b),j} \ge \sum_{k=0}^{I-2} a_{k,j} & i(b) < I \\ b_j + a_{i(b),j} < \sum_{k=0}^{I} a_{k,j} + a_{i(b),j} \le \sum_{k=0}^{i(b)} a_{k,j} & i(b) > I \end{cases}.$$

In the first row, we used that $a_{i(b),j} \le a_{I-1,j}$. □

Definition 3.10. We define the greedy subset $\mathcal{G} \subset \mathcal{B}$ to be formed by all the lattice points $b \in \mathcal{B}$ such that:

$$\sum_{i=0}^{I} t_{b,i} \leq I + 1 \quad \forall I < n.$$

Theorem 3.3. Let $b \in \mathcal{G}$ and $b' \notin \mathcal{G}$. Then, $b' \notin b - a(b) + \mathcal{A}_{i(b)}$

Proof. Let I be the greatest index such that $\sum_{i=0}^{I} t_{b',i} > I + 1$. As it is the greatest, we must have $t_{b',I+1} = 0$ and $\sum_{i=I+2}^{n} t_{b',i} < n - I - 1$.

On the other hand, $\sum_{i=0}^{I} t_{b,i} \leq I + 1$. Using Corollary 3.2, the previous sum cannot grow in $b - a(b) + \mathcal{A}_{i(b)}$ when $i(b) > I$. If $\sum_{i=0}^{I} t_{b,i} = I + 1$, then $\sum_{i=I+1}^{n} t_{b,i} < n - I$ which implies that there is $i > I$ with $t_{b,i} = 0$ and $i(b) > I$.

Suppose $\sum_{i=0}^{I} t_{b,i} < I + 1$ and $i(b) < I$. Using Corollary 3.2, we have:

$$\sum_{i=I+1}^{n} t_{b,i} \geq n - I \quad \text{and} \quad \sum_{i=I+1}^{n} t_{\overline{b},i} \geq n - I - 1$$

for $\overline{b} \in b - a(b) + \mathcal{A}_{i(b)}$. Therefore,

$$\sum_{i=I+2}^{n} t_{b',i} < n - I - 1 \leq \sum_{i=I+2}^{n} t_{\overline{b},i}$$

meaning that it is not possible that b' has a type function on the range of $b - a(b) + \mathcal{A}_{i(b)}$. $\square$

Definition 3.11. Let $I_b \in \{0, \ldots, n\}$ be the index satisfying:

$$I_b = \begin{cases} \max\{i \in \{0, \ldots, n\} \mid t_{b,i} \geq 2\} & b \text{ lies in a non-mixed cell} \\ 0 & b \text{ lies in a mixed cell} \end{cases}$$

Let $g_b = |\{i < I_b \quad t_{i,b} = 0\}|$ be the number of zeros that t_b has before I_b.

Lemma 3.1. Let $b \in \mathcal{G}$ and suppose that $g_b = 0$. Then, b lies in a mixed cell.

Proof. Suppose that b lies in a non-mixed cell. This would mean that there is no zero before I_b implying that $\sum_{i=0}^{I_b} t_{b,i} = \sum_{i=0}^{I_b - 1} t_{b,i} + t_{b,I_b} \geq I_b + 2$. $\square$

Lemma 3.2. If $t_{b,I} = 0$ and $b \in \mathcal{G}$, $\sum_{i=0}^{I} t_{b,i} < I + 1$.

Proof. Otherwise, $\sum_{i=0}^{I-1} t_{b,i} \geq I + 1$ implying $b \notin \mathcal{G}$. $\square$

Theorem 3.4. Let $\mathcal{G}$ be the greedy subset and $b \in \mathcal{G}$ such that $g_b = K$ for $K > 0$. Then, there is $b' \in \mathcal{G}$ with $g_{b'} = K - 1$ such that for some $\overline{b} \in b' - a(b') + \mathcal{A}_{i(b)}$, $\varphi_{\overline{b}} = \varphi_b$. As a consequence, we reach b from b'.

Proof. Consider t_b to be the type vector of b and suppose that t_b has two or more zeros after I_b. Then,

$$\sum_{i=I_b+1}^{n} t_{b,i} \le n - I_b - 2,$$

implying that $\sum_{i=0}^{I_b} t_{b,i} \ge I_b + 2$, and $b \notin \mathcal{G}$.

If t_b has one zero after I_b, it implies that $i(b) > I_b$. If $g_b > 0$, it needs to have at least one zero before I_b. Therefore, the type vector contains a sequence of the form

$$(\dots, \overbrace{0}^{I'}, 1, \dots, 1, \overbrace{t_{b,I}}^{I}, \dots)$$

for some $I' < I \le I_b$ with $t_{b,I} \ge 2$. Consider the type function:

$$\varphi_{b'}(j) = \begin{cases} \varphi_b(j) - 1 & I' < \varphi_b(j) \le I \\ \varphi_b(j) & \text{otherwise} \end{cases}$$

The corresponding type vector $t_{b'}$ contains a sequence:

$$(\dots, \overbrace{1}^{I'}, 1, \dots, 1, \overbrace{t_{b,I_b} - 1}^{I}, \dots).$$

Using Lemma 3.2, $\sum_{i=0}^{I'} t_{b,i} < I' + 1$, therefore we will have:

$$\sum_{i=0}^{I'} t_{b',i} \le I' + 1.$$

The same will hold for all the partial sums from I' to I_b implying there is $b' \in \mathcal{G}$ with type function $\varphi_{b'}$. Using Corollary 3.2, φ_b is in the range of type functions in $b' - a(b') + \mathcal{A}_{i(b')}$. As long as $i(b') < n$, we can find $b' \in \mathcal{G}$ such that:

$$(b - b' + a(b'))_j \le a_{\varphi_{b'}(j),j} \le a_{j,i(b')},$$

so $b \in b' - a(b') + \mathcal{A}_{i(b')}$.

If $i(b) = i(b') = n$, we must have $a(b) = a(b') = 0$, so we reach a point $\bar{b} \in b' - a(b') + \mathcal{A}_{i(b')}$ in the same cell as b such that:

$$(b - \bar{b})_j < (b - b')_j \quad \forall I' < \varphi_b(j) \le I$$

As $i(\bar{b})$ is always the same, after a finite number of steps, we have $b \in \bar{b} - a(\bar{b}) + \mathcal{A}_{i(\bar{b})}$.

If t_b does not have any zero after I_b, then $i(b) < I_b$. The vector contains a sequence that looks like

$$(\dots, \overbrace{0}^{i(b)}, t_{b,i(b)+1}, \dots, t_{b,I_b}, \dots)$$

for $t_{b,I} \geq 1$ with $i(b) < I \leq I_b$. In this case, consider the type function:

$$\varphi_{b'}(j) = \begin{cases} \varphi_b(j) - 1 & i(b) < \varphi_b(j) \leq I \\ \varphi_b(j) & \text{otherwise} \end{cases},$$

with type vector $(\ldots, \overbrace{t_{b,i(b)+1}}^{i(b)}, \ldots, \overbrace{t_{b,I_b}}^{I_b-1}, \overbrace{0}^{I_b}, \ldots)$. For these functions, we derive

$$\sum_{i=i(b')+1}^{n} t_{b',i} \geq n - i(b) + \sum_{\substack{t_{b,i}\geq 2 \\ i>i(b')}} (t_{b',i} - 1) \implies$$

$$\sum_{i=0}^{i(b)} t_{b',i} \leq i(b) - \sum_{t_{b,i}\geq 2\; i>i(b')} (t_{b',i} - 1) \leq i(b) + 1$$

which implies that

$$\sum_{i=0}^{i(b)} t_{b',i} + t_{b',i(b')+1} \leq i(b) - \sum_{\substack{t_{b,i}\geq 2 \\ i>i(b')}} (t_{b',i-1}) + t_{b',i(b')+1} \leq i(b') + 1$$

This argument holds for bounding the partial sums for $I > i(b)$ so there is $b' \in \mathcal{G}$ with type function $\varphi_{b'}$ and φ_b is in the range of type functions in $b' - a(b') + \mathcal{A}_{i(b')}$. In this case, it is not possible that $i(b') = n$. The same argument as in the previous case holds in order to say that $b \in b' - a(b') + \mathcal{A}_{i(b')}$. □

Theorem 3.3 and Theorem 3.4 imply that if we start the greedy algorithm from the lattice points in mixed cells, we will reach exactly the lattice points in $\mathcal{G}$. This actually reduces the size of the Canny-Emiris matrices.

Corollary 3.3. The size of the matrix $\mathcal{H}_{\mathcal{G}}$:

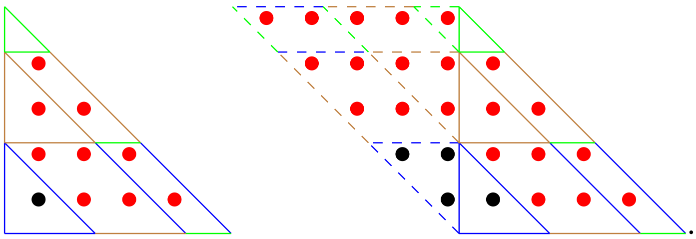
$$\sum_{\varphi_b:\{1,\ldots,n\}\to\{0,\ldots,n\}} \prod_{j=1}^{n} a_{\varphi_b(j),j}$$

where the sum is over the functions that satisfy $\varphi_b^{-1}(\{0,\ldots,I\}) \leq I+1$ for all $I < n$.

Proof. Each type function φ_b corresponds to a cell $D \in S(\rho)$. The lattice points $b \in D$ satisfy Definition 3.9. Therefore, for each j, there are $a_{\varphi_b(j),j}$ possible values of b_j. The product over all of them gives the desired count. □

Example 3.5. Let f_0, f_1, f_2 be three homogeneous polynomials of degrees $2, 2, 1$ respectively. We choose $v = (-1, -2)$ and $\delta = (-3/4, -3/4)$ and define an admissible mixed subdivision $S(\rho)$ in the Minkowski sum Δ of their Newton polytopes Δ_i. Let $\mathcal{B}$ be the set of lattice points in $\Delta + \delta$. Consider a system of polynomials whose Newton polytopes are n-zonotopes generated by the vectors $w_1 = (1, 0)$ and $w_2 = (-1, 1)$

and let $a_{0,1} = a_{0,2} = a_{1,1} = a_{1,2} = 2$ and $a_{2,1} = a_{2,2} = 1$ be the bounds of the supports as in Section 4.. Let $S(\overline{\rho})$ be the mixed subdivision in the Minkowski sum $\overline{\Delta}$ of $\overline{\Delta_i}$ of this system given by the same v, δ as the previous, and let $\overline{\mathcal{B}}$ be the set of lattice points in $\overline{\Delta} + \delta$.



It turns out that the mixed subdivision $S(\rho)$ embeds into $S(\overline{\rho})$, i.e. all the cells of $S(\rho)$ are contained in a cell of $S(\overline{\rho})$. This implies that $\mathcal{B} = \overline{\mathcal{B}} \cap \Delta$. As the greedy reduction applies to the second system, it must apply to the first as well. We get a 9×9 matrix $\mathcal{H}_{\mathcal{G}}$ for the homogeneous system, excluding the black lattice point in the figure.

Similar to Example 3.5, consider multihomogeneous polynomial systems and embed them into n-zonotopes. Let $n_1, \ldots, n_s \in \mathbb{N}_{>0}$ be natural numbers and let $\oplus_{l=1}^{s} \mathbb{Z}^{n_l}$ be the lattice. Each multihomogeneous polynomial system can be written as:

$$F_i = \sum_{a \in \mathcal{A}_i} u_{i,a} \chi^a, \quad i = 0, \ldots, n$$

where the supports are:

$$\mathcal{A}_i = \{(b_{jl})_{l=1,\ldots,s}^{j=1,\ldots,n_l} \in \oplus_{l=1}^{s} \mathbb{Z}^{n_l} \mid b_{jl} \geq 0, \sum_{j=0}^{n_l} b_{jl} \leq d_{i,l}\}$$

where $\mathbf{d}_i = (d_{i,1}, \ldots, d_{i,s})$ is the multidegree of F_i. Each of these supports can be embedded into the following sets of supports:

$$\overline{\mathcal{A}_i} = \{(b_{jl}) \in \oplus_{j=1}^{s} \mathbb{Z}^{n_j} \mid 0 \leq \sum_{J=j}^{n_l} b_{Jl} \leq d_{i,l}\} \quad l = 1, \ldots, s \quad j = 1, \ldots, n_l.$$

Let Δ_i, $\overline{\Delta_i}$ be the Newton polytopes of each of the systems and $\Delta, \overline{\Delta}$ be their respective Minkowski sums.

Lemma 3.3. The Newton polytopes $\overline{\Delta_i}$ of the system of polynomials with supports in $\overline{\mathcal{A}_i}$ are n-zonotopes whose line segments $(w_{j,l})_{l=1,\ldots,s}^{j=1,\ldots,n_l}$ are given by the columns of the matrix:

$$W = \begin{bmatrix} W_1 & \ldots & 0 \\ \vdots & \ddots & \\ 0 & \ldots & W_s \end{bmatrix}, \quad W_l = \begin{pmatrix} 1 & -1 & 0 & \ldots & \ldots \\ 0 & 1 & -1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & -1 \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$$

where the square blocks W_l are of size n_l for $l = 1, \ldots, s$. Moreover, $\overline{\mathcal{H}} = \cup_{l=1}^{s} \cup_{j=1}^{n_l} \{\langle x, w_{j,l} \rangle = 0\} \subset \prod_{l=1}^{s} \mathbb{R}^{n_l}$ is the hyperplane arrangement associated to $\overline{\Delta}$.

Proof. Let $b \in \oplus_{j=1}^{s} \mathbb{Z}^{n_j}$ be a lattice point. As the columns of W form a basis of the lattice, we can write $b = \sum_{l=1}^{s} \sum_{j=1}^{n_l} \lambda_{j,l} w_{j,l}$ and these coefficients are precisely $\lambda_{j,l} = \sum_{J=j}^{n_l} b_{J,l}$. Then,

$$b \in \overline{\mathcal{A}}_i \iff 0 \leq \lambda_{j,l} \leq d_{i,l} \quad l = 1, \ldots, s \; j = 1, \ldots, n_l.$$

The normal vectors to the faces of $\overline{\Delta}$ are given by the columns $(\eta_{j,l})_{l=1,\ldots,s}^{j=1,\ldots,n_l}$ of the matrix:

$$H = \begin{bmatrix} H_1 & \ldots & 0 \\ \vdots & \ddots & \\ 0 & \ldots & H_s \end{bmatrix}, \quad H_l = \begin{pmatrix} 1 & 0 & 0 & \ldots & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \ldots & 1 & 1 & 0 \\ 1 & \ldots & 1 & 1 & 1 \end{pmatrix}.$$

One can check that $\langle w_{j,l}, \eta_{j',l'} \rangle \neq 0$, if and only if, $l = l'$ and $j = j'$. Therefore, $v \in \overline{\mathcal{H}}$, if and only if, it belongs to the span of $\sum_{l=1}^{s} n_l - 1$ columns of H, and this will only happen if $\langle v, w_{j,l} \rangle = 0$ for some pair j, l. $\square$

Remark 3.8. As a consequence of Lemma 3.3, we can apply the results of Section 4. to the system with supports $\overline{\mathcal{A}_i}$. The matrix H gives the normals to our polytopes, so we can use it in the sense of Remark 3.6.

Let $v \notin \overline{\mathcal{H}}$ and suppose that we take $\langle v, w_{j,l} \rangle < 0$, for $l = 1, \ldots, s$ and $j = 1, \ldots, n_l$. Consider $S(\overline{\rho})$ to be the admissible mixed subdivision of $\overline{\Delta}$ given by v as in Section 3.. Let $S(\rho)$ be the mixed subdivision given by the same vector in Δ. Using $\langle v, w_{j,l} \rangle < 0$, one can check that this mixed subdivision is also admissible as v does not belong to the hyperplane arrangement $\mathcal{H}$ associated to Δ. Let $\mathcal{B}, \overline{\mathcal{B}}$ be the sets of lattice points in translated cells of Δ and $\overline{\Delta}$, respectively.

We can see the polytopes Δ_i as a product of simplices $\Delta_{i,1} \times \cdots \times \Delta_{i,s}$ in each of the factors of $M_{\mathbb{R}} = \prod_{l=1}^{s} \mathbb{R}^{n_l}$.

Theorem 3.5. The mixed subdivision $S(\rho)$ coincides with $S(\overline{\rho}) \cap \Delta$.

Proof. We check the result for the cells of $S(\theta_1)$ and the argument can be repeated for all the mixed subdivisions in the corresponding incremental chain. The vector $v \in \prod_{l=1}^{s} \mathbb{R}^{n_l}$ has to satisfy that:

$$v_{1,l} < 0 \quad v_{j+1,l} - v_{j,l} < 0 \; j = 1, \ldots, n_l - 1,$$

which can also be written as:

$$v_{n_l,l} < v_{n_l-1,l} < \ldots v_{1,l} < 0.$$

This means that the mixed subdivision lifts the vertices of $\Delta_{0,l}$ in the order

$$0, d_{0,l} w_{1,l}, \ldots, d_{0,l} w_{n_l,l}$$

from higher to lower. As in Proposition 3.2, this means that the hyperplane

$$\{\langle x, w_{j,l}\rangle = d_{0,l}\} \quad l = 1,\dots,s \quad j = 1,\dots,n_l,$$

divides the cells of $S(\rho)$ between those with $d_{0,l}w_{j,l} \subset D_0$ and those with $d_{0,l}w_{j,l} \not\subset D_0$, respectively. This cell structure is the same as the one given in $S(\overline{\rho})$. Therefore, $S(\rho)$ coincides with the intersection of $S(\overline{\rho})$ with Δ. □

Therefore, if we apply the greedy algorithm to the multihomogeneous system with supports in the $\mathcal{A}_i$, we will obtain the same greedy subset $\mathcal{G} \subset \mathcal{B}$, with the restriction on the type vectors given in Definition 3.10. In particular, the domain of the type functions will now be a multiset in each group of variables:

$$\varphi_b : \{\{1,\dots,n_1\},\dots,\{1,\dots,n_s\}\} \to \{0,\dots,n\}.$$

The following proposition gives conditions to guarantee that the type function φ_b corresponds to a lattice point $b \in \mathcal{B}$.

Proposition 3.3. A lattice point $b \in \overline{\mathcal{B}}$ belongs to $\mathcal{B}$ iff its type function satisfies:

$$\varphi_b(j) \le \varphi_b(j'), \quad \forall j < j' \quad j,j' = 1,\dots,n_l, \quad l = 1,\dots,s.$$

Proof. Suppose that there is φ_b for $b \in \mathcal{B}$ such that for some $l \in \{1,\dots,s\}$ and some pair $j < j'$ in $\{1,\dots,n_l\}$ the function satisfies $\varphi_b(j) > \varphi_b(j')$. Using the definition of the type functions and the matrix H, one sees that:

$$\sum_{k=0}^{\varphi_b(j)-1} d_{k,l} \le \sum_{\overline{j}=1}^{j} b_{\overline{j},l} \quad \sum_{k=0}^{\varphi_b(j')} d_{k,l} > \sum_{\overline{j}=1}^{j'} b_{\overline{j},l} \implies \sum_{\overline{j}=j}^{j'} b_{\overline{j},l} < 0.$$

Therefore, there must be $\overline{j} \in \{j,\dots,j'\}$ such that $b_{\overline{j},l} < 0$ and $b \notin \mathcal{B}$. On the other hand, if we find $b_{j,l} < 0$, we can use the same argument to say that the type function φ_b cannot satisfy the previous restriction. □

Corollary 3.4. The size of the matrix $\mathcal{H}_{\mathcal{G}}$ for multihomogeneous systems is:

$$\sum_{\varphi_b} \prod_{l=1}^{s} \prod_{k=0}^{n} \binom{d_{k,l}}{\overline{n}_{k,l,\varphi_b}}$$

where $\overline{n}_{l,k,\varphi_b} = |\{j \in \{1,\dots,n_l\} \quad \varphi_b(j) = k\}|$ and φ_b satisfies the restrictions of Corollary 3.3 and Proposition 3.3.

Proof. Let $D \in S(\rho)$ be the cell associated to a type function φ_b for $b \in \mathcal{G}$. We can consider that this cell has a decomposition:

$$D = \sum_{k=0}^{n} \sum_{l=1}^{s} D_{k,l}$$

where $D_{k,l}$ is a cell of $\Delta_{k,l}$. The number of lattice points in D corresponds to the product over the number of lattice points in each of the $D_{k,l}$. As a face of $\Delta_{k,l}$, $D_{k,l}$ is a simplex of degree $d_{k,l}$ and dimension the number of $j \in \{1, \dots, n_l\}$ such that $\varphi_b(j) = k$.

The count follows by noticing that the lattice points in a translated simplex of degree d and dimension n of size length are contained in a simplex of degree $d - n$ and same dimension. Therefore, there are $\binom{d}{n}$ of them. □

There exist exact determinantal resultant formulas for some multihomogeneous cases, obtained by using the Weyman complex and other tecniques [BFT18; Ben+21; DE03; EM12; SZ94]. Our approach does not improve those cases, but the use of type functions might be easier to generalize to a general case. We can give an example of the size of these matrices with respect to some of the existsing formulas.

Example 3.6. For the polynomial system of Example 3.1, there are exact formulas of Sylvester type [DE03] which give a matrix of size 6, smaller than that of size 8.

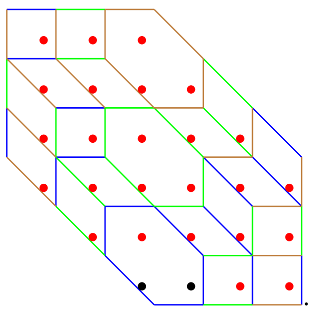
We could also exploit the incremental algorithm for constructing the Canny-Emiris formula [CE95], but we would be losing the combinatorial properties. Therefore, we would not have a proof of the formula for such matrices or we wouldn't be able to guarantee that they have a non-zero determinant as in Remark 3.4. Moreover, such implementation requires the precomputation of mixed volumes.

Apart from the treated cases, we could consider other systems for which the mixed subdivision can be embedded in an n-zonotope and impose restrictions on the type functions accordingly. We could also try to drop the hypothesis that $a_{0,j} \leq \cdots \leq a_{n-1,j}$: the examples show that, for that case, the reduction in the cells that are not in $\mathcal{G}$ is lower. We also expect to measure is when the Newton polytopes are m-zonotopes for $m > n$. In such cases, the examples show that there will still be some reduction.

Example 3.7. Here an example for

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0,0), (1,0), (-1,1), (1,1), (-1,2), (0,2)\} \subset \mathbb{Z}^2$$

and our choice of the mixed subdivision would give:



In this case, there is a reduction on the lattice points of the cells not in $\mathcal{G}$ (lattice points in black), but not all the lattice points can be excluded.

5. Tropical refinement of mixed subdivisions

This section can be read independently with respect to the rest of sections of this article. We describe, in much broader generality than we need, the refinement of mixed subdivisions. In particular, we draw the full picture of when a coherent mixed subdivision refines another one, by only changing the lifting function in Δ_i. In terms of the previous notation, we would like to know whether $S(\theta_i) \preceq S(\theta_{i+1})$ for some $i = 0, \ldots, n$. Instead of studying a given mixed subdivision, we define a dual of such object by introducing tropical geometry. After proving such result using tropical geometry, the family of lifting functions given in Section 3. will satisfy the refinement.

Remark 3.9. As in this paper we are mainly interested in affine lifting functions, we restrict to such case. However, the following results could be reproduced for any piecewise affine lifting function.

The general context of tropical geometry consists of working over rings of polynomials over $\mathbb{R}$ with the tropical operations:

$$x \oplus y = \min(x, y) \quad x \otimes y = x + y$$

Definition 3.12. A tropical polynomial is the expression:

$$\text{trop}(f)(x) = \oplus_{a \in \mathcal{A}} \omega_a x^{\otimes a} = \min_{a \in \mathcal{A}} (\omega_a + ax)$$

for $x \in \mathbb{R}^n$ where $\mathcal{A}$ is the support of f. A tropical hypersurface $V(\text{trop}(f)))$ in $\mathbb{R}^n$ is the set of points where the previous minimum is attained, at least, twice.

Remark 3.10. We can consider the coefficients ω_a to be the values of a lifting function. If the lifting is affine, we have $\omega_a = \langle v, a \rangle$ for some vector $v \in N_{\mathbb{R}}$. Therefore, the tropical polynomial with coefficients ω_a would be:

$$\min_{a \in \mathcal{A}} (a(x + v))$$

Definition 3.13. A tropical system $\mathcal{T}_r$ is formed by $r + 1$ tropical polynomials with supports $P_0, \ldots, P_r \subset M$:

$$\text{trop}(f_i^{\omega_i})(x) = \bigoplus_{a_i \in P_i} \omega_{i,a} \otimes x^{\otimes a} = \min_{a \in P_i} \left(\omega_{i,a} + a \cdot x \right)$$

where the coefficients of the system are given by some lifting function of the P_i. In some references like [MS15], it is important to specify a valuation in the field but here we can suppose it to be trivial.

In our context, as in Remark 3.2, we have a family of tropical systems $\mathcal{T}_i$ for $i = 0, \ldots, n$ of the supports:

$$\mathcal{A}_0, \ldots, \mathcal{A}_{i-1}, \sum_{j=i}^{n} \mathcal{A}_j \subset M$$

The last tropical polynomial is formed by imposing 0 coefficients, therefore, it is defined by:

$$\min_{a \in \sum_{j=r}^{n} \Delta_j} \langle a, x \rangle$$

which corresponds to the normal fan of $\sum_{j=r}^{n} \Delta_j$. This coincides with the assumptions for $S(\theta_i)$ in Remark 3.2.

Proposition 3.4. The expression $\min_{a \in \mathcal{A}} \langle a, x \rangle$ is achieved twice in the $(n-1)$-dimensional cones of the normal fan of $\Delta = \mathrm{conv}(\mathcal{A})$.

Proof. A j-th dimensional cone $\mathcal{N}_F \subset M_{\mathbb{R}}$ is a of the normal fan of Δ corresponds to a $n-j$-dimensional face of Δ. Take $v \in \mathcal{N}_F$, then $\min_{a \in \Delta} \langle a, x \rangle$ is the same for all $a \in F$, which is a face. Therefore, it is achieved, at least twice. On the other hand, if the minimum is achieved at least twice at v, then consider the convex hull

$$\mathrm{conv}\{a_i \in \Delta \quad \min\langle a_i, v \rangle \text{ is achieved}\}$$

and it is a positive dimensional face F of Δ, therefore v is in the a cone of dimension at most $(n-1)$ in $\mathcal{F}$. $\square$

Proposition 3.5. The expression $\min_{a \in \mathcal{A}} \langle a, x + v \rangle$ is achieved twice in the $(n-1)$-skeleton of the normal fan of Δ translated after $v \in N_{\mathbb{R}}$.

Proof. The same proof as the previous works after translating by v. $\square$

In this context, we can see the tropical system $\mathcal{T}_i$ as the superposition in $N_{\mathbb{R}}$ of the normal fans $\mathcal{F}_0, \ldots, \mathcal{F}_n$ centered at different points $v_i \in N_{\mathbb{R}}$ which correspond to each of the lifting functions $\omega_i : \Delta_i \to \mathbb{R}$.

Definition 3.14. A polyhedral complex $\mathcal{P}$ is a union of cells (bounded or unbounded) in $N_{\mathbb{R}}$ such that:

- Every face of a cell in $\mathcal{P}$ is also in $\mathcal{P}$.
- The (possibly empty) intersection of two cells in $\mathcal{P}$ is also in $\mathcal{P}$.

Fans are a good example of polyhedral complexes. Thereofre, a tropical system defines a polyhedral complex.

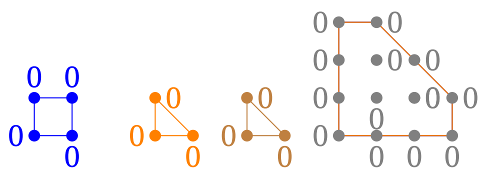
Proposition 3.6. Let $\mathcal{A}_0, \ldots, \mathcal{A}_n$ be a family of supports and $\omega : \sum_{i=0}^{n} \mathcal{A}_i \to \mathbb{R}$ be a lifting function. The polyhedral complex defined by tropical system $\mathcal{T}$ taking the values of ω as coefficients is dual to the mixed subdivision $S(\omega)$.

This duality happens in the following sense: the j-dimensional cells of the polyhedral complex correspond to the $(n-j)$-dimensional cells of the mixed subdivision.

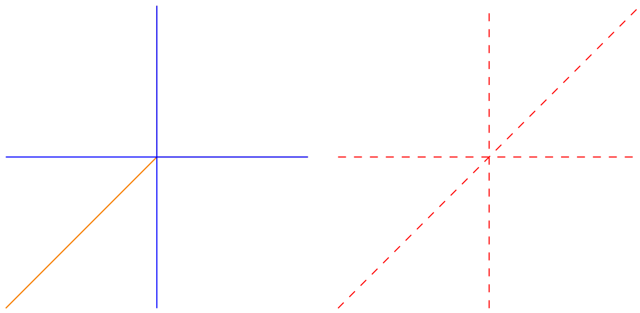
Proof. Let p be a 0-dimensional cell of the polyhedral complex defined by $\mathcal{T}$. As it is the intersection of cones of each of the fans $\mathcal{F}_i$, there is a cell of $S(\rho)$ corresponding to the sum of the faces associated to each of the fans. On the other hand, an n-cell D on the mixed subdivision corresponds to a point p, which is the intersection of the normal cones of each of the summands D_i. Each of the faces of D corresponds to a cell of the polyhedral complex in which p is contained. □

Denote by $\mathbb{H}_i$, the hyperplane arrangement in $\mathbb{R}^n$ associated to the tropical system $\mathcal{T}_i$. Before stating the main theorem, we will put an example of the refining construction.

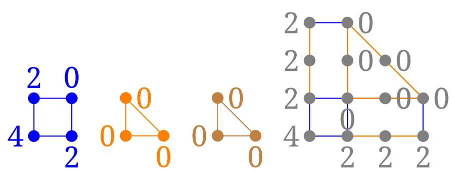
Example 3.8. Let $\mathcal{A}_0 = \{(0,0),(1,0),(0,1),(1,1)\}$, $\mathcal{A}_1 = \mathcal{A}_2 = \{(0,0),(1,0),(0,1)\}$ with corresponding convex hulls $\Delta_0, \Delta_1, \Delta_2$. Start with the trivial mixed subdivision:



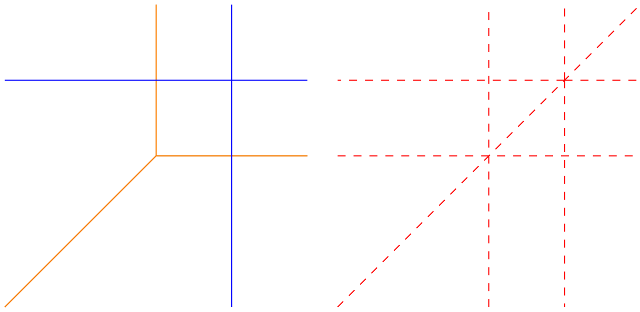
In this case, the corresponding tropical system is given by the inner normal fan to the Minkowski sum, which corresponds to the superposition of the normal fans of each summand.



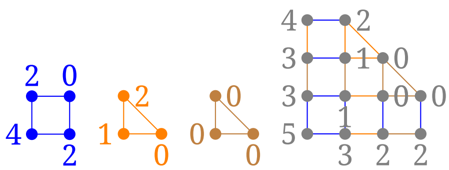
The dashed drawing represents the central hyperplane arrangement which we will denote as $\mathbb{H}_0$. Any lifting of Δ_0 will refine the subdivision. However, we can see that refinement corresponds to moving the point $(0,0)$ of the blue fan to an adjacent chamber $\mathbb{H}_0$. Take $(2,2)$ as a normal vector. This means lifting Δ_0 after an affine function of type $c - 2x - 2y$. We can choose any constant c as it will give the same lifting. I choose $c = 4$ in order to get positive values in the lifting. Now, the subdivision looks like:



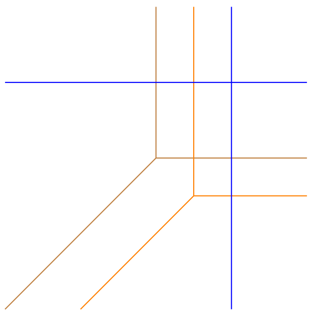
and the corresponding tropical system $\mathcal{T}_1$ and the corresponding (not central) hyperplane arrangement $\mathbb{H}_1$ look like:



Now, we claim that moving the orange fan close enough, we will be refining the mixed subdivision. In particular, moving the orange fan to each of the adjacent cells on the hyperplane arrangement corresponds to all the possible ways to refine the previous mixed subdivision. For instance, if we take the traslation given by the vector $(1, -1)$, which would be the normal vector to the affine lifting $c - x + y$ with $c = 1$. The mixed subdivision looks like:



and the tropical system after the traslation vector $(1, -1)$, corresponds to:



We now recapitulate the notation used so far. Let $\omega_i : \mathcal{A}_i \to \mathbb{R}$ be the lifting function. As in Theorem 3.2, $S(\theta_i)$ be the mixed subdivisions of the candidate incremental chain given by the lifting functions $(\omega_0, \ldots, \omega_{i-1}, 0, \ldots, 0)$. Let $\mathcal{T}_i$ be the tropical systems dual to each of the mixed subdivisions $S(\theta_i)$ for $i = 0, \ldots, n$. Let $\mathbb{H}_i$ be the hyperplane arrangement associated to each of the tropical systems.

We now construct the tools needed for proving the refinement result.

Definition 3.15. We say that a ray r of the normal fan $\mathcal{F}_i$ *preserves adjacencies* if it is adjacent to the same cells in $\mathcal{T}_i$ and $\mathcal{T}_{i-1}$.

Lemma 3.4. Let $S(\theta_i)$ be a mixed subdivision of $\Delta_0, \ldots, \Delta_{i-1}, \sum_{j=i}^{n} \Delta_j$ for $i = 0, \ldots, n$. The lifting of Δ_i will give $S(\theta_i) \preceq S(\theta_{i+1})$, if and only if, each ray of $\mathcal{F}_i$ preserves the adjacencies after the translation.

Proof. Suppose there is a ray r that doesn't preserve an adjacencies. Then, take the 0-dimensional cell of the corresponding polyhedral complex where this adjacency fails and it must correspond to an n-cell of $S(\theta_{i+1})$ that is not contained in the cell of $S(\theta_i)$ corresponding to such adjacency.

On the other hand, take a cell C of $S(\theta_{i+1})$ that is not contained in any of the cells of $S(\theta_i)$ and, as we only lifted the polytope Δ_i, the corresponding dual cell on the polyhedral complex has to fail to be adjacent to the same rays. □

At this point, we have all the ingredients to state and prove the tropical refinement result.

Theorem 3.6. (Tropical refinement) Let $i = 1, \ldots, n$. The mixed subdivision $S(\theta_i)$ refines $S(\theta_{i-1})$, if and only if, the normal vector to the lifting function $\omega_{i-1} : \mathcal{A} \to \mathbb{R}$ lives in a chamber of $\mathbb{H}_i$ adjacent to $0 \in \mathbb{R}^n$.

Proof. Consider p as a point (0-dimensional cell) in the polyhedral complex that is dual to an n-cell D of $S(\theta_{i-1})$. Let v be the normal vector to the lifting function $\omega_i : \mathcal{A}_i \to \mathbb{R}$. We have to prove that v lies in an adjacent cell to 0 in $\mathbb{H}_i$, if and only if, D is contained in a cell D' of $S(\theta_k)$.

Firstly, suppose there was not such cell D'. This would mean that the adjacencies would not be preserved and we can find a ray r in $\mathcal{F}_i$ where this property is failing. Consider the ray of a fan $\mathcal{F}_k$ for $k = 0, \ldots, i-1$ where this adjacency has changed and this means that we have crossed a hyperplane containing such ray in the previous fan.

On the other hand, if there is such cell D', then the lifting of Δ_i preserves adjacencies. However, if we had moved v to a non-adjacent cell to 0, we would have crossed a hyperplane therefore, we would be able to find rays in such hyperplane where the adjacencies are not preserved. □

This result extends the proposition 2.11 on [DJS22, Proposition 2.11] and gives a full picture of refinement of mixed subdivisions. Therefore, we naturally understand all the ways to refine a given mixed subdivision $S(\theta_i)$ with affine lifting functions on Δ_i.

Corollary 3.5. The chambers of the hyperplane arrangement $\mathbb{H}_i$ are in one to one correspondence to all the possible ways to refine $S(\theta_i)$. In particular, if $S(\theta_i)$ is tight, the chambers of $\mathbb{H}_i$ correspond to tight mixed subdivisions.

In the context of Theorem 3.2, in the direction of $v \notin \mathbb{H}_\Delta$, the function $\langle \lambda_i v, x \rangle$ will reach the hyperplane arrangement $\mathbb{H}_i$ when $\lambda_i = \lambda_{i-1}$. Therefore, for any $0 < \lambda_i < \lambda_{i-1}$, the subdivision $S(\theta_{i+1})$ will refine $S(\theta_i)$ for $i = 0, \ldots, n$.

Theorem 3.7. The mixed subdivision $S(\rho)$ in Definition 3.5 is admissible.

Proof. All the lattice points with row content 0 are 0-mixed. Therefore, $S(\theta_0)$ satisfies $ii)$ in Definition 3.5. Let D be an n-cell of $S(\theta_i)$. If $\dim D_i = 0$, then the fundamental subfamily of $\mathcal{A}_D$ is at most $\{i\}$ as shown in Remark 3.3. We show that, for our choice of the lifting function, the rest of cells D satisfy $ii)$ in Definition 3.5.

Let $D \in S(\theta_i)$ such that $\dim D_i > 0$. Suppose that this cell contains a lattice point $b \in \mathcal{B}$ that has row content i but is not i-mixed. Therefore, this lattice point b will be in a cell of $S(\rho)$ with a 0-dimensional j-th component for some $j < i$. Take $C \supset D$ in $S(\theta_j)$ containing the previous lattice point b. If $\dim C_j > 0$, then the lifting function $\omega_j = \lambda_j \langle v, x \rangle$ takes the same value in all the points of C_j. Therefore, the vector v is normal to C_j and has to be contained in the hyperplane arrangement associated to Δ. As this is not the case, $\dim C_j = 0$ and consequently $\dim D_j = 0$, contradicting the initial hypothesis. $\square$

This proves that the family of lifting functions that we have defined, always provides an admissible mixed subdivision.

Chapter 4

Toric Sylvester forms

The constructions in the previous chapter are based on the idea of using Sylvester-type formulas for computing the resultant. These formulas provide a very natural multivariate generalization to the classical construction of the resultant of two univariate polynomials, due to Sylvester [Syl18] and are characterized by the fact that each entry of the matrix corresponds to a single coefficient of the system. However, there are other formulas for the resultant in which the entries of the matrix can be other polynomials in the coefficients. Examples of such formulas appear in the very classical works of Bezout [Bez79] and Dixon [Dix09], Morley and Coble [MC27] and others.

In [Jou97], Jouanolou compiled these formulas and added some more of his own. We can extract the following idea from his work: if one wants to find more compact formulas than those of Sylvester-type, a key ingredient will be to add inertia forms [Hur13] i.e. polynomials in the saturation of the given ideal. The literature for computing these forms in different degrees includes the works of Hurwitz, Mertens, Van der Waerden and Zariski [Zar37].

To be more specific, consider the ideal $I = (F_0, \dots, F_n)$ where F_i is the generic homogeneous polynomial of degree d_i in the graded polynomial ring $C = A[x_0, \dots, x_n]$, where $\deg(x_i) = 1$ for all $i = 0, \dots, n$ and where A stands for the universal ring of coefficients of the F_i's. The saturation of the ideal I with respect to the irrelevant ideal $\mathfrak{m} = (x_0, \dots, x_n)$, which we denote by $I^{\text{sat}} = I : \mathfrak{m}^\infty$, is the ideal of inertia forms.

As the elements in I are trivially inertia forms, I^{sat}/I is the natural quotient to study. It turns out that the Jacobian determinant of the F_i's is a generator, as an A-module, of the graded component of I^{sat}/I in degree $\delta = d_0 + \cdots + d_n - (n+1)$ and their resultant is a generator of I^{sat}/I in degree 0. In order to unravel the structure of I^{sat}/I in degrees smaller than δ, Jouanolou introduced and studied the formalism of Sylvester forms [Jou97]. His ideas were based on the fact that for each $\mu = (\mu_0, \dots, \mu_n) \in \mathbb{N}^{n+1}$ such that $|\mu| := \sum_i \mu_i < \min_i d_i$, each polynomial F_i can be decomposed as

$$F_i = \sum_{j=0}^{n} x_j^{\mu_j+1} F_{i,j} \tag{4.1}$$

and one can consider the determinant $\det(F_{i,j})_{0\leq i,j\leq n}$. This latter is called a *Sylvester form* of the F_i's and denoted by Sylv_μ. Independently of the choice of decompositions (4.1), the class of Sylv_μ modulo I, which is denoted by sylv_μ, gives a nonzero element in $(I^{\mathrm{sat}}/I)_{\delta-|\mu|}$. Moreover, $(I^{\mathrm{sat}}/I)_{\delta-|\mu|}$ is a free A-module which can be generated by the Sylvester forms of degree $\delta-|\mu|$. This result is a consequence of a duality property between Sylvester forms and monomials; namely, for all $\nu < \min_i d_i$ we have an isomorphism of A-modules

$$(I^{\mathrm{sat}}/I)_{\delta-\nu} \simeq \mathrm{Hom}_A(C_\nu, A).$$

More explicitly, this isomorphism corresponds to the equalities

$$x^{\mu'}\,\mathrm{sylv}_\mu = \begin{cases} \mathrm{sylv}_0 & \text{if } \mu = \mu' \\ 0 & \text{if } \mu \neq \mu' \end{cases}$$

where sylv_0 is a generator of $(I^{\mathrm{sat}}/I)_\delta$. We note that up to a nonzero multiplicative constant, sylv_0 is equal to the class of the Jacobian determinant of the F_i's; see [Jou97, §3.10].

The definition and main properties of Sylvester forms have been recently extended to the case of $n+1$ generic multi-homogeneous polynomials, i.e. of polynomials defining hypersurfaces over a product of projective spaces of total dimension n; see [BCN22]. In this chapter, we reproduce the results of [BC22], in which we develop the theory of Sylvester forms in the general setting of homogeneous polynomials in the coordinate ring of a projective toric variety X_Σ. In addition, to illustrate the importance of these forms in elimination theory, we also provide applications to the construction of elimination matrices for overdetermined polynomial systems and to the computation of sparse resultants and toric residues.

1. The σ-positive property

As part of the assumptions of our construction, we define a property of toric varieties which we introduced "ad-hoc" for proving the results in [BC22]. At the end, we will try to motivate that this property can be interesting in other contexts.

Notation 4.1. In what follows, we use the notation introduced in Assumptions 2.2. Namely, given lattice polytopes $\Delta_0, \ldots, \Delta_n$, we consider the normal fan Σ of the Minkowski sum $\Delta = \sum_{i=0}^n \Delta_i$ and consider the projective toric variety X_Σ, which (up to resolving singularities) is smooth and the polytopes $\Delta_0, \ldots, \Delta_n$ correspond to nef Cartier divisors in X_Σ. Let $R = \mathbf{k}[x_\rho\ \rho \in \Sigma(1)]$ be the Cox ring of X_Σ. We choose a maximal smooth cone $\sigma \in \Sigma$ and denote by $x_1, \ldots, x_n$ the variables associated to the rays $\rho \in \sigma(1)$ and by $z_1, \ldots, z_r$ the remaining variables of R. Denote by $u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+r}$ the generators of the rays associated to $x_1, \ldots, x_n, z_1, \ldots, z_r$, respectively. According to this choice of σ, we write a matrix of the map π in the form

$$\pi = \begin{pmatrix} \mathcal{P} & \mathrm{Id}_r \end{pmatrix}, \tag{4.2}$$

where $\mathcal{P}$ is a block matrix $(\mathcal{P}_{j,k})_{1 \leq j \leq r, 1 \leq k \leq n}$ whose rows correspond to the relations between u_{n+j} and the basis given by $u_1, \ldots, u_n$ for $j = 1, \ldots, r$.

In order to introduce Sylvester forms later on, we need the following property which is not standard.

Definition 4.1. For $\sigma \in \Sigma(n)$, the projective toric variety X_Σ is called *σ-positive* if σ is a maximal smooth cone such that a matrix of the map π defined in (2.9) can be written as in (4.2) with the additional condition that $\mathcal{P}_{j,k} \geq 0$ for $j = 1, \ldots, r$ and $k = 1, \ldots, n$.

The above property amounts to require that the vectors $-u_{n+j}$ belong to σ for all $j = 1, \ldots, r$; see Figure 4.1.

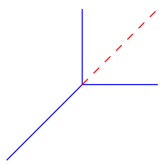


Figure 4.1: An example of the σ-positive property.

A first observation is that not all smooth toric varieties are σ-positive for some $\sigma \in \Sigma(n)$, as shown in the following example.

Example 4.1. Let Σ be the complete smooth fan in $N_{\mathbb{R}} = \mathbb{R}^2$ with the following rays:

$$\rho_1 = (1,0) \; \rho_2 = (0,1) \; \rho_3 = (-1,1) \; \rho_4 = (-1,0) \; \rho_5 = (-1,-1) \; \rho_6 = (0,-1).$$

It is straightforward to check that for every $\sigma \in \Sigma(2)$, there is $\rho \notin \sigma(1)$ such that $-u_\rho \notin \sigma$.

On the other hand, most of the projective toric varieties that are of interest in applications are σ-positive for some smooth maximal cone σ. For instance, this property is preserved under the product of toric varieties. To be more precise, recall that the product of two toric varieties is defined by the product fan; see [CLS12, Theorem 2.4.7]. Any cone of this fan is of the form $\sigma_1 \times \sigma_2$, where its elements are considered as pairs (u, v) for $u \in \sigma_1$ and $v \in \sigma_2$. Moreover, $\dim \sigma_1 \times \sigma_2 = \dim \sigma_1 + \dim \sigma_2$.

Lemma 4.1. If X_1 (resp. X_2) is a toric variety which is σ_1-positive (resp. σ_2-positive) for some maximal cone σ_1 in a fan Σ_1, (resp. σ_2 in a fan Σ_2), then the product $X_1 \times X_2$ is $(\sigma_1 \times \sigma_2)$-positive.

Proof. Any ray ρ of the product fan is generated by an element of the form $(u_{\rho_1}, 0)$ or $(0, u_{\rho_2})$, where ρ_1 is a ray of σ_1 and ρ_2 is a ray of σ_2. By assumption, $-u_{\rho_1}$ and $-u_{\rho_2}$ can be written as a positive combination of elements in either σ_1 or σ_2; therefore, they belong to $\sigma_1 \times \sigma_2$. $\square$

Example 4.2. The projective space $\mathbb{P}^n$ is σ-positive as the map π can be written as $\pi = (1 \cdots 1)$ for any choice of the maximal cone σ. Therefore, any product of projective spaces is σ-positive by Lemma 4.1. Another classical family of smooth toric varieties are Hirzebruch surfaces $\mathcal{H}_b \subset \mathbb{R}^2$: for each $r \in \mathbb{Z}_{>0}$, these varieties correspond to the fans Σ_b with rays

$$\rho_1 = (1, 0)\ \rho_2 = (0, 1)\ \rho_3 = (-1, -b)\ \rho_4 = (0, -1).$$

Hirzebruch surfaces are smooth and σ-positive with respect to the smooth maximal cone $\sigma = \langle \rho_1, \rho_2 \rangle$ as π can be written as

$$\pi = \begin{pmatrix} 1 & r & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Following Kleinschmidt's classification of smooth toric varieties of Picard rank 2 [Kle88], we can give a larger family of toric varieties having the σ-positive. He proved that all these varieties can be constructed as a projectivization of toric vector bundles over the projective space; see [CLS12, Theorem 7.3.7].

Theorem 4.1. Let X_Σ be a smooth projective toric variety such that $\mathrm{Pic}(X_\Sigma) = \mathbb{Z}^2$. Then, there are $s, r \geq 1$ and $0 \leq a_1 \leq \cdots \leq a_r$ such that:

$$X_\Sigma \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$$

Using the notation above, the rays $u_0, u_1, \ldots, u_s$ are the generators of the fan providing the projective space $\mathbb{P}^s$ where $u_1, \ldots, u_s$ is a canonical basis of $\mathbb{Z}^s$ and $u_0 = -\sum_{i=1}^{s} u_i$. Similarly, we can define $e_1, \ldots, e_r$ as a basis of $\mathbb{R}^r$ and $e_0 = -\sum_{i=1}^{r} e_i$. Then, the generators of the rays of X_Σ are:

$$v_0, \ldots, v_s, e_0, \ldots, e_r$$

where $v_0 = u_0 + a_1 e_1 + \cdots + a_r e_r$ and $v_j = u_j$ for all $j = 1, \ldots, s$; see [CLS12, Example 7.3.5]. The relations between the generators of the rays are:

$$e_0 + \cdots + e_r = 0 \quad v_0 + \cdots + v_s = a_1 e_1 + \cdots + a_r e_r \tag{4.3}$$

Corollary 4.1. Let X_Σ be a smooth projective toric variety such that $\mathrm{Pic}(X_\Sigma) = \mathbb{Z}^2$. Then, X_Σ has the σ-positive property with respect to the cone

$$\sigma = \mathrm{Cone}(v_1, \ldots, v_s, e_0, \ldots, e_{r-1}).$$

Proof. The rays that are not in σ are generated by e_r and v_0. Using (4.3), we get:

$$-e_r = \sum_{j=0}^{r-1} e_j$$

and

$$-v_0 = v_1 + \cdots + v_s - a_1 e_1 - \cdots - a_{r-1} e_{r-1} + a_r \sum_{j=0}^{r-1} e_j = v_1 + \cdots + v_s + (a_r - a_1) e_1 + \dots (a_r - a_{r-1}) e_{r-1}$$

deriving the σ-positive property. $\square$

More generally, Batyrev classified all smooth projective toric varieties whose fan splits [Bat91]. Namely, these varieties can be constructed from a series of projectivizations of vector bundles, similarly as above. This leads us to making the following conjecture.

Conjecture 4.1. All smooth toric varieties X_Σ whose fan splits have the σ-positive property for some maximal smooth cone $\sigma \in \Sigma(1)$.

In what follows, we prove the existence of certain decompositions of homogeneous polynomials that we will use in Section 3. for defining toric Sylvester forms. For the sake of clarity, we denote with a lowercase letter f any polynomial in the Cox ring R of a toric variety X_Σ, whose coefficient ring is a field, in contrast with generic polynomials that we denoted above with a capital letter (see also Notation 4.3).

Theorem 4.2. Let X_Σ be a projective toric variety of dimension n such that X_Σ is σ-positive with respect to a smooth cone $\sigma \in \Sigma(n)$. Let J be an ideal of the Cox ring R of X_Σ, generated by homogeneous polynomials $f_0, \dots, f_n$ of degrees $\alpha_0, \dots, \alpha_n$, respectively, whose polytopes $\Delta_0, \dots, \Delta_n$ are written as in (2.34) and only depend on $(a_{i,n+j})_{j=1,\dots,r} \in \mathbb{Z}^r$. Let $\nu \in \mathrm{Cl}(X_\Sigma)$ be a nef Cartier class and let Δ_ν be the corresponding polytope, written as in (2.13), for some $(\nu_{n+j})_{j=1,\dots,r} \in \mathbb{Z}^r$ which satisfies

$$0 \le \nu_{n+j} < \min_{i=0,\dots,n} a_{i,n+j} \text{ for all } j = 1, \dots, r \tag{4.4}$$

Then, the two following properties hold:

(i) $R_\nu = (R/J)_\nu$.

(ii) For every $x^\mu \in R_\nu$ and $f_i \in R_{\alpha_i}$ and $i = 0, \dots, n$, there exists a decomposition of the form

$$f_i = z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1} f_{i,0}^\mu + x_1^{\mu_1+1} f_{i,1}^\mu + \cdots + x_n^{\mu_n+1} f_{i,n}^\mu \tag{4.5}$$

where the $f_{i,j}^\mu$, $i, j = 0, \dots, n$, are homogeneous polynomials in R.

Proof. The graded quotient map $R_\nu \to (R/J)_\nu$ is surjective. If there is a nonzero polynomial of degree ν in J, there must be a monomial $x^\mu \in R_\nu$ that is divided by some monomial $x^{\mu_i} \in R_{\alpha_i}$ of degree α_i for some $i \in \{0, \dots, n\}$, i.e. the degrees of the generators of J. If that is the case, then $x^\mu = x^{\tilde{\mu}} x^{\mu_i}$ for some monomial $x^{\tilde{\mu}}$ of degree $\nu - \alpha_i \in \mathrm{Cl}(X_\Sigma)$. However, using (4.4), we see that $\nu_{n+j} - a_{i,n+j} < 0$ and by

Remark 2.3, there cannot be any monomials of this degree in R. Thus, the kernel of the previous map is zero, proving (i).

We turn to the proof of (ii). Recall from (2.21) that every monomial of degree α_i (and thus every monomial in f_i) can be written as $x^{\mathbf{F}m+a_i}$ for some lattice point $m \in \mathcal{A}_i$. Thus we fix a monomial $x^{\mathbf{F}m_0+a_i}$ in the support of f_i for some $m_0 \in \mathcal{A}_i$.

Given $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n} z_1^{\mu_{n+1}} \dots z_r^{\mu_{n+r}} \in R_\nu$, we are going to show that if $x^{\mathbf{F}m_0+a_i} \in R_{\alpha_i}$ is not divisible by the monomial $z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}$, then it must be divisible by one of the monomials $x_1^{\mu_1+1}, \dots, x_n^{\mu_n+1}$. Indeed, if this property holds, every monomial in f_i for $i = 0, \dots, n$ must be divided by either $x_1^{\mu_1+1}, \dots, x_n^{\mu_n+1}$ or $z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}$ and the decompositions (4.5) follow.

Using that $a_{i,k} = 0$ for $k = 1, \dots, n$, the $n+r$ components of $\mathbf{F}m_0 + a_i$ are:

$$\begin{cases} \langle u_k, m_0 \rangle & k \in \{1, \dots, n\} \\ \langle u_{n+j}, m_0 \rangle + a_{i,n+j} & j \in \{1, \dots, r\}. \end{cases} \tag{4.6}$$

Thus, the fact that $x^{\mathbf{F}m_0+a_i}$ is not divisible by $z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}$ implies that

$$\langle u_{n+j_0}, m_0 \rangle + a_{i,n+j_0} \leq \mu_{n+j_0} \text{ for some } j_0 \in \{1, \dots, r\}.$$

From here, using (4.4), we get that $\langle u_{n+j_0}, m_0 \rangle + \nu_{n+j_0} < \mu_{n+j_0}$. On the other hand, the monomial $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n} z_1^{\mu_{n+1}} \dots z_r^{\mu_{n+r}}$ is of degree ν and hence by (2.19), we have $\nu_{n+j_0} = \mu_{n+j_0} + \sum_{k=1}^n \mathcal{P}_{j_0,k}\mu_k$, which implies that

$$\langle u_{n+j_0}, m_0 \rangle + \sum_{k=1}^{n} \mathcal{P}_{j_0,k}\mu_k < 0.$$

Finally, we use the relation (2.18) between u_{n+j_0} and the generators of σ to derive the inequality

$$\sum_{k=1}^{n} \mathcal{P}_{j_0,k}(\mu_k - \langle u_k, m_0 \rangle) < 0.$$

As X_Σ is σ-positive and all the $\mathcal{P}_{j_0,k}$'s are non-negative integers for all $k = 1, \dots, n$ (and they are not all equal to 0 as $u_{n+j} \neq 0$), there exists $k_0 \in \{1, \dots, n\}$ such that $\mu_{k_0} - \langle u_{k_0}, m \rangle < 0$. As the exponent of x_{k_0} in $x^{\mathbf{F}m_0+a_i}$ is precisely $\langle u_{k_0}, m_0 \rangle$, we deduce that $x_{k_0}^{\mu_{k_0}+1}$ divides $x^{\mathbf{F}m_0+a_i}$. □

Corollary 4.2. Assume that the projective toric variety X_Σ is σ-positive for some $\sigma \in \Sigma(n)$. If the polytopes Δ_i in (2.34) are n-dimensional for all $i = 0, \dots, n$, then Theorem 4.2 holds for $(\nu_{n+j})_{j=1,\dots,r} = 0 \in \mathbb{Z}^r$.

Proof. If there are $i_0 \in \{0, \dots, n\}$ and $j_0 \in \{1, \dots, r\}$ such that $a_{i_0,n+j_0} = 0$, then for every $m \in \mathcal{A}_{i_0} = \Delta_{i_0} \cap \mathbb{Z}^n$, we have the inequality $\langle u_{n+j_0}, m \rangle \geq 0$. Using the relation (2.18), we get

$$\sum_{k=1}^{n} \mathcal{P}_{j_0,k} \langle u_k, m \rangle \leq 0 \quad \forall m \in \mathcal{A}_{i_0}.$$

As X_Σ is σ-positive and the $\mathcal{P}_{j_0,k}$ are non-negative integers for $k = 1, \dots, n$ (not all equal to zero as $u_{n+j_0} \neq 0$), there must be some $k_0 \in \{1, \dots, n\}$ such that $\langle u_{k_0}, m \rangle \leq 0$ for all $m \in \mathcal{A}_{i_0}$. On the other hand, we also have the inequality $\langle u_{k_0}, m \rangle \geq 0$ for all $m \in \mathcal{A}_{i_0}$ due to the presentation in (2.34) and using that $a_{i_0,k} = 0$ for all $k = 1, \dots, n$. Thus, the lattice points in Δ_{i_0} must satisfy $\langle u_{k_0}, m \rangle = 0$ and thus Δ_{i_0} cannot be n-dimensional.

Therefore, if the Δ_i are n-dimensional for all $i = 0, \dots, n$, we have

$$0 < \min_{i=0,\dots,n} a_{i,n+j} \quad j = 1, \dots, r,$$

which proves that $(\nu_{n+j})_{j=1,\dots,r} = 0 \in \mathbb{Z}^r$ satisfies the hypotheses of Theorem 4.2. $\square$

Finally, we note that that if X_Σ is assumed to be σ-positive, then Theorem 4.2 can be easily extended to the setting of generic homogeneous sparse polynomials in (2.21) and yield a decomposition of the F_i for $i = 0, \dots, n$, over $X_\Sigma \times_{\mathbf{k}} \operatorname{Spec}(A)$.

Remark 4.1. If X_Σ does not have the σ-positive property, but one can find another way to decompose the polynomials F_i for $i = 0, \dots, n$ as in (4.5), then the results presented in the next sections hold similarly. One such example is the construction of the form Δ_σ with a nonzero residue, as detailed in [CCD97, Theorem 0.2], which relies on the polynomials F_i corresponding to $\mathbb{Q}$-ample divisors.

2. A duality theorem

Let X_Σ be a projective toric variety of dimension n which admits a maximal smooth cone $\sigma \in \Sigma(n)$. In this section, we consider the ideal generated by $n+1$ generic homogeneous sparse polynomials (2.21) and analyze some graded components of its saturation via a duality property. For that purpose, we take again the notation of the resultant setting (2.33): $F_0, \dots, F_n$ are the generic homogeneous polynomials of degree $\alpha_0, \dots, \alpha_n$, respectively; they are of the form

$$F_i = \sum_{x^\mu \in R_{\alpha_i}} c_{i,\mu} x^\mu \in C = A[x_1, \dots, x_n, z_1, \dots, z_r]. \tag{4.7}$$

As a preliminary result, we first show that $F_0, \dots, F_n$ form a regular sequence outside $V(\mathfrak{b}) \subset \operatorname{Spec}(C)$.

Lemma 4.2. For every maximal cone $\tau \in \Sigma(n)$ and for every $i = 0, \dots, n$, there is a lattice point $m_{i,\tau} \in \mathcal{A}_i$ and $L \in \mathbb{Z}_{>0}$ such that $x^{\mathbf{F}m_{i,\tau}+a_i}$ divides $(\tilde{x}^\tau)^L$ where $\tilde{x}^\tau$ is defined in (2.10).

Proof. The exponents of of $x^{\mathbf{F}m+a_i}$ are:

$$\begin{cases} \langle u_k, m \rangle & k \in \{1, \dots, n\} \\ \langle u_{n+j}, m \rangle + a_{i,n+j} & j \in \{1, \dots, r\}. \end{cases} \tag{4.8}$$

Thus, using (2.14), we can find $m_{i,\tau} \in \mathcal{A}_i$ such that for $\rho_j \in \tau(1)$, we have $\langle u_j, m_{i,\tau}\rangle + a_{i,j} = 0$. Moreover, we can choose L that bounds above $\langle u_j, m_{i,\tau}\rangle + a_{i,j}$ for $\rho_j \notin \tau(1)$. Therefore, $x^{\mathbf{F}m_{i,\tau}+a_i}$ divides $(\tilde{x}^\tau)^L$. $\square$

Lemma 4.3. The homogeneous generic polynomials $F_0, \dots, F_n$ define a regular sequence in the localization ring $C_{\tilde{x}^\tau}$ for any $\tau \in \Sigma(n)$.

Proof. We claim that F_0 is a nonzero divisor in C. This follows from Dedekind-Mertens Lemma [BJ14, Corollary 2.8], which says that a polynomial F is a nonzero divisor in $A[x_1, \dots, x_n]$ if its content ideal is a nonzero divisor in A. The content ideal is generated by the coefficients $c_{0,\mu}$ for $x^\mu \in R_{\alpha_0}$ and they are all nonzero divisors. Therefore, F_0 is a nonzero divisor also in $C_{\tilde{x}^\tau}$ for all $\tau \in \Sigma(n)$.

By Lemma 4.2, we can always find $m_{i,\tau} \in \mathcal{A}_i$ such that $x^{\mathbf{F}m_{i,\tau}+a_i}$ is invertible in the localization ring $C_{\tilde{x}^\tau}$ and let $c_{i,\tau}$ be the coefficient in A associated to this monomial. Then, similarly to [BCN22, Lemma 3.2], for any $t \in \{0, \dots, n-1\}$ there is an isomorphism of $(A^t_\tau[x_1, \dots, x_n, z_1, \dots, z_r])$-algebras

$$\big(A[x_1, \dots, x_n, z_1, \dots, z_r]/\langle F_0, \dots, F_t\rangle\big)_{\tilde{x}^\tau} \xrightarrow{\sim} (A^t_\tau[x_1, \dots, x_n, z_1, \dots, z_r])_{\tilde{x}^\tau}$$

where $A^t_\tau = \mathbf{k}[c_{i,\mu} \quad c_{i,\mu} \neq c_{i,\tau} \quad 0 \leq i \leq t]$ i.e., $A = A^t_\tau[c_{i,\tau} \quad 0 \leq i \leq t]$. This map sends $c_{i,\tau}$ to $\frac{-F_i + c_{i,\tau} x^{\mathbf{F}m_{i,\tau}+a_i}}{x^{\mathbf{F}m_{i,\tau}+a_i}}$ for $i = 0, \dots, t$, and leaves the rest of coefficients and variables invariant. Applying again the Dedekind-Mertens Lemma as above, we deduce that the polynomial F_{t+1} is a nonzero divisor in $(A^t_\tau[x_1, \dots, x_n, z_1, \dots, z_r])_{\tilde{x}^\tau}$, and therefore in the ring $\big(A[x_1, \dots, x_n, z_1, \dots, z_r]/\langle F_0, \dots, F_t\rangle\big)_{\tilde{x}^\tau}$. $\square$

Next, we consider the two canonical spectral sequences associated with the Čech-Koszul double complex $\mathcal{C}^\bullet_\mathfrak{b}(K_\bullet(F))$, where $K_\bullet(F)$ denotes the Koszul complex of the sequence of homogeneous polynomials $F_0, \dots, F_n$ in C. The terms of the Koszul complex are graded free C-modules and we denote their homology modules by H_p for simplicity in the notation. If we start taking homologies horizontally, the second page is:

$$\begin{array}{ccccc}
H^0_\mathfrak{b}(H_{n+1}) & H^0_\mathfrak{b}(H_n) & H^0_\mathfrak{b}(H_{n-1}) & \cdots & H^0_\mathfrak{b}(H_0) = I^{\mathsf{sat}}/I \\
0 & 0 & 0 & \cdots & H^1_\mathfrak{b}(H_0) \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & H^n_\mathfrak{b}(H_0) \\
0 & 0 & 0 & \cdots & H^{n+1}_\mathfrak{b}(H_0)
\end{array}.$$

The vanishing of the local cohomology modules $H^i_{\mathfrak{b}}(H_j)$ for $i > 0$ and $j > 0$ follows from Lemma 4.3 which shows that the F_i's form a regular sequence outside $V(\mathfrak{b})$. In addition, we deduce that H_p are geometrically supported on $V(\mathfrak{b})$ for all $p > 0$ by a classical property of Koszul complexes, and hence that $H^0_{\mathfrak{b}}(H_p) = H_p$ for all $p > 0$.

On the other hand, if we start taking homologies vertically, we obtain the following first page:

$$\begin{array}{ccccc}
0 & & 0 & \cdots & 0 \\
 & & & & \\
0 & & 0 & \cdots & 0 \\
 & & & & \\
\vdots & & \vdots & & \vdots \\
 & & & & \\
H^n_{\mathfrak{b}}(C(-\sum_j \alpha_j)) & \to & H^n_{\mathfrak{b}}(\oplus_{k,k'} C(-\sum_{j\neq k,k'} \alpha_j)) & \cdots & H^n_{\mathfrak{b}}(C) \\
 & & & & \\
H^{n+1}_{\mathfrak{b}}(C(-\sum_j \alpha_j)) & \to & H^{n+1}_{\mathfrak{b}}(\oplus_{k,k'} C(-\sum_{j\neq k,k'} \alpha_j)) & \cdots & H^{n+1}_{\mathfrak{b}}(C)
\end{array}$$

using that $K_j(F) = \bigoplus_{|J|=j}^{J\subset\{0,\dots,r\}} C(-\sum_{k\in J} \alpha_k)$. We note that the vanishing of the two first rows follows from (2.24) and the vanishing of $H^p_{\mathfrak{b}}(C)$ for all $p > n+1$ is a consequence of Grothendieck's vanishing theorem [Gro57, Theorem 3.6.5].

Notation 4.2. The support Supp S of a graded module S is the subset of $\nu \in \mathrm{Cl}(X_\Sigma)$ such that $S_\nu \neq 0$. We denote by Γ_1 the support of the modules on the main diagonal, except on the last row, and by Γ_0 the support of the modules in the diagonal under Γ_1, except on the last row again, i.e.

$$\Gamma_i = \mathrm{Supp}(\oplus_{p=0}^n H^p_{\mathfrak{b}}(K_{p+i-1}(F))) \quad i = 0, 1. \tag{4.9}$$

In addition, we define Γ_{Res} to be the support of all the cohomology modules that are appearing above the diagonal in the first page of the second spectral sequence, i.e. $\Gamma_{\mathrm{Res}} = \mathrm{Supp}(\oplus_{i<j} H^i_{\mathfrak{b}}(K_j(F)))$. Moreover, from now on, we denote by δ the divisor class $\alpha_0 + \cdots + \alpha_n - K_X$ where K_X denotes the anticanonical divisor of X_Σ.

Remark 4.2. In the above analysis of the two spectral sequences associated to F, we proved that $K_\bullet(F)_\alpha$ is an acyclic complex of A-modules for all $\alpha \notin \Gamma_{\mathrm{Res}}$.

The comparison of the two above spectral sequences leads to the following duality theorem.

Theorem 4.3. Let X_Σ be a projective toric variety which admits a maximal smooth cone $\sigma \in \Sigma(n)$ and let $\nu \in \mathrm{Cl}(X_\Sigma)$ be a nef Cartier divisor. If $\delta - \nu \notin \Gamma_0 \cup \Gamma_1$, then

$$(I^{\mathrm{sat}}/I)_{\delta-\nu} \simeq \mathrm{Hom}_A((C/I)_\nu, A).$$

Proof. From the comparison of the two spectral sequences associated to the double complex $\mathcal{C}^\bullet_\mathfrak{b}(K_\bullet(F))$, for all $\nu \in \mathrm{Cl}(X_\Sigma)$ such that $\delta-\nu \notin \Gamma_0\cup\Gamma_1$, we get an isomorphism

$$(I^{\mathrm{sat}}/I)_{\delta-\nu} \simeq \mathrm{Ker}\left(H^{n+1}_\mathfrak{b}(C(-\sum_j \alpha_j)) \to H^{n+1}_\mathfrak{b}(\oplus_k C(-\sum_{j\neq k}\alpha_j))\right)_{\delta-\nu}.$$

Moreover, using toric Serre duality (2.25) and the relation between sheaf and local cohomology modules (2.23), we obtain

$$H^{n+1}_\mathfrak{b}(C(-\sum_j \alpha_j))_{\delta-\nu} \simeq H^n(X_\Sigma,-\nu-K_X) \simeq H^0(X_\Sigma,\nu)^\vee \simeq \mathrm{Hom}_A(C_\nu,A).$$

By the same argument, we also have $H^{n+1}_\mathfrak{b}(\oplus_k C(-\sum_{j\neq k}\alpha_i))_{\delta-\nu} \simeq \mathrm{Hom}_A(I_\nu,A)$. Using the first isomorphism, we get the duality property. □

Corollary 4.3. Let X_Σ be a projective toric variety which admits a maximal smooth cone $\sigma \in \Sigma(n)$. Let $\Delta_0,\dots,\Delta_n$ be lattice polytopes as in (2.34) corresponding to the polynomials $F_0,\dots,F_n$. Let $\nu \in \mathrm{Cl}(X_\Sigma)$ be a nef Cartier class and Δ_ν be the corresponding polytope, written as in (2.13), satisfying $0 \le \nu_{n+j} < \min_{i=0,\dots,n} a_{i,n+j}$ for $j=1,\dots,r$. Assume also that $\delta-\nu \notin \Gamma_0\cup\Gamma_1$. Then,

$$(I^{\mathrm{sat}}/I)_{\delta-\nu} \simeq \mathrm{Hom}_A(C_\nu,A).$$

In particular, $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ is a free A-module whose rank is equal to the rank of C_ν, equivalently $\mathrm{HF}(R,\nu)$.

Proof. Using Theorem 4.2 $i)$ (which does not require the σ-positive property), we can derive that $(C/I)_\nu = C_\nu$. □

Remark 4.3. We notice that the case $\nu=0$, which corresponds to the isomorphism $(I^{\mathrm{sat}}/I)_\delta \simeq A$, appears in [CDS97] in the case the polytopes $\Delta_0,\dots,\Delta_n$ are scaled copies of the same ample polytope.

To close this section, we prove that if we consider a proper subset of the polynomials that generate I, then the corresponding ideal must be saturated at δ. We will need this property in the next section.

Lemma 4.4. Assume that the polytopes $\Delta_0,\dots,\Delta_n$ are n-dimensional. Let T be a proper subset of $\{0,\dots,n\}$ and consider the ideal $I_T=(F_i,\, i\in T)$. Then, $(I_T^{\mathrm{sat}})_\delta = (I_T)_\delta$.

Proof. Consider the cohomology groups $H^i_\mathfrak{b}(K_j(F_T))$ where $K_j(F_T)$ denotes the Koszul complex associated to I_T. Then, by (2.23),

$$H^i_\mathfrak{b}(K_j(F_T))_\delta = \bigoplus_{|J|=j}^{J\subset T} H^i_\mathfrak{b}\big(C(-\sum_{k\in J}\alpha_k)\big)_\delta = \bigoplus_{|J|=j}^{J\subset T} H^{i-1}\big(X_\Sigma,\sum_{k\notin J}\alpha_k - K_X\big).$$

Using Serre duality (2.25), each of the summands is of the form:

$$H^{i-1}(X_\Sigma, \sum_{k\notin J} \alpha_k - K_X) \simeq H^{n-i+1}(X_\Sigma, -\sum_{k\notin J} \alpha_k).$$

As $J \subset T$ is a proper subset, $\sum_{k\notin J} \alpha_k$ is nef and its associated polytope is n-dimensional. Therefore, we can apply Theorem 2.10, implying that $H^i_{\mathfrak{b}}(K_j(F_T))_\delta = 0$ for all $i \geq 2$. If $i = 0, 1$, we can use (2.24). Therefore, comparing the two spectral sequences of the Čech-Koszul double complex, we get $\left(I_T^{\text{sat}}/I_T\right)_\delta = 0$. $\square$

3. Toric Sylvester forms

We take again the notation of Section 2.. As a consequence of Corollary 4.3, some graded components of I^{sat}/I are free A-modules and hence a natural question is to provide explicit A-bases for them. This is precisely the goal of this section. We first describe the graded component $(I^{\text{sat}}/I)_\delta$, which essentially follows from [CCD97]. Then, we introduce Sylvester forms to deal with the other cases. In what follows, we assume that the projective toric variety X_Σ is σ-positive with respect to a maximal smooth cone $\sigma \in \Sigma(n)$.

Along the same lines as [CCD97], a nonzero element in $(I^{\text{sat}}/I)_\delta \simeq A$ can be constructed as follows. Using Corollary 4.2, if the polytopes $\Delta_0, \dots, \Delta_n$ are n-dimensional, one can decompose each polynomial as

$$F_i = z_1 \cdots z_r F_{i,0} + x_1 F_{i,1} + \cdots + x_n F_{i,n}, \tag{4.10}$$

and consider the determinant

$$\text{Sylv}_0 = \det \left(F_{i,j}\right)_{0\leq i,j\leq n}.$$

This homogeneous polynomial is called the *toric jacobian*; we will denote its class modulo I by sylv_0. Observe that, by construction, Sylv_0 is a linear form with respect to the coefficients of each F_i, $i = 0, \dots, n$.

Lemma 4.5. Assume that $\Delta_0, \dots, \Delta_n$ are n-dimensional polytopes. Let $P \in I^{\text{sat}}_\delta$ be any homogeneous polynomial whose class in $(I^{\text{sat}}/I)_\delta$ is nonzero. Then, for all $i = 0, \dots, n$, P must have degree ≥ 1 with respect to the coefficients of F_i.

Proof. For simplicity, suppose that P does not depend on the coefficients of F_0. For any maximal cone $\tau \in \Sigma(n)$, consider the monomial $x^{\mathbf{F}m_{0,\tau}+a_0}$ for some $m_{0,\tau} \in \mathcal{A}_i$, which is invertible in $C_{\tilde{x}^\tau}$ by Lemma 4.2. Let $c_{0,\tau}$ be the coefficient of $x^{\mathbf{F}m_{0,\tau}+a_0}$ in F_0 and consider P as an element of $C_{\tilde{x}^\tau}$. As $P \in I^{\text{sat}}$, there must be $L \in \mathbb{Z}_{>0}$ such that:

$$(\tilde{x}^\tau)^L P = G_0 F_0 + \cdots + G_n F_n \in I.$$

However, as P does not involve $c_{0,\tau}$, we can change this coefficient in $C_{\tilde{x}^\tau}$ by

$$\frac{c_{0,\tau}x^{\mathbf{F}m+a_0} - F_0}{x^{\mathbf{F}m+a_0}}$$

without changing P. Therefore, $(\tilde{x}^\tau)^L P$ belongs to the ideal generated by $F_1, \ldots, F_n$ in $C_{\tilde{x}^\tau}$. Up to multiplying by some power $L' \geq L$, we must have that $(\tilde{x}^\tau)^{L'} P$ belongs to the ideal generated by $F_1, \ldots, F_n$ in C. As the above conslusion holds for every $\tau \in \Sigma(n)$, we deduce that $P \in (F_1, \ldots, F_n)^{\text{sat}}_\delta$. Now, using that the polytopes $\Delta_0, \ldots, \Delta_n$ are n-dimensional, Lemma 4.4 implies that $P \in (F_1, \ldots, F_n)_\delta$, contradicting that the class of P modulo I is nonzero. $\square$

Proposition 4.1. If the polytopes $\Delta_0, \ldots, \Delta_n$ are n-dimensional, the element Sylv_0 belongs to $(I^{\text{sat}})_\delta$. Moreover, sylv_0 is independent of the choices of decompositions (4.10). In addition, if $\delta \notin \Gamma_0 \cup \Gamma_1$, then sylv_0 is a generator of $(I^{\text{sat}}/I)_\delta$ which is a free A-module of rank 1.

Proof. Note that if $\tau \in \Sigma(n)$, then either $\tau \neq \sigma$, in which case there is $k \in \{1, \ldots, n\}$ such that x_k divides $\tilde{x}^\tau$ or $\tau = \sigma$, in which case $\tilde{x}^\tau = z_1 \cdots z_r$. Using the invariance of the determinant under column operations and using the decomposition in (4.10), we get

$$x_k \,\text{Sylv}_0 = \det\begin{pmatrix} \cdots & x_k F_{0,k} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & x_k F_{n,k} & \cdots \end{pmatrix} = \det\begin{pmatrix} \cdots & F_0 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & F_n & \cdots \end{pmatrix} \in I,\ k = 1, \ldots, n. \tag{4.11}$$

The same holds for the monomial $z_1 \cdots z_r$. Therefore, we deduce that $\text{Sylv}_0 \in I^{\text{sat}} = (I : \mathfrak{b}^\infty)$. In order to prove that sylv_0 has degree δ, we find the degree of each entry (i, j) of the matrix defined by the $F_{i,j}$'s. In (4.10), we divided the monomials of degree α_i by a monomial of degree

$$\begin{cases} \pi(e_j) & \text{if the monomial is } x_k \text{ for } k = 1, \ldots, n, \\ \pi(\sum_{j=1}^r e_{n+j}) & \text{if the monomial is } z_1 \cdots z_r, \end{cases}$$

where $\{e_j\}_{j=1}^{n+r}$ is the canonical basis of $\mathbb{Z}^{\Sigma(1)}$. On the other hand, the anticanonical class K_X coincides with the degree of the monomial $x_1 \cdots x_n z_1 \cdots z_r$ (see [CLS12, Theorem 8.2.3]), which is equal to $\pi(\sum_{j=1}^{n+r} e_j)$. Therefore, the degree of each the summands constituting the determinant is equal to:

$$\sum_{i=0}^{n} \left(\alpha_i - \pi(e_{\tau(i)})\right) = \left(\sum_{i=0}^{n} \alpha_i\right) - K_X = \delta, \tag{4.12}$$

where $e_0 = \sum_{k=n+1}^{n+r} e_k$ and τ is any permutation of $\{0, \ldots, n\}$.

The fact that sylv_0 is nonzero and the independence from the choice of the decompositions in (4.10) are consequences of the global transformation law; see [CCD97, Remark 2.12 iii), iv)].

If $\delta \notin \Gamma_0 \cup \Gamma_1$, $(I^{\text{sat}}/I)_\delta$ is a free A-module of rank one. By Lemma 4.5, any generator g of $(I^{\text{sat}}/I)_\delta$ must have degree greater or equal than 1 with respect to the coefficients of F_i for $i = 0, \dots, n$. On the other hand, the construction of sylv_0 indicates that for all $i = 0, \dots, n$, the degree of sylv_0 with respect to the coefficients of F_i is smaller or equal to 1. Thus, if we write $\text{sylv}_0 = cg$ for some $c \in A$, the degree of c with respect to A must be zero, implying that $c \in \mathbf{k}$. This implies that sylv_0 is also a generator of $(I^{\text{sat}}/I)_\delta$ as an A-module. $\square$

In order to use [CCD97, Remark 4.12 iv)], we need to be able to specialize to values in the field of complex numbers $\mathbb{C}$. Therefore, from now on, we assume that the field $\mathbf{k}$ is a *subfield of the complex numbers.* Assuming $\delta \notin \Gamma_0 \cup \Gamma_1$, Theorem 2.10 implies that the Sylvester form sylv_0 corresponds to the unique lattice point in the interior of the polytope Δ_Σ associated to the anticanonical divisor K_X, i.e.

$$(I^{\text{sat}}/I)_\delta \simeq H^{n+1}_{\mathfrak{b}}(C(-\sum \alpha_i))_\delta \simeq H^n(X_\Sigma, -K_X) \simeq \oplus_{m \in \text{Relint}(\Delta_\Sigma)} A\chi^{-m}.$$

So far, we proved that the toric Jacobian sylv_0 yields an A-basis of $(I^{\text{sat}}/I)_\delta \simeq A$. The next step is to construct an A-basis of $(I^{\text{sat}}/I)_{\delta-\nu}$ when it is a free A-module.

Definition 4.2. Let X_Σ be a projective toric variety which is σ-positive for some $\sigma \in \Sigma(n)$. Assume that the polytopes $\Delta_0, \dots, \Delta_n$ are n-dimensional. Let $\nu \in \text{Cl}(X_\Sigma)$ be a nef Cartier class and Δ_ν be the corresponding polytope written as in (2.13) and satisfying $0 \leq \nu_{n+j} < \min_{i=0,\dots,n} a_{i,n+j}$ for $j = 1, \dots, r$. According to Theorem 4.2, for any $x^\mu \in R_\nu$ and for any $i \in \{0, \dots, n\}$ the polynomial F_i can be decomposed as

$$F_i = z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1} F^\mu_{i,0} + x_1^{\mu_1+1} F^\mu_{i,1} + \cdots + x_n^{\mu_n+1} F^\mu_{i,n}. \tag{4.13}$$

We define the *toric Sylvester form* Sylv_μ as the determinant

$$\text{Sylv}_\mu = \det(F^\mu_{i,j})_{0 \leq i,j \leq n}.$$

The class of Sylv_μ modulo I is denoted by sylv_μ. Observe that, as with Sylv_0, the Sylvester forms are linear in the coefficients of F_i for $i = 0, \dots, n$.

If we are given two different monomials $x^\mu, x^{\mu'} \in C_\nu$, there must be some $k \in \{1, \dots, n\}$ such that $\mu_k \neq \mu'_k$. Otherwise, using (2.19), we can derive that $x^\mu = x^{\mu'}$. With this, we can introduce the following *lexicographical monomial order*.

Definition 4.3. Given two monomials x^μ and $x^{\mu'}$ of degree ν, we say $\mu < \mu'$ if $k_0 = \min\{k \in \{1, \dots, n\} \quad \mu_k \neq \mu_k\}$ satisfies $\mu_{k_0} < \mu'_{k_0}$.

Theorem 4.4. Let X_Σ be a projective toric variety which is σ-positive for some $\sigma \in \Sigma(n)$ and that the polytopes $\Delta_0, \dots, \Delta_n$ are n-dimensional. Let $\nu \in \text{Cl}(X_\Sigma)$ be a class satisfying the hypotheses of Theorem 4.2. Then, for every $x^\mu \in R_\nu$, Sylv_μ belongs to $(I^{\text{sat}})_{\delta-\nu}$ and its class sylv_μ is a nonzero element in $(I^{\text{sat}}/I)_{\delta-\nu}$. Moreover, for $x^\mu, x^{\mu'} \in R_\nu$, we have

$$x^{\mu'}\,\text{sylv}_\mu = \begin{cases} \text{sylv}_0 & \mu = \mu' \\ 0 & \mu < \mu' \end{cases} \tag{4.14}$$

As a consequence, the Sylvester forms $\{\mathrm{sylv}_\mu\}_{x^\mu \in R_\nu}$ are linearly independent in $(I^{\mathrm{sat}}/I)_{\delta-\nu}$.

Proof. The fact that Sylv_μ is of degree $\delta - \nu$ follows by analyzing the degree of each summand in $\det(F^\mu_{i,j})$ as in (4.12). Moreover, we can use the same argument as in (4.11), to see that $x_k^{\mu_k+1}\,\mathrm{Sylv}_\mu \in I$ for all $k = 1, \ldots, n$ and $z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}\,\mathrm{Sylv}_\mu \in I$. This proves that $\mathrm{Sylv}_\mu \in I^{\mathrm{sat}}_{\delta-\nu}$. Consider two distinct monomials $x^\mu, x^{\mu'} \in R_\nu$ such that $\mu < \mu'$, then there is $k_0 \in \{1, \ldots, n\}$ such that:

$$x^{\mu'}\,\mathrm{Sylv}_\mu = \frac{x^{\mu'}}{x_{k_0}^{\mu_{k_0}+1}} x_{k_0}^{\mu_{k_0}+1}\,\mathrm{Sylv}_\mu \in I$$

and hence $x^{\mu'}\,\mathrm{sylv}_\mu = 0 \in (I^{\mathrm{sat}}/I)_{\delta-\nu}$. On the other hand, we have:

$$x^\mu\,\mathrm{Sylv}_\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} z_1^{\mu_{n+1}} \cdots z_r^{\mu_{n+r}} \det(F^\mu_{i,j}) = \det(x_j^{\mu_j} F^\mu_{i,j})$$

but at the same time, the decomposition

$$F_i = z_1 \cdots z_r z_1^{\mu_{n+1}} \cdots z_r^{\mu_{n+r}} F^\mu_{i,0} + x_1 x_1^{\mu_1} F^\mu_{i,1} + \cdots + x_n x_n^{\mu_2} F^\mu_{i,n}$$

gives the Sylvester form sylv_0, implying that $x^\mu\,\mathrm{sylv}_\mu = \mathrm{sylv}_0$ and that $\mathrm{Sylv}_\mu \notin I$. From these two facts, we can derive that the Sylvester forms are nonzero in $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ and linearly independent. Namely, if we have a relation $\sum_{x^\mu \in R_\nu} \lambda_\mu\,\mathrm{sylv}_\mu = 0$ for some $\lambda_\mu \in A$, then multiplying by the monomials $x^\mu \in R_\nu$ in decreasing order with respect to $<$, we derive that $\lambda_\mu = 0$ for all $x^\mu \in R_\nu$. □

We notice that the relation between Sylvester forms and monomials stated in Theorem 4.4 can also be deduced from the global transformation law in [CCD97]. As the decomposition we provided in Theorem 4.2 differs from the one provided in [BCN22, Section 4] for the multihomogeneous case, we can see that, in general, the Sylvester forms and the monomials of degree ν do not form a pairing. In particular, we can see that there is a matrix $\mathcal{D} = (\mathcal{D}_{\mu,\mu'})_{x^\mu, x^{\mu'} \in R_\nu}$ whose entries are polynomials in A ordered with respect to $<$ and satisfy that:

$$x^{\mu'}\,\mathrm{sylv}_\mu = \mathcal{D}_{\mu,\mu'}\,\mathrm{sylv}_0\,. \tag{4.15}$$

Note that $\mathcal{D}_{\mu,\mu'}$ can be computed using the global transformation law and noting that:

$$\mathcal{D}_{\mu,\mu'} = \mathrm{Residue}_{(F_0,\ldots,F_n)}(x^{\mu'}\,\mathrm{sylv}_\mu) = \mathrm{Residue}_{(x_1^{\mu_1+1},\ldots,x_n^{\mu_n+1}, z_1^{\mu_{n+1}+1}\cdots z_r^{\mu_{n+r}+1})}(x^{\mu'}) \tag{4.16}$$

This last residue is zero, if and only if, $x^{\mu'}$ belongs to the ideal

$$(x_1^{\mu_1+1}, \ldots, x_n^{\mu_n+1}, z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}).$$

Otherwise, as the residue does not depend on A, it must be a nonzero element in $\mathbf{k}$, which is also independent of the decomposition (4.5) giving rise to sylv_μ. Theorem

4.4 implies that the matrix $\mathcal{D}$ is lower triangular with ones in the diagonal. Therefore, $\mathcal{D}$ is invertible and its inverse has entries in $\mathbf{k}$. If $X_\Sigma = \mathbb{P}^n$, the decomposition in (4.13) coincides with the one given in [Jou97] and we can see that $\mathcal{D}$ is the identity matrix.

In the next theorem, we prove that the Sylvester forms yield an A-basis of $(I^{\mathsf{sat}}/I)_\delta$, when it is a free A-module. This result is the key to the applications we discuss in the following sections.

Theorem 4.5. Under the assumptions of Theorem 4.4 and if $\delta - \nu \notin \Gamma_0 \cup \Gamma_1$ (see Notation 4.2), $\{\mathrm{sylv}_\mu\}_{x^\mu \in C_\nu}$ is an A-basis of $(I^{\mathsf{sat}}/I)_{\delta-\nu}$. Moreover, the classes sylv_μ do not depend on the choice of the decompositions in (4.13).

Proof. In Theorem 4.4, we proved that the set of forms $\{\mathrm{sylv}_\mu\}_{x^\mu \in R_\nu}$ is linearly independent. Moreover, as in [BCN22, Theorem 4.9], consider the canonical basis of $\mathrm{Hom}(C_\nu, A)$ which is dual to the monomial basis of C_ν. Namely, to each monomial $x^\mu \in C_\nu$, we associate the map:

$$X^\mu : C_\nu \to A$$

which sends x^μ to one and every other monomial to 0. Moreover, consider the A-linear isomorphisms

$$\phi : A \to (I^{\mathsf{sat}}/I)_\delta \quad c \to c\,\mathrm{sylv}_0$$

and

$$\mathcal{D}_\nu : C_\nu \to C_\nu \quad x^{\overline{\mu}} \to \sum_{x^{\mu'} \in R_\nu} \mathcal{D}_{\mu', \overline{\mu}} x^{\mu'}$$

where ϕ is an isomorphism by Proposition 4.1 and $\mathcal{D}_\nu$ is an isomorphism because the matrix $\mathcal{D}$ is invertible. Therefore, by (4.15), the composition $\phi \circ X^\mu \circ \mathcal{D}_\nu$ corresponds to multiplying the monomials in C_ν by sylv_μ and realizes the isomorphism $(I^{\mathsf{sat}}/I)_{\delta-\nu} \simeq \mathrm{Hom}(C_\nu, A)$. This proves that the Sylvester forms $\{\mathrm{sylv}_\mu\}_{x^\mu \in R_\nu}$ yield an A-basis of $(I^{\mathsf{sat}}/I)_{\delta-\nu}$. Moreover, this also implies that the classes sylv_μ are independent of the decompositions (4.13) since the maps $\phi \circ X^\mu \circ \mathcal{D}_\nu$ are themselves independent of these decompositions.

$\square$

4. Application to toric elimination matrices

An important motivation for studying the structure of the saturation of an ideal generated by generic sparse polynomials is for applications in elimination theory, in particular for solving sparse polynomial systems. In this section, we introduce a family of matrices whose construction involves toric Sylvester forms. It yields new compact elimination matrices that can be used for solving 0-dimensional sparse polynomial systems via linear algebra methods. We refer the reader to [EM99;

BT22; Tel20] for a thorough exposition of such methods that we will not discuss in this paper.

In what follows, X_Σ will denote a projective toric variety which is assumed to be smooth and σ-positive for some maximal cone $\sigma \in \Sigma(n)$, and we will consider a generic sparse polynomial system defined by homogeneous polynomials $F_0, \ldots, F_n$ as defined in (2.21). We require X_Σ to be smooth because we will use the Grothendieck-Serre formula (2.27). This setting covers many cases that are of interest for applications. We notice that the smoothness assumption is not very restrictive as X_Σ can be replaced by one of its desingularization varieties (see e.g. [CLS12, Chapters 10, 11]), but the preservation of the σ-positive property under desingularization is not obvious.

Notation 4.3. The elimination matrices we will consider are universal with respect to the coefficients of the F_i's, so we introduce the following notation to study rigorously their properties under specialization of these coefficients. Recall that I denotes the ideal in C generated by $F_0, \ldots, F_n$.

Any specialization (i.e. ring morphism) $\theta : A \to \mathbf{k}$ induces a surjective map $C \to R$ where $R = \mathbf{k}[x_\rho : \rho \in \Sigma(1)]$ (this map leaves invariant the variables x_ρ). For all $i = 0, \ldots, n$, we define $f_i = \theta(F_i) \in R$, we denote by $I(f)$ the homogeneous ideal $(f_0, \ldots, f_n)$ of R and set $B(f) = R/I(f)$. Moreover, we also set $B^{\mathsf{sat}} = C/I^{\mathsf{sat}}$, $B(f)^{\mathsf{sat}} = R/I(f)^{\mathsf{sat}}$ and $B^{\mathsf{sat}}(f) = C/I^{\mathsf{sat}}(f)$ (observe that $I(f)^{\mathsf{sat}}$ and $I^{\mathsf{sat}}(f)$ are in general not the same ideals). Finally, for any matrix $\mathbb{M}$ with coefficients in A, we denote by $\mathbb{M}(f)$ its specialization by $\theta : A \to \mathbf{k}$. We will refer as $V(I(f))$ to the zero set of the polynomial system defined by $I(f)$ over $(X_\Sigma)_{\overline{\mathbf{k}}}$ where $\overline{\mathbf{k}}$ denotes an algebraic closure of $\mathbf{k}$. Recall that in Section 3., we assumed that $\mathbf{k}$ is a subfield of $\mathbb{C}$. Thus, we can consider that $V(I(f))$ are the zeros of $I(f)$ over $\mathbb{C}$.

In what follows, we will consider Pic(X_Σ) instead of Cl(X_Σ) as all Weil divisors are Cartier in a smooth variety (see [CLS12, Proposition 4.2.6]).

Hybrid elimination matrices We begin by describing precisely what we mean by an elimination matrix $\mathbb{M}$ associated to the polynomials $F_0, \ldots, F_n$. It is a matrix whose columns are filled with coefficients of some homogeneous forms that are of the same degree and that all belong to the saturated ideal $I^{\mathsf{sat}} \subset C$. Thus, its entries are polynomials in A. Moreover, it is required that for any specialization map $\theta : A \to \mathbf{k}$ the following two properties hold:

i) The corank of $\mathbb{M}(f)$ is equal to zero, if and only if, $f_0 = \cdots = f_n = 0$ has no solution in X_Σ.

ii) If the number of solutions of $f_0 = \cdots = f_n = 0$ (over $\overline{\mathbf{k}}$) is finite in X_Σ and equals κ, then the corank of $\mathbb{M}(f)$ is κ.

We note that the first property yields a certificate of existence of a common root of the f_i's, which is related to sparse resultants, a topic we will address in the next section. The second property is mainly required for solving 0-dimensional polynomial systems by means of linear algebra techniques based on eigen-computations. In this approach, the common roots of the f_i's are extracted from the cokernel of $\mathbb{M}(f)$ (see e.g. [BT21]).

A very classical family of elimination matrices is obtained by filling columns with all the multiples of the F_i's of a certain degree. These matrices are usually called Macaulay-type matrices and are widely used for solving 0-dimensional polynomial systems (see for instance [BT22]). To be more precise, these matrices, that we will denote by $\mathbb{M}_\alpha$, are presentation matrices of the A-module B_α, i.e. are matrices of the maps

$$\begin{array}{rcl} \left(\oplus_{i=0}^n C(-\alpha_i)\right)_\alpha & \to & C_\alpha \\ (G_0,\dots,G_n) & \mapsto & \displaystyle\sum_{i=0}^n G_iF_i. \end{array} \tag{4.17}$$

Of course, some conditions on $\alpha \in \mathrm{Pic}(X_\Sigma)$ are required in order to guarantee that $\mathbb{M}_\alpha$ is an elimination matrix; we refer to [EM99] and to [Tel20, Chapter 5] for more details. Applying results we proved in the previous sections, we are going to extend the family of Macaulay-type matrices by using toric Sylvester forms. We recall that Sylvester forms belong to I^{sat} by Theorem 4.4.

Definition 4.4. Let α be such that $\left(I^{\mathrm{sat}}/I\right)_\alpha$ is a free A-module generated by Sylvester forms, so that $\left(I^{\mathrm{sat}}/I\right)_\alpha \simeq \oplus_{x^\mu \in C_{\delta-\alpha}} A$ (see Corollary 4.3 and Theorem 4.5), and consider the map

$$\begin{array}{rcl} \left(\oplus_{i=0}^n C(-\alpha_i)\right)_\alpha \oplus \left(\displaystyle\bigoplus_{x^\mu \in C_{\delta-\alpha}} A\right) & \to & C_\alpha \\ (G_0,\dots,G_n)\oplus(\dots,l_\mu,\dots) & \mapsto & \displaystyle\sum_{i=0}^n G_iF_i + \sum_{x^\mu\in C_{\delta-\alpha}} l_\mu \,\mathrm{Sylv}_\mu\,. \end{array} \tag{4.18}$$

Its matrix (in canonical bases) is called a *hybrid elimination matrix* and denoted by $\mathbb{H}_\alpha$.

The matrices $\mathbb{H}_\alpha$ are called *hybrid* because they are composed of two blocks, one from the classical Macaulay-type matrices and another one built from toric Sylvester forms; see Example 4.3. In particular, $\mathbb{M}_\alpha = \mathbb{H}_\alpha$ if $(I^{\mathrm{sat}}/I)_\alpha = 0$, so that the family of matrices $\mathbb{H}_\alpha$ extends the family of Macaulay-type matrices $\mathbb{M}_\alpha$. Thus, from now on we will use the notation $\mathbb{H}_\alpha$ instead of $\mathbb{M}_\alpha$. Our next step is to prove that these matrices are elimination matrices.

Main properties In this section, we first prove that the matrices $\mathbb{H}_\alpha$ introduced in Definition 4.4 are elimination matrices. Then, we give an illustrative example and also provide another criterion to construct the matrices $\mathbb{H}_\alpha$ without relying on the computation of the supports Γ_0 and Γ_1 (see Notation 4.2).

First, suppose given a specialization map (see Notation 4.3) and a degree α. From the results of Section 2. and Section 3., and also Definition 4.4, we deduce that the image of the matrix $\mathbb{H}_\alpha(f)$ is $I^{\text{sat}}(f)_\alpha$, so that its corank is $\mathrm{HF}(B^{\text{sat}}(f), \alpha)$. Therefore, a natural question is to compare this Hilbert function of $B^{\text{sat}}(f)$ with the one of $B(f)^{\text{sat}}$ in degrees for which hybrid matrices $\mathbb{H}_\alpha$ are defined (see Definition 4.4). We recall that we use the notation of Section 2. and we assume that the toric variety X_Σ is smooth and σ-positive for a maximal cone $\sigma \in \Sigma(n)$.

Lemma 4.6. Let $\alpha \notin \Gamma_0 \cup \Gamma_1 \subset \mathrm{Pic}(X_\Sigma)$ and suppose given specialized polynomials $f_0, \dots, f_n$ defining a 0-dimensional subscheme in X_Σ, possibly empty, of κ points, counted with multiplicity. Then,

$$\mathrm{HF}(B(f)^{\text{sat}}, \alpha) = \mathrm{HF}(B^{\text{sat}}(f), \alpha) = \kappa.$$

Proof. This proof goes along the same lines as [BCN22, Lemma 2.7]. First, one observes that $I(f) \subset I^{\text{sat}}(f) \subset I(f)^{\text{sat}}$ so that $B(f)^{\text{sat}}$, $B^{\text{sat}}(f)$ and $B(f)$ have the same Hilbert polynomial, which is the constant κ by our assumption.

Now, $H^i_{\mathfrak{b}}(B(f)^{\text{sat}}) = 0$ for $i = 0$ and for all $i > 1$ since $V(I(f))$ is finite. Applying Grothendieck-Serre formula, it follows that $\mathrm{HF}(B(f)^{\text{sat}}, \alpha) = \kappa$ for all α such that $H^1_{\mathfrak{b}}(B(f)^{\text{sat}})_\alpha = 0$. Analyzing the two spectral sequences associated to the Čech-Koszul complex of $f_0, \dots, f_n$, we get that the above vanishing holds for all $\alpha \notin \Gamma_0 \cup \Gamma_1$.

Similarly, Grothendieck-Serre formula and the finiteness of $V(I(f))$ imply that $\mathrm{HF}(B^{\text{sat}}(f), \alpha) = \kappa$ for all α such that $H^0_{\mathfrak{b}}(B(f)^{\text{sat}})_\alpha = H^1_{\mathfrak{b}}(B(f)^{\text{sat}})_\alpha = 0$. By [Cha13, Proposition 6.3], the vanishing of these modules can be derived from the similar vanishing conditions $H^0_{\mathfrak{b}}(B^{\text{sat}})_\alpha = H^1_{\mathfrak{b}}(B^{\text{sat}})_\alpha = 0$. These latter conditions hold for all $\alpha \notin \Gamma_0 \cup \Gamma_1$, which concludes the proof. $\square$

Remark 4.4. As a consequence of the above lemma, the canonical map from I^{sat}_α to $I(f)^{\text{sat}}_\alpha$, which is induced by a specialization θ, is surjective, i.e. generators of $I(f)^{\text{sat}}_\alpha$ can be computed by means of universal formulas.

Theorem 4.6. Assume that the toric variety X_Σ is smooth and σ-positive for a maximal cone $\sigma \in \Sigma(n)$. Then, for any $\alpha \notin \Gamma_0 \cup \Gamma_1 \subset \mathrm{Pic}(X_\Sigma)$ satisfying that $\left(I^{\text{sat}}/I\right)_\alpha \simeq \oplus_{x^\mu \in C_{\delta-\alpha}} A$, the matrix $\mathbb{H}_\alpha$ is an elimination matrix, i.e. it satisfies:

i) $\mathrm{corank}(\mathbb{H}_\alpha(f)) = 0$ if and only if $V(I(f))$ is empty in X_Σ,

ii) If $V(I(f))$ is a finite subscheme of degree κ in X_Σ, then $\mathrm{corank}(\mathbb{H}_\alpha(f)) = \kappa$.

Proof. We first prove i). If $V(I(f))$ is empty, equivalently $B(f)^{\text{sat}} = 0$ (which follows by the Grothendieck-Serre formula requiring the smoothness of X_Σ), then $\mathrm{HF}(B^{\text{sat}}(f), \alpha) = 0$ by Lemma 4.6. If $V(I(f)) \neq \emptyset$, then the f_i's have a common solution, say the point $p \in X_\Sigma$ (over $\mathbf{k}$) with defining ideal I_p (radical and maximal in R). Therefore, since $I^{\text{sat}}(f) \subset I(f)^{\text{sat}} \subset I_p$ and $\mathrm{HF}(R/I_p, \beta) = 1$ for all $\beta \in \mathrm{Pic}(X_\Sigma)$ by the maximality of I_p, we deduce that $\mathrm{HF}(R/I^{\text{sat}}(f), \alpha) \neq 0$ for any α. The proof of ii) follows from Lemma 4.6. $\square$

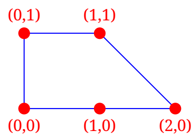


Figure 4.2: The polytopes Δ_i corresponding to the generic sparse homogeneous polynomials in Example 4.3 with the lattice points marked in red.

Example 4.3. Let $\mathbb{Z}^2$ be the lattice and X_Σ be the Hirzebruch surface $\mathcal{H}_1$ described in Example 4.2. Consider the following polytope presentations:

$$\Delta_i = \{m \in \mathbb{R}^2 : \langle m,(1,0)\rangle \geq 0,\ \langle m,(0,1)\rangle \geq 0,\ \langle m,(-1,-1)\rangle \geq -2,\ \langle m,(0,-1)\rangle \geq -1\},$$

for $i = 0,1,2$. $\mathcal{H}_1$ has the σ-positive property for $\sigma = \langle(1,0),(0,1)\rangle$. The class in $\mathrm{Pic}(\mathcal{H}_1) = \mathbb{Z}^2$ corresponding to these polytopes is $\alpha_i = (2,1)$, $i = 0,1,2$, and we write the corresponding generic homogeneous sparse polynomials as:

$$F_0 = a_0 z_1^2 z_2 + a_1 x_1 z_1 z_2 + a_2 x_1^2 z_2 + a_3 x_2 z_1 + a_4 x_1 x_2 \qquad \text{resp. } F_1, F_2 \text{ with coefficients } b_j, c_j, j = 0,\dots,4. \tag{4.19}$$

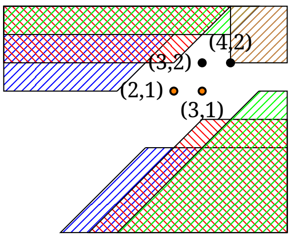


Figure 4.3: This is the picture of the regions $\Gamma_0, \Gamma_1, \Gamma_{\text{Res}}, \Gamma \subset \mathrm{Pic}(X_\Sigma) = \mathbb{Z}^2$ (the latter being defined in Section 3., (4.32)). The blue region corresponds to Γ_0, the red region corresponds to Γ_1, the green region corresponds to Γ_{Res} and the brown region corresponds to Γ. We marked in orange those α with $(I^{\text{sat}}/I)_\alpha \neq 0$. We derived the local cohomology of $\mathcal{H}_1$ from [Alt+20]; see also [EMS00; Bot11].

Figure 4.3 describes the supports $\Gamma_0, \Gamma_1, \Gamma_{\text{Res}}$. We deduce that elimination matrices $\mathbb{H}_\alpha$ are obtained for $\alpha \in \{(4,2),(3,2),(3,1),(2,1)\}$. In the cases $\alpha = (4,2)$ and $\alpha = (3,2)$, we get two Macaulay-type matrices. The two other cases give the following matrices:

- Case $\alpha = (3,1)$. This matrix corresponds to $\alpha = \delta$ and in this case, we are introducing a Sylvester form. This form is Sylv_0 and can be computed, as before, by a determinant that we write as:

$$\det\begin{pmatrix} a_1 z_1 z_2 + a_2 x_1 z_2 + a_4 x_2 & a_3 z_1 & a_0 z_1 \\ b_1 z_1 z_2 + b_2 x_1 z_2 + b_4 x_2 & b_3 z_1 & b_0 z_1 \\ c_1 z_1 z_2 + c_2 x_1 z_2 + c_4 x_2 & c_3 z_1 & c_0 z_1 \end{pmatrix} = [130] z_1^3 z_2 + [230] x_1 z_1^2 z_2 + [430] x_2 z_1^2,$$

where $[ijk] = \det\begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix}$. Therefore, the elimination matrix $\mathbb{H}_{(3,1)}$ is of the form:

$$\mathbb{H}_{(3,1)} = \begin{pmatrix} a_0 & a_1 & a_2 & 0 & a_3 & a_4 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & a_3 & a_4 \\ b_0 & b_1 & b_2 & 0 & b_3 & b_4 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & b_3 & b_4 \\ c_0 & c_1 & c_2 & 0 & c_3 & c_4 & 0 \\ 0 & c_0 & c_1 & c_2 & 0 & c_3 & c_4 \\ [130] & [230] & 0 & 0 & [430] & 0 & 0 \end{pmatrix}.$$

This type of matrices for $\alpha = \delta$ were already known from [CDS97] as the Δ_i's are all equal and ample in $\mathcal{H}_1$.

- Case $\alpha = (2,1)$. We obtain the following matrix $\mathbb{H}_{(2,1)}$ which is built from two different Sylvester forms:

$$\mathbb{H}_{(2,1)} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ [013] & [023]+[014] & [024] & 0 & 0 \\ [023] & [024]+[123] & [124] & 0 & 0 \end{pmatrix}$$

that correspond to the monomial basis $\{z_1, x_1\}$ of C_ν for $\nu = (1,0)$. As far as we know, these matrices did not appear in the existing literature.

Example 4.4. Consider again Example 4.3 with the same F_0, F_1 as in (4.19) but suppose now that $\alpha_2 = (1,1)$ and thus the corresponding generic homogeneous sparse polynomial F_2 is:

$$F_2 = c_0 z_1 z_2 + c_1 x_1 z_2 + c_3 x_2. \tag{4.20}$$

In this case, the Newton polytopes Δ_i's are not scaled copies of a fixed ample class and α_2 is not even ample in $\mathcal{H}_1$. However, the polytopes Δ_i are n-dimensional. Therefore, Corollary 4.2 and Theorem 4.6 imply that $\mathbb{H}_\delta$ is an elimination matrix for $\delta = (2,1)$. The corresponding Sylvester form is

$$\det\begin{pmatrix} a_1 z_1 z_2 + a_2 x_1 z_2 + a_4 x_2 & a_3 z_1 & a_0 z_1 \\ b_1 z_1 z_2 + b_2 x_1 z_2 + b_4 x_2 & b_3 z_1 & b_0 z_1 \\ c_1 z_2 & c_3 & c_0 \end{pmatrix} = [130] z_1^2 z_2 + [230] x_1 z_1 z_2 + [430] x_2 z_1,$$

where $[ijk] := \det\begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix}$, with the convention that $c_i = 0$ if this coefficient does not appear in F_2. Then, the corresponding elimination matrix is

$$\mathbb{H}_{(2,1)} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & 0 & c_3 & 0 \\ 0 & c_0 & c_1 & 0 & c_3 \\ [013] & [023]+[014] & 0 & [024] & 0 \end{pmatrix}.$$

This example illustrates that we obtain the same type of matrices as in [CDS97] but under different assumptions: we are assuming that the polytopes $\Delta_0, \ldots, \Delta_n$ are n-dimensional and X_Σ has the σ-positive property, and in [CDS97] it is assumed that the α_i's are scaled copies of the same ample class.

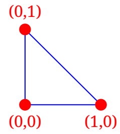


Figure 4.4: The polytope corresponding to the generic sparse homogeneous polynomial F_2 in Example 4.4.

As illustrated in Example 4.3, the construction of elimination matrices $\mathbb{H}_\alpha$ requires the computation of the support of the local cohomology modules $H^i_{\mathfrak{b}}(R)$. This task can be delicate, although several results are known; for instance, see [Alt+20] for the cases where the fan Σ splits or the rank of Pic(X_Σ) is 2 or 3, or see also [EMS00; Bot11]. In order to avoid such computations, the next result yields some sufficient conditions to get hybrid elimination matrices.

We recall that we are using the notation in Section 2.. In particular, for $i = 0, \ldots, n$, we write $\alpha_i \in$ Pic(X_Σ) for the classes associated to the homogeneous polynomial system, K_X for the anticanonical divisor and we set $\delta = \alpha_0 + \cdots + \alpha_n - K_X$.

Theorem 4.7. Assume that the toric variety X_Σ is smooth and σ-positive for some maximal cone $\sigma \in \Sigma(n)$. Moreover, assume that the polytopes $\Delta_0, \ldots, \Delta_n$ are n-dimensional. If $\alpha \in$ Pic(X_Σ) satisfies either of the two following properties:

i) $\alpha = \delta + \nu$ with ν a nef class or,

ii) $\alpha = \delta - \nu$, where ν is a nef class satisfying the hypotheses of Theorem 4.2 and for all $i = 0, \ldots, n$, $\alpha_i - \nu$ is a nef class that corresponds to an n-dimensional polytope,

then $\mathbb{H}_\alpha$ is an elimination matrix. In addition, it is purely of Macaulay-type if and only if α satisfies i) but not ii).

Proof. First, recall that the notation $K_j(F)$ stands for the terms of the Koszul complex associated to $F_0, \ldots, F_n$. We will also denote by J subsets of $\{0, \ldots, n\}$. For both cases, our strategy is to show that $\alpha \notin \Gamma_0 \cup \Gamma_1$ and $(I^{\text{sat}}/I)_\alpha = \oplus_\mu A$ in order to apply Theorem 4.6.

We begin with the case i) and pick $\alpha = \delta + \nu$ with ν a nef class. We have

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta+\nu} \simeq H^i_{\mathfrak{b}}(\oplus_{|J|=j} C(-\sum_{l\in J}\alpha_l))_{\delta+\nu} \simeq \oplus_{|J|=j} H^i_{\mathfrak{b}}(C)_{\delta+\nu-\sum_{l\in J}\alpha_l} \quad i \geq 0, j = 0, \ldots, n+1.$$

Recall that the local cohomology functors commute with direct sums. Using (2.23) and (2.25) for $i \geq 2$, we get:

$$H^i_{\mathfrak{b}}(C)_{\delta+\nu-\sum_{l\in J}\alpha_l} \simeq H^{i-1}(X_\Sigma, \sum_{l\notin J}\alpha_l - K_X + \nu) \simeq H^{n-i+1}(X_\Sigma, -\sum_{l\notin J}\alpha_l - \nu).$$

If $J \neq \{0,\dots,n\}$ then the class of $\sum_{l\notin J}\alpha_l + \nu$ is nef and its associated polytope is n-dimensional. In this case, we can apply Theorem 2.10 to deduce that

$$H^{n-i+1}(X_\Sigma, -\sum_{l\notin J}\alpha_l - \nu) = 0 \text{ for } i \geq 2.$$

As $H^i_{\mathfrak{b}}(C) = 0$ for $i = 0, 1$ (see (2.24)), it follows that

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta+\nu} = 0 \text{ for all } j \neq n+1$$

and hence, by definition of Γ_0 and Γ_1 (see (4.9)), that $\delta+\nu \notin \Gamma_0\cup\Gamma_1$. As a consequence, Theorem 4.3 shows that:

$$(I^{\text{sat}}/I)_{\delta+\nu} = \text{Hom}_A((C/I)_{-\nu}, A).$$

As ν is nef (and also effective), Remark 2.3 implies that $(C/I)_{-\nu} = 0$ for all $\nu \neq 0$. On the other hand, Corollary 4.2 implies that and $(C/I)_0 = A$. Therefore, $(I^{\text{sat}}/I)_{\delta+\nu} = 0$ for all ν, except $\nu = 0$ where we have $(I^{\text{sat}}/I)_\delta \simeq A$. From here, Theorem 4.6 implies i).

We proceed similarly to prove ii) and pick $\alpha = \delta - \nu$. We have:

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta-\nu} \simeq H^i_{\mathfrak{b}}(\oplus_{|J|=j}C(-\sum_{l\in J}\alpha_l))_{\delta-\nu} \simeq \oplus_{|J|=j}H^i_{\mathfrak{b}}(C)_{\delta-\nu-\sum_{l\in J}\alpha_l} \quad i\geq 0, j = 0,\dots,n+1$$

and using (2.23) and (2.25), for all $i \geq 2$ we get:

$$H^i_{\mathfrak{b}}(C)_{\delta-\nu-\sum_{l\in J}\alpha_l} \simeq H^{i-1}(X_\Sigma, \sum_{l\notin J}\alpha_l - K_X - \nu) \simeq H^{n-i+1}(X_\Sigma, \nu - \sum_{l\notin J}\alpha_l).$$

Our assumptions imply that the $\alpha_i - \nu$ are nef and their associated polytopes are n-dimensional for $i = 0,\dots,n$. Hence, if $J \neq \{0,\dots,n\}$, the classes $\sum_{l\notin J}\alpha_l - \nu$ are also nef and their associated polytopes are n-dimensional. Applying Theorem 2.10, we deduce that

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta-\nu} = 0 \text{ for all } j \neq n+1,$$

hence $\delta - \nu \notin \Gamma_0 \cup \Gamma_1$. Applying Theorem 4.3, we deduce that

$$(I^{\text{sat}}/I)_{\delta-\nu} = \text{Hom}_A((C/I)_\nu, A).$$

Finally, since ν satisfies the hypotheses of Theorem 4.2, we deduce from this theorem that $(C/I)_\nu = C_\nu$ and $(I^{\text{sat}}/I)_{\delta-\nu}$ is a free A-module, which concludes the proof. $\square$

Corollary 4.4. Assume that the toric variety X_Σ is smooth and σ-positive for some maximal cone $\sigma \in \Sigma(n)$. If the polytopes $\Delta_0, \dots, \Delta_n$ are all n-dimensional, then $\mathbb{H}_\delta$ is an elimination matrix.

Proof. Apply Theorem 4.7, i) with $\nu = 0$. $\square$

Example 4.5. Taking again Example 4.3, we observe that several elimination matrices are obtained from Theorem 4.7. Indeed, the matrix $\mathbb{H}_{(4,2)}$ is of Macaulay-type and corresponds to case i) in this theorem. The matrix $\mathbb{H}_{(2,1)}$ corresponds to case ii) while the matrix $\mathbb{H}_{(3,1)}$ corresponds to both cases i) and ii) ($\nu = 0$). However, the matrix $\mathbb{H}_{(3,2)}$ does not belong to either of the two cases. Using the explicit computation of Γ_0 and Γ_1 that we showed in Figure 4.3, we can derive that $(I^{\text{sat}}/I)_{(3,2)} = \text{Hom}((C/I)_{-(0,1)}, A)$ where $(C/I)_{-(0,1)} = 0$ and thus, $\mathbb{H}_{(3,2)}$ is also an elimination matrix.

Overdetermined sparse polynomial systems In this section we extend the construction of hybrid elimination matrices to the case of homogeneous polynomial systems that are defined by $r+1$ equations with $r \geq n$. Such systems often appear in practical applications and are referred to as overdetermined polynomial systems.

Notation 4.4. We assume that the projective toric variety X_Σ is smooth and σ-positive for some maximal cone σ. In what follows, $F_0, \dots, F_r$ are generic homogeneous sparse polynomials corresponding to nef classes $\alpha_0, \dots, \alpha_r$, I denotes the ideal they generate and $B = C/I$ the corresponding quotient ring. For each subset $T \subset \{0, \dots, r\}$ of cardinality $n+1$, we set $I_T = (F_i : i \in T)$, $B_T = C/I_T$ and $\delta_T = \sum_{i \in T} \alpha_i - K_X$. We denote by $\text{Sylv}_{T,\mu}$ the Sylvester forms that can be formed from $\{F_i\}_{i \in T}$; see Section 3.. We also denote by $K_\bullet(F)$ the Koszul complex of $F_0, \dots, F_r$ and by $K_{T,\bullet}(F)$ the Koszul complex of $\{F_i\}_{i \in T}$.

The following result is a generalization of [BCP23, Chapter 3, Proposition 3.23] which deals with the particular case $X_\Sigma = \mathbb{P}^n$.

Theorem 4.8. Using the previous notation, suppose that there exists a subset $S \subset \{0, \dots, r\}$ of cardinality $n+1$ and a nef class $\nu \in \text{Pic}(X_\Sigma)$ satisfying the hypotheses of Theorem 4.2 such that

$$\begin{aligned} &\forall i \in S \quad j \notin S \quad \alpha_i - \alpha_j \text{ is nef and} \\ &\qquad \forall i \in S \quad \alpha_i - \nu \text{ is nef and corresponds to an } n\text{-dimensional polytope}. \end{aligned} \tag{4.21}$$

Then, the set of Sylvester forms

$$\{\text{sylv}_{T,\mu} : T \subset \{0, \dots, r\} \text{ such that } |T| = n+1 \text{ and } x^\mu \in C_{\delta_T - \delta_S + \nu}\}$$

yields a generating set of the A-module $(I^{\text{sat}}/I)_{\delta_S - \nu}$.

Proof. First, we use Serre duality and Theorem 2.10 in order to compute the local cohomology modules $H^i_{\mathfrak{b}}(K_j(F))_{\delta_S-\nu}$, for $i=0,\dots,n+1$ and $j=0,\dots,n$, similarly to what we did in Theorem 4.7. Namely, for $i\geq 2$ we get

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta_S-\nu}\simeq \oplus_{|T|=j}H^i_{\mathfrak{b}}(C(-\sum_{k\in T}\alpha_k))_{\delta_S-\nu}\simeq \oplus_{|T|=j}H^{n+1-i}(X_\Sigma,\sum_{k\in T}\alpha_k-\sum_{k'\in S}\alpha_{k'}+\nu).$$

As we assumed that $j<n+1$, we can show that the previous cohomology module is of the form $H^{n+1-i}(X_\Sigma,-\alpha)$ for α a sum of nef divisors whose corresponding polytope is n-dimensional. Namely, the elements in $S\cap T$ cancel each other, and the rest of elements $k'\in S$ can be either (i) paired up with α_k for $k\in T$ satisfying that $\alpha_k-\alpha_{k'}$ is nef, (ii) paired up with ν satisfying that $\alpha_{k'}-\nu$ is nef and the corresponding polytope is n-dimensional, or (iii) they are nef themselves. Therefore, applying Theorem 2.10 for $i\geq 2$ and Remark 2.24 for $i=0,1$, we deduce:

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta_S-\nu}\simeq 0\quad i=0,\dots,n+1,\ j<n+1. \tag{4.22}$$

As a consequence, from the comparison of the two spectral sequences that are considered in Theorem 4.3, we obtain the following transgression map, which is an isomorphism of graded modules:

$$\tau: H_{n+1}(K_\bullet(F),H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu}\xrightarrow{\sim} H^0_{\mathfrak{b}}(B)_{\delta_S-\nu}.$$

For any $T\subset\{0,\dots,r\}$ of cardinalty $n+1$, let τ_T be the corresponding transgression map for $K_{T,\bullet}(F)$ and B_T. For each of these Koszul complexes, we have a canonical morphism of complexes $K_{T,\bullet}(F)\to K_\bullet(F)$ that induces:

$$L_\bullet(F)=\bigoplus_{|T|=n+1}K_{T,\bullet}(F)\to K_\bullet(F).$$

It follows that there is a commutative diagram:

$$\begin{tikzcd}
\oplus_{|T|=n+1}H_{n+1}(K_{T,\bullet}(F),H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu} \arrow[r] \arrow[d, "\bigoplus_T \tau_T"'] & H_{n+1}(K_\bullet(F),H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu} \arrow[d, "\tau"] \\
\oplus_{|T|=n+1}H^0_{\mathfrak{b}}(B_T)_{\delta_S-\nu} \arrow[r] & H^0_{\mathfrak{b}}(B)_{\delta_S-\nu}
\end{tikzcd}. \tag{4.23}$$

As the two vertical arrows are isomorphisms, in order to show that the bottom arrow is surjective, it is enough to show that the top arrow is surjective. For that purpose, we observe that $L_{n+1}(F)=K_{n+1}(F)$ by construction and also

$$\oplus_{|T|=n+1}H_{n+1}(K_{T,\bullet}(F),H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu}=\ker(H^{n+1}_{\mathfrak{b}}(L_{n+1}(F))\to H^{n+1}_{\mathfrak{b}}(L_n(F)))_{\delta_S-\nu}.$$

However, by the same argument as in (4.22), $H^{n+1}_{\mathfrak{b}}(L_n(F))_{\delta_S-\nu}=0$, so

$$\oplus_{|T|=n+1}H_{n+1}(K_{T,\bullet}(F),H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu}\simeq H^{n+1}_{\mathfrak{b}}(K_{n+1}(F))_{\delta_S-\nu}. \tag{4.24}$$

On the other hand,

$$H_{n+1}(K_\bullet(F),H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu}\simeq$$
$$\ker(H^{n+1}_{\mathfrak{b}}(K_{n+1})_{\delta_S-\nu}\to H^{n+1}_{\mathfrak{b}}(K_n)_{\delta_S-\nu})/\operatorname{im}(H^{n+1}_{\mathfrak{b}}(K_{n+2})_{\delta_S-\nu}\to H^{n+1}_{\mathfrak{b}}(K_{n+1})_{\delta_S-\nu}).$$

As above, $H_{\mathfrak{b}}^{n+1}(K_n)_{\delta_S-\nu} = 0$ and so

$$H_{n+1}(K_\bullet(F), H_{\mathfrak{b}}^{n+1}(C))_{\delta_S-\nu} \simeq H_{\mathfrak{b}}^{n+1}(K_{n+1})_{\delta_S-\nu}/\operatorname{im}(H_{\mathfrak{b}}^{n+1}(K_{n+2})_{\delta_S-\nu} \to H_{\mathfrak{b}}^{n+1}(K_{n+1})_{\delta_S-\nu}). \tag{4.25}$$

By comparing (4.24) and (4.25), we see that the top map in the diagram (4.23) is surjective, as we wanted to prove. It follows that the basis of Sylvester forms of $\oplus_{|T|=n+1} H_{\mathfrak{b}}^0(B_T)_{\delta_S-\nu}$ is a set of generators of $H_{\mathfrak{b}}^0(B)_{\delta_S-\nu} = (I^{\mathrm{sat}}/I)_{\delta_S-\nu}$. $\square$

We are now ready to extend the construction of hybrid elimination matrices to overdetermined homogeneous polynomial systems.

Theorem 4.9. We denote by $\mathbb{H}_\alpha$ the matrix of the following map:

$$\begin{array}{rcl} (\oplus_{i=0}^n C(-\alpha_i))_\alpha \displaystyle\bigoplus_{\substack{T\subset\{0,\ldots,r\},\,|T|=n+1,\\ x^\mu\in C_{\delta_T-\alpha}}} A & \to & C_\alpha \\ (G_0,\ldots,G_n)\oplus(\ldots,l_{T,\mu},\ldots) & \mapsto & \displaystyle\sum_{i=0}^n G_iF_i + \sum_{\substack{T\subset\{0,\ldots,r\}\\ |T|=n+1}} \sum_{x^\mu\in C_{\delta_T-\alpha}} l_{T,\mu}\,\mathrm{Sylv}_{T,\mu} \end{array} \tag{4.26}$$

where $\alpha = \delta_S - \nu$ and where $l_{\mu,T} \in A$ for all μ and T. Under the assumptions of Theorem 4.8, $\mathbb{H}_\alpha$ is an elimination matrix, where $\alpha = \delta_S - \nu$.

Proof. The proof goes along the same lines as the proof of Theorem 4.6 for the case $r = n$. $\square$

Example 4.6. Taking again the notation and the polynomials F_0, F_1, F_2 of Example 4.3, we add another polynomial of degree $\alpha_3 = (2,1)$ in $\mathcal{H}_1$ and write it in homogeneous coordinates as

$$F_3 = d_0z_1^2z_2 + d_1x_1z_1z_2 + d_2x_1^2z_2 + d_3x_2z_1 + d_4x_1x_2.$$

Following Theorem 4.9, the matrix $\mathbb{H}_{\delta_S}$ for $\delta_S = (3,1)$ is

$$\begin{pmatrix}
a_0 & a_1 & a_2 & 0 & a_3 & a_4 & 0 \\
0 & a_0 & a_1 & a_2 & 0 & a_3 & a_4 \\
b_0 & b_1 & b_2 & 0 & b_3 & b_4 & 0 \\
0 & b_0 & b_1 & b_2 & 0 & b_3 & b_4 \\
c_0 & c_1 & c_2 & 0 & c_3 & c_4 & 0 \\
0 & c_0 & c_1 & c_2 & 0 & c_3 & c_4 \\
d_0 & d_1 & d_2 & 0 & d_3 & d_4 & 0 \\
0 & d_0 & d_1 & d_2 & 0 & d_3 & d_4 \\
[130]_{abc} & [230]_{abc} & 0 & 0 & [430]_{abc} & 0 & 0 \\
[130]_{abd} & [230]_{abd} & 0 & 0 & [430]_{abd} & 0 & 0 \\
[130]_{acd} & [230]_{acd} & 0 & 0 & [430]_{acd} & 0 & 0 \\
[130]_{bcd} & [230]_{bcd} & 0 & 0 & [430]_{bcd} & 0 & 0
\end{pmatrix}$$

where $[ijk]_{abc} = \begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix}$, and $[ijk]_{abd}, [ijk]_{acd}, [ijk]_{bcd}$ defined accordingly. It is an elimination matrix for the overdetermined polynomial system defined by the polynomials F_0, F_1, F_2 and F_3.

We conclude this section with a comment on the computational impact of the hybrid elimination matrices obtained in Theorem 4.9. Indeed, these matrices are intended for solving overdetermined 0-dimensional polynomial systems via eigenvalue and eigenvector computations, applicable over projective spaces, multi-projective spaces, or more broadly, smooth projective toric varieties that are σ-positive for a given maximal cone σ. In comparison with the more classical Macaulay-type matrices, hybrid elimination matrices are more compact. In particular, these matrices have a smaller number of rows, which is a key ingredient with respect to computational complexity.

Indeed, this number of rows is controlled by the vanishing of the local cohomology modules at certain degrees, including the control of the saturation index of the homogeneous ideal $I(f)$ generated by general polynomials $f_0, \ldots, f_r$ of degrees $\alpha_0, \ldots, \alpha_r$. In the case of hybrid elimination matrices, the situation is similar with the difference that now one considers the homogeneous ideal generated by $f_0, \ldots, f_r$ and their toric Sylvester forms, whose saturation index is smaller than the one of $I(f)$.

To be more concrete, we considered some specific polynomial systems for which we report, in Table 4.1, the number of rows of hybrid elimination matrices (4.18) and of Macaulay matrices (4.17). We considered systems of 4 generic homogeneous polynomials in four different settings of Newton polytopes and degrees, all corresponding to 3-dimensional varieties. As expected, we observe that hybrid elimination matrices have a significantly smaller number of rows compared to Macaulay elimination matrices.

Type of system	degree α		number of rows	
	Classical	Hybrid	Classical	Hybrid
Polynomials of deg. 2 in $\mathbb{P}^3$	5	3	56	20
Polynomials of deg. 10 in $\mathbb{P}^3$	37	27	9880	4060
Polynomials of deg. $(2,1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$	(6,3)	(4,2)	112	45
Polynomials of deg. $\Delta \times [0,1]$ in $\mathcal{H}_1 \times \mathbb{P}^1$	3($\Delta \times [0,1]$)	2($\Delta \times [0,1]$)	88	36

Table 4.1: The first column describes the type of system of 4 homogeneous polynomials which is considered. The second column provides the degree α for which the classical Macaulay-type matrices and the hybrid elimination matrices are constructed. The third column gives the corresponding number of rows of these two matrices. The Newton polytope Δ in the last row corresponds to the degrees of the polynomials considered in Example 4.3 .

We remark that the number of columns of hybrid elimination matrices may increase fast when the number of equations is large compared to the dimension of the ground projective toric variety.

5. Sylvester forms and sparse resultants

For the results of this section, we assume the setting of the homogeneous resultant in Section 3. and the fact that, for some $\alpha \in \mathrm{Pic}(X_\Sigma)$, $\mathrm{Res}_\mathcal{A}$ can be computed as the determinant $\det(K_\bullet(F)_\alpha)$. In order to incorporate Sylvester forms in this setting construction we proceed as follows.

As in Definition 4.4, let α be such that $\left(I^{\mathrm{sat}}/I\right)_\alpha$ is a free A-module generated by Sylvester forms, i.e. $\left(I^{\mathrm{sat}}/I\right)_\alpha \simeq \oplus_{x^\mu \in C_{\delta-\alpha}} A$. We define the complex $K_\bullet^{\mathrm{sat}}(F)_\alpha$ as the graded strand $K_\bullet(F)_\alpha$ of the Koszul complex, where the map on the right, namely $(K_1)_\alpha \to C_\alpha$ (see (2.36)), is replaced by the defining map (4.18) of the hybrid elimination matrices. More precisely, $K_\bullet^{\mathrm{sat}}(F)_\alpha$ the following graded complex of free A-modules

$$
\begin{aligned}
C(-\sum \alpha_i)_\alpha \xrightarrow{(\partial_{n+1})_\alpha} \dots \xrightarrow{(\partial_3)_\alpha} \oplus_{k,k'} C(-\alpha_k - \alpha_{k'})_\alpha \\
\xrightarrow{(\partial_2)_\alpha \oplus 0} \oplus_k C(-\alpha_k)_\alpha \oplus_{x^\mu \in C_{\delta-\alpha}} A \xrightarrow{(\partial_1)_\alpha \oplus \tau_\alpha} C_\alpha, \quad (4.27)
\end{aligned}
$$

where the map $(\partial_1)_\alpha \oplus \tau_\alpha$ is the map (4.18), τ_α denoting the map from $\oplus_{x^\mu \in C_{\delta-\alpha}} A$ to C_α corresponding to the Sylvester forms. By definition, we notice that $H_i(K_\bullet^{\mathrm{sat}}(F)_\alpha) = H_i(K_\bullet(F)_\alpha)$ for all $i \geq 2$. Moreover, we also see that $H_1(K_\bullet^{\mathrm{sat}}(F)_\alpha) \simeq H_1(K_\bullet(F)_\alpha)$, because τ_α is injective by property of the Sylvester forms, and that $H_0(K_\bullet^{\mathrm{sat}}(F)_\alpha) = (B^{\mathrm{sat}})_\alpha$.

Theorem 4.10. Assume that X_Σ is a smooth projective toric variety which is σ-positive for a maximal cone σ and that the classes $\alpha_0, \dots, \alpha_n$ are ample. For every $\alpha \notin \Gamma_{\mathrm{Res}}$, $K_\bullet^{\mathrm{sat}}(F)_\alpha$ is an acyclic complex of free A-modules. Moreover, if $\alpha = \delta - \nu$ as in Theorem 4.7 ii), then $\det(K_\bullet^{\mathrm{sat}}(F)_\alpha)$ is equal to $\mathrm{Res}_\mathcal{A}$ up to a nonzero multiplicative scalar in **k**.

Proof. By construction, the acyclicity of $K_\bullet^{\mathrm{sat}}(F)_\alpha$ follows from the acyclicity of the usual Koszul complex (see above). Moreover, since $H_0(K_\bullet^{\mathrm{sat}}(F)_\alpha) = (B^{\mathrm{sat}})_\alpha$, we deduce that $\det(K_\bullet^{\mathrm{sat}}(F)_\alpha)$ and $\mathrm{Res}_\mathcal{A}$ are two polynomials in A that vanish under the same specializations in **k**. In order to show that they are the same polynomial, we will proceed by comparing their degrees with respect to the coefficients of F_i for $i = 0, \dots, n$. For the sake of simplicity, we proceed by computing the degree of these polynomials with respect to the coefficients of F_0, and denote this degree as $\deg_{F_0}$. As proved in [GKZ94, Appendix A], the determinant of a complex of vector spaces $V_\bullet : V_{n+1} \to \dots \to V_1 \to V_0$ is given by the formula

$$
\det(V_\bullet) = \bigotimes_i \bigwedge^{\dim(V_i)} V_i^{(-1)^i}. \quad (4.28)
$$

Regarding the degree computation for $K_\bullet(F)$, the terms of the Koszul complex are **k**-vector spaces tensored with A, thus we can apply (4.28) to this complex of A-

modules. The degree of $\bigwedge^{\dim(K_j(F))} K_j(F)_\alpha$ with respect to the coefficients of F_0 is:

$$\sum_{J\subset\{0,\dots,n\},\,|J|=j}^{0\in J} \mathrm{HF}(R,\alpha-\sum_{j\in J}\alpha_j). \tag{4.29}$$

For $\alpha \gg 0$, we have $(I^{\mathrm{sat}}/I)_\alpha = 0$ and $\mathrm{HF}(R,\alpha) = \mathrm{HP}(R,\alpha)$. Therefore, the degree of the determinant of the complex $K_\bullet^{\mathrm{sat}}(F)_\alpha$ coincides with the degree of the resultant and we can compute:

$$\begin{gathered}\deg_{F_0}(\mathrm{Res}_{\mathcal{A}}) = \deg_{F_0}\det(K_\bullet^{\mathrm{sat}}(F)_\alpha) = \\ \sum_{J\subset\{0,\dots,n\}}^{0\in J}(-1)^{|J|}\,\mathrm{HF}(R,\alpha-\sum_{j\in J}\alpha_j) = \sum_{J\subset\{0,\dots,n\}}^{0\in J}(-1)^{|J|}\,\mathrm{HP}(R,\alpha-\sum_{j\in J}\alpha_j).\end{gathered} \tag{4.30}$$

As the degree of the resultant with respect to the coefficients of F_0 is constant (and equal to the mixed volume of the polytopes $\Delta_1,\dots,\Delta_n$), the last term in (4.30) is a constant polynomial in α, so when we evaluate it at any α, it will always be equal to $\deg_{F_0}(\mathrm{Res}_{\mathcal{A}})$. Therefore, for $\alpha = \delta-\nu$ as in the statement, we have $(I^{\mathrm{sat}}/I)_{\delta-\nu} = \mathrm{Hom}_A(C_\nu, A) \neq 0$ and we can check that the difference of degrees between the previous alternate sum and the degree of the classical Koszul complex is compensated by adding $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ at the term K_1 of $K_\bullet(F)$ (and thus counted with sign -1 in the determinant of the complex):

$$\begin{gathered}\deg_{F_0}\det(K_\bullet(F)_{\delta-\nu}) - \deg_{F_0}(\mathrm{Res}_{\mathcal{A}}) = \\ \sum_{J\subset\{0,\dots,n\}}^{0\in J}(-1)^{|J|}\big(\mathrm{HF}(R,\delta-\nu-\sum_{j\in J}\alpha_j) - \mathrm{HP}(R,\delta-\nu-\sum_{j\in J}\alpha_j)\big).\end{gathered} \tag{4.31}$$

Using Grothendieck-Serre formula (2.27), we deduce that this coincides with the quantity

$$\sum_{J\subset\{0,\dots,n\}}^{0\in J}(-1)^{|J|}\sum_{i=0}^{n+1}(-1)^i\dim_{\mathbf{k}} H^i_{\mathfrak{b}}(R)_{\delta-\nu-\sum_{j\in J}\alpha_j}.$$

Under the hypotheses of Theorem 4.7 ii), and using Theorem 2.10, we get that all the summands in the above sum vanish except if $i = n+1$ and $J = \{0,\dots,n\}$. In this latter case, we have $H^{n+1}_{\mathfrak{b}}(R)_{-K_X-\nu}$, which is counted with the sign $(-1)^{2(n+1)} = 1$ and has the same dimension as the rank of the free A-module $H^{n+1}_{\mathfrak{b}}(C)_{-K_X-\nu}$. Recalling the duality theorem, which holds under the hypotheses of Theorem 4.7 ii), we have:

$$H^{n+1}_{\mathfrak{b}}(C)_{-K_X-\nu} \simeq (I^{\mathrm{sat}}/I)_{\delta-\nu} \simeq \mathrm{Hom}_A(C_\nu, A),$$

which concludes the proof as the degree of each of the Sylvester forms Sylv_μ for $x^\mu \in R_\nu$ with respect to the coefficients of F_0 is 1. $\square$

From the above result, we can also identify cases where the matrices $\mathbb{H}_\alpha$ are square matrices, and therefore their determinant (in the usual sense of the determinant of a matrix) is equal to the sparse resultant, up to a nonzero multiplicative constant. For this purpose, we consider

$$\Gamma = \mathrm{Supp}\left(\oplus_{k,k'} C(-\alpha_k - \alpha_{k'})\right) \tag{4.32}$$

to be the support of the term $K_2(F)$ in the Koszul complex; see Figure 4.3 for an example.

Corollary 4.5. Let X_Σ be a smooth projective toric variety which is σ-positive for a maximal cone σ. Assume that $\Delta_0, \ldots, \Delta_n$ correspond to ample divisors. Then, for any $\alpha \notin \Gamma \cup \Gamma_{\mathrm{Res}} \cup \Gamma_0 \cup \Gamma_1$ we have $\det(\mathbb{H}_\alpha) = \mathrm{Res}_{\mathcal{A}}$, up multiplication by a nonzero scalar.

Proof. If $\alpha \notin \Gamma_{\mathrm{Res}} \cup \Gamma_0 \cup \Gamma_1$, then $(I^{\mathrm{sat}}/I)_\alpha$ is free and $\mathrm{Res}_{\mathcal{A}} = \det(K_\bullet^{\mathrm{sat}}(F)_\alpha)$ as in Theorem 2.36. If $\alpha \notin \Gamma$, then the complex $K_\bullet^{\mathrm{sat}}(F)_\alpha$ has only two terms and therefore $\det(K_\bullet^{\mathrm{sat}}(F)_\alpha) = \det(\mathbb{H}_\alpha)$. □

Remark 4.5. Computing the determinant of a complex can be done using some techniques such as Cayley determinants (see [GKZ94, Appendix A]), but it is not very practical. However, Theorem 4.10 yields new expressions of the sparse resultant as a ratio of two determinants if $\alpha \notin \mathrm{Supp} \oplus_{k,l,m} C(-\alpha_k - \alpha_l - \alpha_m)$; see [CDS97, Corollary 2.4] for a combinatorial characterization of such case.

We close this section with a comment and an example related to the well-known Canny-Emiris formula. For Macaulay-type matrices of the form $\mathbb{M}_\alpha$, the Canny-Emiris formula gives a way to choose a nonzero minor of maximal size; see [CE93] for the formula and [DJS22] for a proof that this minor is nonzero. It remains an open problem to see whether the conditions in the proof of the Canny-Emiris formula [DJS22] coincide with the Cayley determinant for such a choice of a minor. In the case of hybrid elimination matrices $\mathbb{H}_\alpha$, a similar formula has been explored in [DE01] for $n = 2$ and $\alpha = \delta$.

Example 4.7. Let's consider the four matrices provided in Example 4.3, which correspond to the cases $\alpha \in \{(4,2), (3,2), (3,1), (2,1)\}$. The last three are square matrices while the first one is not. We have drawn the region Γ in brown in Figure 4.3, in order to indicate the elements that provide a square matrix, as well as Γ_{Res}, in green, for the acyclicity of the complex. For the Macaulay-type matrices, we can combinatorially describe a maximal minor of $\mathbb{M}_{(4,2)}$ using the Canny-Emiris formula; see

[CE93; DJS22]. The matrix $\mathbb{M}_{(3,2)}$ is square,

$$\mathbb{M}_{(3,2)} = \begin{pmatrix} a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 \\ b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 \\ c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & 0 & 0 \\ 0 & c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & 0 \\ 0 & 0 & c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 \end{pmatrix},$$

and it might be obtained using a greedy approach to the same formula (see the results of Chapter 3 or [CP93]), but as far as we know, there was no known certificate of its existence as a resultant formula. The hybrid matrices for $\alpha = (3,1), (2,1)$ are square. More generally, for non-square hybrid matrices, a procedure for choosing a minor is known when $n = 2$ and $\alpha = \delta$; see [DE01].

6. Toric residue of the product of two forms

Another topic for which Sylvester forms are of interest is the computation of toric residues. These objects were initially introduced by Cox as a way to relate the residue of a family of $n+1$ forms to the integral of a certain differential form in a toric variety X_Σ (see [Cox96]). Being given $F_0, \dots, F_n$ generic homogeneous polynomials as in (2.21), and denoting by $K(A)$ the quotient field of the universal ring of coefficients A, Cox proved the existence of a residue map

$$\text{Residue}_F : B_\delta \to K(A)$$

(recall that $I = (F_0, \dots, F_n)$ and $B = C/I$) which has the following property: for any specialization $\theta : A \to \mathbf{k}$ (see Notation 4.3) such that the specialized system $f_0 = \cdots = f_n = 0$ has no solution in X_Σ, the residue map $\text{Residue}_f : (R/I(f))_\delta \to k$ is an isomorphism. Cox defined residue maps through trace maps of Čech cohomology, but they can be characterized through the fact that, if there is no solution in X_Σ, $\rho(\text{sylv}_0)$ is sent to $\pm 1 \in K$, so we can assume $\text{Residue}_F(\text{sylv}_0) = \pm 1$. Many authors contributed formulas based on elimination matrices and resultants to compute residues [KS05; DK05; CCD97; CDS97] and also used them in other applications such as polynomial interpolation [Sop07] or mirror symmetry [BM02]. In particular, in [DK05] an explicit formula for computing the toric residue of a form of degree δ as a quotient of two determinants “à la Macaulay” is proved.

If a form G of degree δ can be written as a product $G = PQ$, a natural question is to ask whether one can take advantage of this factorization in the computation of the residue of $G = PQ$ with respect to the polynomial system defined by $F_0, \dots, F_n$. In the case $X_\Sigma = \mathbb{P}^n$, Jouanolou proved that this is possible by exploiting the duality between the degrees $\delta-\nu$ and ν of P and Q, respectively; see [Jou97, Proposition 3.10.27]. Notice that in [Jou97] the residue is defined as a map to A, and not to $K(A)$, by multiplying with $\mathrm{Res}_{\mathcal{A}}$ in the image. In the case of ample divisors, the product of the residue and the resultant lies in A; see [CDS97, Theorem 1.4]. In what follows, we explore the generalization of Jouanolou's formula to a general smooth projective toric variety X_Σ which is σ-positive for a maximal cone σ, using toric Sylvester forms.

Let $\mathbb{H}_{\delta-\nu}$ be an elimination matrix that satisfies the assumptions of Theorem 4.7 ii), and let $\mathcal{H}_{\delta-\nu}$ be a nonzero maximal minor of $\mathbb{H}_{\delta-\nu}$ which contains the entire block built with Sylvester forms. Now, being given two generic forms $P \in C_\nu$ and $Q \in C_{\delta-\nu}$, we consider the matrix

$$\Theta_{\delta-\nu} = \begin{pmatrix} & \mathcal{H}_{\delta-\nu} & \mathbf{q} \\ 0 & (\mathbf{p})^T\mathcal{D} & 0 \end{pmatrix} \tag{4.33}$$

where $\mathbf{p}$, respectively $\mathbf{q}$, stands for the vector of coefficients of P, respectively Q, and $\mathcal{D}$ is the matrix defined in (4.15). Recall that by the construction of the matrix $\mathbb{H}_{\delta-\nu}$, the matrix $\mathcal{H}_{\delta-\nu}$ is built as the join of a Macaulay-type block-matrix and another column-block matrix built from Sylvester forms. Thus, the row $(\mathbf{p})^T\mathcal{D}$ is aligned with the column-block built from Sylvester forms; see Example 4.6 for an illustration.

We first prove that the residue of the product of two monomials can be computed as a quotient. In what follows, we denote by $\mathcal{H}_{\mu,\xi}$ the submatrix of $\mathcal{H}_{\delta-\nu}$ that is obtained by deleting the column corresponding to the monomial $x^\mu \in R_\nu$ and the row corresponding to the monomial $x^\xi \in C_{\delta-\nu}$.

Lemma 4.7. Assume that X_Σ is a smooth projective toric variety which is σ-positive for a maximal cone σ. Let $F_0,\dots,F_n$ be a system of homogeneous polynomials in C as in (2.21), then for two monomials $x^\mu \in R_\nu$ and $x^\xi \in R_{\delta-\nu}$,

$$\mathrm{Residue}_F(x^{\mu+\xi}) = \frac{\sum_{x^{\mu'}\in C_\mu}(-1)^{\mu'}(-1)^{\xi}\mathcal{D}_{\mu,\mu'}\det(\mathcal{H}_{\mu',\xi})}{\det(\mathcal{H}_{\delta-\nu})},$$

where $(-1)^{\mu'}$ (resp. $(-1)^\xi$) is set to 1 if the relative position of the monomial x^μ (resp. x^ξ) in the columns (resp. rows) of $\mathcal{H}_{\delta-\nu}$ is even, otherwise it is set to -1.

Proof. Let H^ξ be the matrix obtained by multiplying the row of $\mathcal{H}_{\delta-\nu}$ corresponding to x^ξ by the monomial x^ξ itself. Then, by expanding the determinant along this row, one gets:

$$x^\mu x^\xi \det(\mathcal{H}_{\delta-\nu}) = x^\mu \det(H^\xi) = x^\mu(\sum G_iF_i + \sum_{\mu'\in C_\nu}(-1)^{\mu}(-1)^{\xi}c_{\mu',\xi}\,\mathrm{Sylv}_{\mu'}).$$

Then, using the matrix $\mathcal{D}$, we get that

$$x^\mu x^\xi \det(\mathcal{H}_{\delta-\nu}) = \sum x^\mu G_iF_i + \sum_{x^\mu\in C_\mu}(-1)^{\mu'}(-1)^{\xi}\mathcal{D}_{\mu,\mu'}c_{\mu',\xi}\,\mathrm{Sylv}_0 \mod I.$$

Taking residues, we deduce that

$$\mathrm{Residue}_F(x^{\mu+\xi})\,\det(\mathcal{H}_{\delta-\nu}) = \sum_{x^\mu \in C_\mu} (-1)^{\mu'}(-1)^{\xi}\mathcal{D}_{\mu,\mu'} c_{\mu,\xi}.$$

Finally, from the expansion of the determinant $\det(H^\xi)$, we get that $c_{\mu',\xi} = \det(\mathcal{H}_{\mu',\xi})$. $\square$

We are now ready to prove the claimed formula for the residue of the product of two forms.

Theorem 4.11. Assume that X_Σ is a smooth projective toric variety which is σ-positive for a maximal cone σ. Let $F_0, \dots, F_n$ be a system of homogeneous polynomials in C as in (2.21), and suppose given two forms $P \in C_\nu$ and $Q \in C_{\delta-\nu}$, then

$$\mathrm{Residue}_F(PQ) = \frac{\det(\Theta_{\delta-\nu})}{\det(\mathcal{H}_{\delta-\nu})}.$$

Proof. Write $P = \sum_{x^\mu \in C_\nu} p_\mu x^\mu$ and $Q = \sum_{x^\xi \in C_{\delta-\nu}} q_\xi x^\xi$. Then, by linearity of residues, we have:

$$\mathrm{Residue}_F(PQ) = \sum_{x^\mu \in C_\nu,\, x^\xi \in C_{\delta-\nu}} p_\mu q_\xi \,\mathrm{Residue}_F(x^{\mu+\xi}) =$$

$$\frac{\sum_{\mu,\xi}\sum_{\mu'}(-1)^{\mu'}(-1)^{\xi}}{p}{}_{\mu}\, q_\xi \mathcal{D}_{\mu,\mu'} \det(\mathcal{H}_{\mu',\xi})\det(\mathcal{H}_{\delta-\nu}). \quad (4.34)$$

The numerator is precisely the expansion of the determinant $\det(\Theta_{\delta-\nu})$ of the matrix defined in (4.33), firstly with respect to the last row and secondly with respect to the last column. $\square$

Example 4.8. In Example 4.3, the elimination matrix $\mathbb{H}_{(2,1)}$ is square, therefore we take

$$\mathcal{H}_{(2,1)} = \mathbb{H}_{(2,1)} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ [013] & [023]+[014] & [024] & 0 & 0 \\ [023] & [024]+[123] & [124] & 0 & 0 \end{pmatrix}.$$

Let $P = p_0 z_1 + p_1 x_1$ and $Q = q_0 z_1^2 z_2 + q_1 z_1 z_2 x_1 + q_2 z_2 x_1^2 + q_3 z_1 x_2 + q_4 x_1 x_2$ be homogeneous forms in $C_{(1,0)}$ and $C_{(2,1)}$, respectively and let $\mathcal{D}$ be the matrix in Remark (4.15) which is of the form $\mathcal{D} = \begin{pmatrix} 1 & 0 \\ \mathcal{D}_{01} & 1 \end{pmatrix}$, then

$$\Theta_{(2,1)} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ c_0 & c_1 & c_2 & c_3 & c_4 & 0 \\ [013] & [023]+[014] & [024] & 0 & 0 & p_0 + \mathcal{D}_{01} p_1 \\ [023] & [024]+[123] & [124] & 0 & 0 & p_1 \\ q_0 & q_1 & q_2 & q_3 & q_4 & 0 \end{pmatrix}.$$

$\mathcal{D}_{01}$ can be computed as in (4.16) and it is nonzero as $z_1 \notin (x_1^2, x_2, z_1 z_2)$. Applying Theorem 4.11, we deduce that $\text{Residue}_F(PQ) = \frac{\det(\Theta_{(2,1)})}{\det(\mathcal{H}_{(2,1)})}$. For the sake of comparison, let us examine the formula we obtain by developing the product of P and Q. In this case, we apply Theorem 4.11 with $\delta = (3,1)$ and $\nu = 0$, so we have to consider the matrix $\Theta_{(3,1)}$ which is of the form:

$$\Theta_{(3,1)} = \begin{pmatrix} a_0 & a_1 & a_2 & 0 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & a_3 & a_4 & 0 \\ b_0 & b_1 & b_2 & 0 & b_3 & b_4 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & b_3 & b_4 & 0 \\ c_0 & c_1 & c_2 & 0 & c_3 & c_4 & 0 & 0 \\ 0 & c_0 & c_1 & c_2 & 0 & c_3 & c_4 & 0 \\ [130] & [230] & 0 & 0 & [430] & 0 & 0 & 1 \\ p_0q_0 & p_0q_1 + p_1q_0 & p_0q_2 + p_1q_1 & p_1q_2 & p_0q_3 & p_0q_4 + p_1q_3 & p_1q_4 & 0 \end{pmatrix}$$

since the product PQ is equal to

$$p_0q_0z_1^3z_2 + (p_0q_1 + p_1q_0)z_1^2z_2x_1 + (p_0q_2 + p_1q_1)z_1z_2x_1^2 + \\ p_0q_3z_1^2x_2 + (p_0q_4 + p_1q_3)z_1x_1x_2 + p_1q_2z_2x_1^3 + p_1q_4x_1^2x_2. \quad (4.35)$$

The expansion of $\det(\Theta_{(3,1)})$ with respect to the last row leads to the same formula as in [DK05, Corollary 3.4].

Chapter 5

Multigraded Castelnuovo-Mumford regularity and Gröbner bases

The multigraded Castelnuovo-Mumford regularity has attracted the interest of many researchers in the last decades either for finding an extension of its definition and main properties [BHS21; HW06; MS04], studying properties of local cohomology in the multigraded case [BC17; CH22], understanding its relation with the Betti numbers and virtual resolutions [AHS21; BES20], considering special properties in the cases of points and curves [Cob24; HV04] or for providing bounds that generalize those existing in classical case [BHS22; MS03; RSM22]. However, finding a multigraded generalization of the Bayer and Stillman criterion (see Theorem 2.13) that can be used to describe the multi-degrees that generate the Gröbner basis has remained an open problem. For the sake of simplicity, we will describe our results in the bigraded case i.e., when the degrees are prescribed in two groups of variables, that we will denote with x's and y's. All the discussion and results that follow extend to the multigraded setting.

Notation 5.1. Let $\mathbf{k}$ be a field of characteristic 0. Let $S = \mathbf{k}[x_0, \dots, x_n, y_0, \dots, y_m]$ be a ring with a (standard) $\mathbb{Z}^2$-grading, such that $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$. We will write the monomials in S as $x^\alpha y^\beta = x_0^{\alpha_0} \cdots x_n^{\alpha_n} y_0^{\beta_0} \cdots y_m^{\beta_m}$ for a vector $(\alpha, \beta) \in \mathbb{Z}^{n+m+2}$. A monomial $x^\alpha y^\beta$ has degree (a, b) if $\sum_{i=0}^{n} \alpha_i = a$ and $\sum_{j=0}^{m} \beta_j = b$. Let $\mathfrak{m}_x$ (resp. $\mathfrak{m}_y$) be the ideal generated by the x (resp. y) variables. Let $\mathfrak{m}_x$ (resp. $\mathfrak{m}_y$) be the ideal generated by the x (resp. y) variables. The ambient biprojective space is $\mathbb{P}^n \times \mathbb{P}^m$ and the irrelevant ideal is $\mathfrak{b} = \mathfrak{m}_x \mathfrak{m}_y$.

Notation 5.2. A polynomial $f = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta \in S$ is bihomogeneous of bi-degree $(a, b) \in \mathbb{Z}^2$ if all of its terms are monomials of bi-degree (a, b). An ideal $I \subset S$ is bihomogeneous if every polynomial $f \in I$ is bihomogeneous. The graded part of bi-degree (a, b) of I is the $\mathbf{k}$-vector space generated by all the polynomials of bidegree (a, b) in I, and denoted as $I_{(a,b)}$. Recall that a *a generic linear x-form* as a general element in $S_{(1,0)}$.

A generalization of the Castelnuovo-Mumford regularity to the bigraded setting, which we also denote as $\operatorname{reg}(I)$, was given by Maclagan and Smith in [MS04], by using local cohomology modules with respect to the irrelevant ideal $\mathfrak{b}$, i.e. the intersection of the ideals generated by each group of variables.

Definition 5.1. [MS04, Definition 1.1] Consider a bihomogeneous ideal $I \subset S$. The bigraded Castelnuovo-Mumford regularity $\operatorname{reg}(I)$ is the subset of $\mathbb{Z}^2$ containing bidegrees (a, b) such that, for all $i \geq 1$ and for all $(a', b') \geq (a - \lambda_x, b - \lambda_y)$, it holds

$$H^i_{\mathfrak{b}}(I)_{(a',b')} = 0,$$

where $\lambda_x + \lambda_y = i - 1$, with $\lambda_x, \lambda_y \in \mathbb{Z}_{\geq 0}$.

The goal of this part of the thesis is to establish a connection between the bigraded Castelnuovo-Mumford regularity and the degrees of the minimal generators of the DRL Gröbner basis after a generic change of coordinates that preserves the bigraded structure, providing the *bigeneric initial ideal*; see Chapter 2 and [ACDN00, Section 1]. Unlike the single graded case, the need of preserving the bihomogeneous structure of the ideal makes this bigeneric initial ideal dependent on the choice of the order of the variables of the different blocks.

Example 5.1. The following example is an adaptation of [BHS21, Example 4.3] (similarly [BES20, Example 1.4]) and corresponds to a smooth hyperelliptic curve of genus 8 embedded in $\mathbb{P}^2 \times \mathbb{P}^1$. It will be our running example throughout the chapter. Consider the standard $\mathbb{Z}^2$-graded ring $\mathbb{C}[x_0, x_1, x_2, y_0, y_1]$ and the ideal:

$$J = (y_0^2 x_0^2 + y_1^2 x_1^2 + y_0 y_1 x_2^2, y_0^3 x_2 + y_1^3(x_0 + x_1))$$

and consider $I = J^{\text{sat}} = (J : \mathfrak{b}^\infty)$ to be its saturation with respect to the irrelevant ideal $\mathfrak{b} = (x_0y_0, x_0y_1, x_1y_0, x_1y_1, x_2y_0, x_2y_1)$. We notice that if we use a monomial order such that:

$$x_0 < x_1 < x_2 < y_0 < y_1, \tag{5.1}$$

the degrees involved in the computation of the bigeneric initial ideal are those appearing in the left image in Figure 5.1. On the other hand, if the monomial order satisfies:

$$y_0 < y_1 < x_0 < x_1 < x_2, \tag{5.2}$$

the degrees involved in the computation of the bigeneric initial ideal are different; see right of Figure 5.1.

The problem of the relation between $\operatorname{reg}(I)$ and the generators of $\operatorname{bigin}(I)$ had already been raised in the work of Aramova, Crona and De Negri [ACDN00] and Römer [Röm01], where the following partial notion of regularity was defined using the Betti numbers.

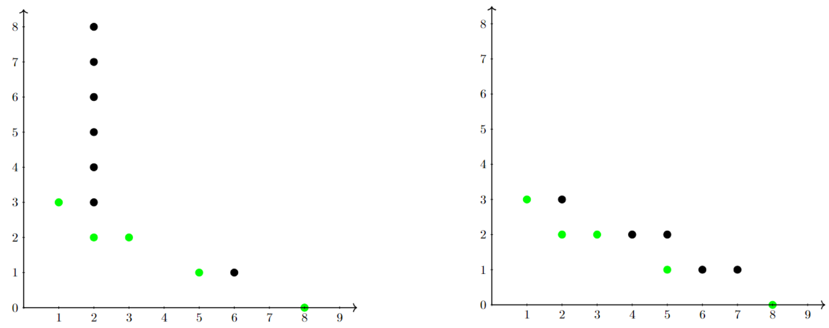


Figure 5.1: In each drawing, the green dots represent the degrees of the generators of I and the black dots represent the degrees of the generators of the bigeneric initial ideal. In the left hand side, we can see the degrees of the generators of bigin(I) using the monomial order in (5.1), while in the right hand side, we can see the degrees of the generators of bigin(I) using the monomial order in (5.2).

Definition 5.2. Let I be a bihomogeneous ideal in S, then $\mathfrak{R}_{\mathrm{x}}(I)$ is the minimal degree $a \in \mathbb{Z}$ such that:

$$\beta_{i,(a'+i+1,b')}(I) = 0$$

for all $i, b' \in \mathbb{Z}_{\geq 0}$ and for all $a' \geq a$. Similarly, one can define $\mathfrak{R}_{\mathrm{y}}(I)$.

With the above definition, Aramova, Crona and De Negri proved that the maximal degree of a minimal generator bigin(I) with respect to the x (resp. y) block of variables is given by $\mathfrak{R}_{\mathrm{x}}(\text{bigin}(I))$ (resp $\mathfrak{R}_{\mathrm{y}}(\text{bigin}(I))$); see [ACDN00, Theorem 2.2] using the same assumptions on the monomial order, Römer showed that:

$$\mathfrak{R}_{\mathrm{x}}(I) = \mathfrak{R}_{\mathrm{x}}(\text{bigin}(I))$$

and thus the description for the x block of variables depends solely on the algebra of I. If J is a monomial ideal satisfying the properties of Lemma 2.2, for instance bigin(I), then $\mathfrak{R}_{\mathrm{x}}(J)$ (resp. $\mathfrak{R}_{\mathrm{y}}(J)$) is the maximal degree of any minimal generator of J with respect to the degrees of x variables (resp. y).

Theorem 5.1 ([ACDN00, Theorem 2.2]). Let I be a bihomogeneous ideal. Then, there is $b \in \mathbb{N}$ and a generator of degree $(\mathfrak{R}_{\mathrm{x}}(\text{bigin}(I)), b)$ in bigin(I). Moreover, no generator of bigin(I) has degree with respect to the x variables bigger than $\mathfrak{R}_{\mathrm{x}}(\text{bigin}(I))$. The same property holds for $\mathfrak{R}_{\mathrm{y}}(\text{bigin}(I))$ and the degrees with respect to the y variables.

Furthermore, Römer proved that, using the relative order of the variables in Eq. (5.4), $\mathfrak{R}_{\mathrm{x}}(I)$ behaves well with respect to the bigeneric initial, i.e.,

$$\mathfrak{R}_{\mathrm{x}}(I) = \mathfrak{R}_{\mathrm{x}}(\text{bigin}(I)). \tag{5.3}$$

A direct consequence of this is that the maximum degree of the generators of bigin(I) with respect to the x variables is $\mathfrak{R}_{\mathrm{x}}(I)$; this is a partial generalization of the Bayer and Stillman criterion to the bihomogeneous setting.

Theorem 5.2 ([Röm01, Proposition 4.2])**.** Let $I \subset S$ be a bihomogeneous ideal. Then, there is $b \in \mathbb{N}$ and a generator of degree $(\mathfrak{R}_\mathrm{x}(I), b)$ in $\mathrm{bigin}(I)$. Moreover, no generator of $\mathrm{bigin}(I)$ has degree with respect to the x variables bigger than $\mathfrak{R}_\mathrm{x}(I)$.

Römer also noted [Röm01, Remark 4.3] that, as we are using the monomial order in (5.4), $\mathfrak{R}_\mathrm{y}(I)$ and $\mathfrak{R}_\mathrm{y}(\mathrm{bigin}(I))$ might be different and so the previous theorem does not hold for the variables in y.

Example 5.2. We continue Example 5.1. From the minimal free resolution of I (see [BHS21, Example 7.1]) we can derive that $\mathfrak{R}_\mathrm{x}(I) = 8$ and $\mathfrak{R}_\mathrm{y}(I) = 3$. Using the results in [ACDN00] and [Röm01], we can derive the maximal degrees of $\mathrm{bigin}_x(I)$ and $\mathrm{bigin}_y(I)$ with respect to each block of variables; see Figure 5.2.

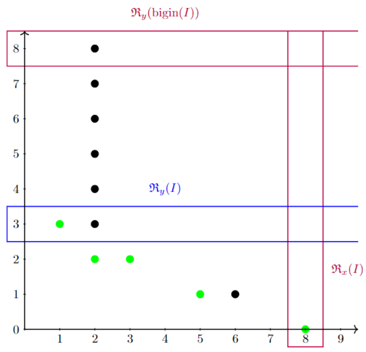


Figure 5.2: The bi-degrees of the generators of $\mathrm{bigin}(I)$, where the bound for the degrees of the x's is given by $\mathfrak{R}_\mathrm{x}(I)$. However, using the monomial order in (5.4), $\mathfrak{R}_\mathrm{y}(I)$ is lower than the bound on the y's given by $\mathfrak{R}_\mathrm{y}(\mathrm{bigin}(I))$.

Assumption 5.1. Consider a degree reverse lexicographical monomial order $<$ (or DRL) such that:

$$x_0 < \cdots < x_n < y_0 \cdots < y_m. \tag{5.4}$$

The above example also shows that the description in terms of $\mathfrak{R}_\mathrm{x}(I)$ only describes the maximal degree of a generator of $\mathrm{bigin}(I)$ with respect to one block of varialbes. Thus, a natural question is whether we can find an algebraic invariant that describes more tightly the bi-degrees involved in the computations based on Gröbner bases, in terms of the algebraic properties of I.

1. First examples and the case of ideals defining empty varieties

The works of Aramova et al. (Theorem 5.1) and Römer (Theorem 5.2) allow us to construct a bound for the bidegrees of the minimal generators of $\mathrm{bigin}(I)$. One of our objectives in this work is to construct coarser regions bounding these bidegrees.

Following the work of Bayer and Stillman, a natural candidate to construct such regions is the bigraded Castelnuovo-Mumford regularity either from I or bigin(I). However, as we illustrate in this section, it is not possible to construct straightforwardly such a generalization: on the one hand, there can be unbounded regions outside of reg(I) and reg(bigin(I)) where there is no minimal generator of bigin(I) (see Example 5.3); on the other hand, there can be minimal generators with bidegrees which are strictly bigger than reg(I); see Example 5.4. The same example also illustrates that reg(I) and reg(bigin(I)) may differ.

We start our discussion by considering the case of ideals defining empty varieties for which we can characterize the regularity in terms of the associated Hilbert function. Let I be a bihomogeneous ideal defining an empty variety of $\mathbb{P}^n \times \mathbb{P}^m$ and consider the associated Hilbert function

$$\begin{array}{rccc} \mathrm{HF}_{S/I}: & \mathbb{Z}^2 & \to & \mathbb{Z}_{\geq 0} \\ & (a,b) & \mapsto & \dim_{\mathbf{k}}(S/I)_{(a,b)} \end{array}. \tag{5.5}$$

The regularity reg(I) is determined by the bidegrees at which this function attains the value zero, i.e.,

$$\mathrm{reg}(I) = \{(a,b) \in \mathbb{Z}^2 \,:\, \mathrm{HF}_{S/I}(a,b) = 0\}. \tag{5.6}$$

This last equality follows, mutatis mutandis, from [BS87a, Lemma 1.7] where a similar result is presented for the homogeneous case.

It is a general property that the Hilbert functions of an ideal I and its bigeneric initial ideal bigin(I) coincide; this also follows similarly from the single-graded case [Eis95, Theorem 15.26]. Therefore, if I defines an empty subvariety of $\mathbb{P}^n \times \mathbb{P}^m$, it holds that

$$\mathrm{reg}(I) = \mathrm{reg}(\mathrm{bigin}(I)).$$

In addition, it is possible to determine the existence of minimal generators of bigin(I) at bidegrees where the Hilbert function of I vanishes.

Theorem 5.3. Let $(a,b) \in \mathbb{Z}^2_{\geq 0}$ be a bidegree such that $\mathrm{HF}_{S/I}(a,b) = 0$. Then, bigin$(I)$ has a minimal generator of bidegree (a,b) if and only if $\mathrm{HF}_{S/I}(a',b') \neq 0$ for every $(a',b') \lneq (a,b)$.

Proof. With respect to DRL, the monomial $x_0^a y_0^b$ is the smallest monomial of bidegree (a,b). As $\mathrm{HF}_{S/I}(a,b) = 0$, we have that $x_0^a y_0^b \in \mathrm{bigin}(I)_{(a,b)}$. If $x_0^a y_0^b$ is not a minimal generator of bigin(I), then there must be $(a',b') \lneq (a,b)$ such that $x_0^{a'} y_0^{b'} \in \mathrm{bigin}(I)$. By Lemma 2.2, this last condition is equivalent to the fact that every monomial of bidegree (a',b') belongs to bigin(I), that is, equivalent to $\mathrm{HF}_{S/I}(a',b') = 0$ for some $(a',b') \lneq (a,b)$.

□

As a consequence of this theorem, we deduce that the region $\mathrm{reg}(I)$ does not contain bidegrees of minimal generators of $\mathrm{bigin}(I)$, except maybe in its minimal bi-degrees with respect to inclusion.

Corollary 5.1. Assume that I defines an empty variety of $\mathbb{P}^n \times \mathbb{P}^m$. If $(a,b) \in \mathrm{reg}(I)$ then $\mathrm{bigin}(I)$ has no minimal generator of bidegree $(a',b') \gneq (a,b)$.

Proof. Follows from Eq. (5.6) and Theorem 5.5. □

Remark 5.1. Using that, in the case of I defining an empty subvariety, $\mathrm{reg}(I)$ and $\mathrm{reg}(\mathrm{bigin}(I))$ coincide, we may also deduce the above corollary from Theorem 2.15.

As mentioned before, unlike in the single graded case, the region $\mathrm{reg}(I)$ does not yield a sharp description of the bidegrees of a minimal set of generators of $\mathrm{bigin}(I)$. We illustrate it with the following example where we observe that the converse of Corollary 5.1 does not hold and also that $\mathrm{reg}(I)$ does not yield any information for infinitely many bidegrees.

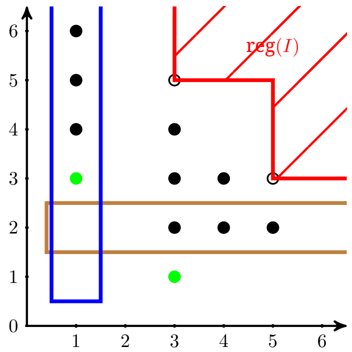


Figure 5.3: The green dots • represent the bidegrees (a,b) of the generators of the ideal I in Example 5.3. The black dots • represent the bidegrees (a,b) of minimal generators of $\mathrm{bigin}(I)$and the white dots those bidegrees for which $\mathrm{HF}_{S/I}(a,b) = 0$. The region $\mathrm{reg}(I)$ is marked in red. In blue (resp. brown), an infinite column (resp. a row) which does not intersect $\mathrm{reg}(I)$.

Example 5.3. Consider the standard $\mathbb{Z}^2$-graded ring $S = \mathbb{C}[x_0,x_1,y_0,y_1]$ and the ideal $I \subset S$ generated by four bihomogeneous polynomials:

$$p_3(x_0,x_1)q_1(y_0,y_1), p_3'(x_0,x_1)q_1'(y_0,y_1), p_1(x_0,x_1)q_3(y_0,y_1), p_1'(x_0,x_1)q_3'(y_0,y_1),$$

where p_i's and p_i''s and general forms of degree i in x_0,x_1 and q_i's and q_i''s and general forms of degree i in y_0,y_1. The ideal I defines an empty variety. In Figure 5.3 we show the bidegrees of a minimal set of generators for $\mathrm{bigin}(I)$, as well the region $\mathrm{reg}(I)$.

At the bidegrees $(3,5)$ and $(5,3)$, Theorem 5.3 applies so that there are minimal generators of $\mathrm{bigin}(I)$. The bidegree $(4,4)$, where $\mathrm{HF}_{S/I}(4,4) \neq 0$, so that $(4,4) \notin \mathrm{reg}(I)$, shows that the converse of Corollary 5.1 does not hold. Also, in Figure 5.3 we highlighted an infinite column and an infinite row that does not intersect $\mathrm{reg}(I)$.

It turns out that if we do not restrict to ideals defining an empty variety, then even Corollary 5.1 is no longer true, i.e. there might be bidegrees $(a, b) \in \operatorname{reg}(I)$ such that there are minimal generators of $\operatorname{bigin}(I)$ in degree $\gneq (a, b)$.

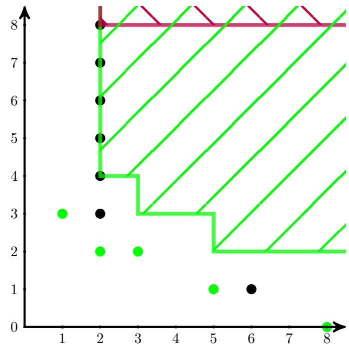


Figure 5.4: The multigraded Castelnuovo-Mumford regularity $\operatorname{reg}(I)$, in green and the multigraded Castelnuovo-Mumford regularity of the bigeneric initial ideal $\operatorname{bigin}(I)$, in purple.

Example 5.4. We continue with Example 5.1. We note that $(2, 4) \in \operatorname{reg}(I)$, nevertheless there are generators of bidegrees $\gneq (2, 4)$; se Figure 5.4. This example also illustrates that, in general, we cannot expect that $\operatorname{reg}(I)$ and $\operatorname{reg}(\operatorname{bigin}(I))$ to coincide, even though we have an inclusion, as in Eq. (2.43). For this reason, there is no straightforward way of using $\operatorname{reg}(I)$ in order to bound the bidegrees of the minimal generators of $\operatorname{bigin}(I)$.

Remark 5.2. To compute the regions in these examples, we used the `Macaulay2` packages `VirtualResolutions` [Alm+20] and `LinearTruncations` [CHN22]. In both packages, the input is assumed to be saturated, which is the case of the ideal I in Example 5.1.

2. The partial regularity region and its main properties

The results and examples presented in Section 1. illustrate the difficulty to establish a direct bihomogeneous analogue of Eq. **(??)** by means of the bigraded Castelnuovo-Mumford regularity region. To unravel this situation, we introduce a new region, denoted by $\operatorname{xreg}(I)$ and explore its properties. Compared to the Castelnuovo-Mumford regularity region, which relies on the vanishing of local cohomology modules with respect to the irrelevant ideal $\mathfrak{b}$ of the product of two projective spaces, this new region relies on the vanishing of local cohomology modules with respect to the ideal $\mathfrak{m}_{\mathbf{x}}$. This construction is inspired by the work of Botbol, Chardin and Holanda [BC17; CH22].

Definition 5.3. Let $I \subset S$ be a bihomogeneous ideal. We denote by $\operatorname{xreg}(I)$, and call it the *partial regularity region*, the region of bidegrees $(a, b) \in \mathbb{Z}^2$ such that for all $i \geq 1$ and $(a', b') \geq (a - i + 1, b)$,

$$H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a',b')} = 0.$$

One of the main results of our work is that, in generic coordinates, $\operatorname{xreg}(I)$ provides a bounding region to bidegrees of the elements in a minimal Gröbner basis which is (partially) tight. This result will be proved in Section 3.. In order to do so, in this section we need to establish two key properties of $\operatorname{xreg}(I)$. First, in Theorem 5.6, we present a criterion to compute $\operatorname{xreg}(I)$ similar to the one proposed by Bayer and Stillman in the classical setting [BS87a, Theorem 1.10]. Second, in Theorem 5.5, we show that, in generic coordinates, the partial regularity region of an ideal and its bigeneric initial ideal agree, that is, $\operatorname{xreg}(I) = \operatorname{xreg}(\operatorname{bigin}(I))$. The results on this section generalize the ones in [BS87a, §1] and most of the proofs follow similar strategies.

Notation 5.3. Let $I \subset S$ be a bihomogeneous ideal.

- We denote by $I^{\mathsf{sat,x}}$ the saturation of I with respect to $\mathfrak{m}_{\mathrm{x}}$, i.e. $(I : \mathfrak{m}_x^\infty)$.
- Given any polynomial $f \in S$, (I, f) will denote the sum of the ideals I and (f).

Lemma 5.1. A generic linear x-form h is not a zero divisor in $S/I^{\mathsf{sat,x}}$. Namely, $(I^{\mathsf{sat,x}} : h) = I^{\mathsf{sat,x}}$.

Proof. The proof follows using the same argument as [Van02, Lemma 3.3]. □

The following lemma shows that local cohomology modules with respect to $\mathfrak{m}_{\mathrm{x}}$ vanish when the degree with respect to the x variables is big enough.

Lemma 5.2. Let $I \subset S$ be a bihomogeneous ideal. Then there is $a_0 \in \mathbb{Z}$ such that for all $b \in \mathbb{Z}$, it holds

$$H^i_{\mathfrak{m}_{\mathrm{x}}}(I)_{(a,b)} = 0 \quad \forall a \geq a_0.$$

Proof. It is classical property that the local cohomology modules $H^i_{\mathfrak{m}_{\mathrm{x}}}(I)$ can be defined and computed using the Čech complex $\mathcal{C}^\bullet_{\mathfrak{m}_{\mathrm{x}}}(I)$. We refer the reader to [BS08] for more details on the construction of this complex and its main properties. In this proof, we will use this complex, together with a minimal free resolution $F_\bullet$ of I, to construct a double complex that we denote by $\mathcal{C}^\bullet_{\mathfrak{m}_{\mathrm{x}}}(F_\bullet)$. We note that this double complex has often been used in the bibliography; see, for example, [BH19, §2].

There are two natural spectral sequences associated with the double complex $\mathcal{C}^\bullet_{\mathfrak{m}_{\mathrm{x}}}(F_\bullet)$, depending on whether we consider the filtrations with respect to the horizontal or the vertical maps. Both sequences converge to the same limit. Considering the filtration given by the horizontal maps, since $F_\bullet$ is a minimal free resolution of I we deduce that the spectral sequence converges to $H^\bullet_{\mathfrak{m}_{\mathrm{x}}}(I)$ in its second page.

Similarly, the second spectral sequence has the terms $H^i_{\mathfrak{m}_{\mathrm{x}}}(F_q)$ in its first page. These terms are direct sums of $H^i_{\mathfrak{m}_{\mathrm{x}}}(S)$, up to the shifts appearing in the minimal free resolution $F_\bullet$. Thus, using Eq. (2.42), we deduce that there exists $a_0 \in \mathbb{Z}$ such that for all $b \in \mathbb{Z}$,

$$H^i_{\mathfrak{m}_{\mathrm{x}}}(F_q)_{(a',b')} = 0 \text{ for all } i, q \text{ and } (a', b') \geq (a_0, b).$$

The proof follows straightforwardly by comparing the limits of the two above spectral sequences. $\square$

Lemma 5.3. Let $I \subset S$ be a bihomogeneous ideal. Let h be generic linear x-form and $(a, b) \in \mathbb{Z}^2_{\geq 0}$. Then, the following are equivalent:

i) $(I : h)_{(a',b')} = I_{(a',b')}$ for all $(a', b') \geq (a, b)$.

ii) $H^1_{\mathfrak{m}_x}(I)_{(a',b')} = 0$ for all $(a', b') \geq (a, b)$.

Proof. Since $n \geq 1$, we have that $H^1_{\mathfrak{m}_x}(I) \cong H^0_{\mathfrak{m}_x}(S/I) = I^{\mathrm{sat,x}}/I$; see for example [BS87a, Remark 1.3]. Hence, $H^1_{\mathfrak{m}_x}(I)_{(a',b')} = 0$, if and only if, $I^{\mathrm{sat,x}}_{(a',b')} = I_{(a',b')}$. By Lemma 5.1, as h is a generic x-form, condition $ii)$ implies condition $i)$ as in this case we have that, for every $(a', b') \geq (a, b)$

$$(I : h)_{(a',b')} = (I^{\mathrm{sat,x}} : h)_{(a',b')} = I^{\mathrm{sat,x}}_{(a',b')} = I_{(a',b')}.$$

To prove the opposite implication, we observe that by Lemma 5.2, for a given bidegree $(a, b) \in \mathbb{Z}^2$, there is a $\lambda_0 \in \mathbb{Z}_{\geq 0}$ such that, for every $\lambda \geq \lambda_0$, we have that

$$I_{(a'+\lambda,b')} = I^{\mathrm{sat,x}}_{(a'+\lambda,b')} \text{ for every } (a', b') \geq (a, b).$$

Either the previous condition holds for every λ_0 and so $H^1_{\mathfrak{m}_x}(I)_{(a',b')} = 0$ for all $(a', b') \geq (a, b)$, or either there is a minimal λ_0 satisfying the previous condition. In the latter case, by minimality of λ_0, we have that $I_{(a'+\lambda_0-1,b')} \neq I^{\mathrm{sat,x}}_{(a'+\lambda_0-1,b')}$ for some $(a', b') \geq (a, b)$. Therefore, there must be bihomogeneous $f \in I^{\mathrm{sat,x}}$ of bidegree $(a' + \lambda_0 - 1, b')$ such that $f \notin I$. However, as $I^{\mathrm{sat,x}}_{(a'+\lambda_0,b')} = I_{(a'+\lambda_0,b')}$, for every x-form $h \in S_{(1,0)}$, we have that $h\,f \in I_{(a'+\lambda_0,b')}$ and so $f \in (I : h)_{(a'+\lambda_0-1,b')}$. If $\lambda_0 \geq 1$, condition $i)$ implies that $(I : h)_{(a'+\lambda_0-1,b')} = I_{(a'+\lambda_0-1,b')}$, so we get a contradiction as $f \notin I$. Hence $\lambda_0 \leq 0$ and so $H^1_{\mathfrak{m}_x}(I)_{(a',b')} = 0$ for every $(a', b') \geq (a, b)$. $\square$

The following lemma shows that the partial regularity region can be computed recursively using colon ideals with respect to generic linear x-forms.

Lemma 5.4. Let h be a generic linear x-form and $(a, b) \in \mathbb{Z}^2_{\geq 0}$, then the following are equivalent.

i) $(a, b) \in \mathrm{xreg}(I)$.

ii) $(I : h)_{(a',b')} = I_{(a',b')}$ for $(a', b') \geq (a, b)$ and $(a, b) \in \mathrm{xreg}(I, h)$.

Proof. First, we observe that, if $(I : h)_{\geq(a,b)} = I_{\geq(a,b)}$, then for every $i \geq 1$, $H^i_{\mathfrak{m}_x}((I : h)_{\geq(a,b)}) = H^i_{\mathfrak{m}_x}(I_{\geq(a,b)})$. Under this assumption, Lemma 2.3 implies that

$$H^1_{\mathfrak{m}_x}(I : h)_{(a',b')} = H^1_{\mathfrak{m}_x}(I)_{(a',b')} \text{ for every } (a', b') \geq (a, b). \tag{5.7}$$

and

$$H^i_{\mathfrak{m}_{\mathbf{x}}}(I:h) = H^i_{\mathfrak{m}_{\mathbf{x}}}(I) \text{ for every } i \geq 2. \tag{5.8}$$

To prove that $i)$ implies $ii)$, let $(a,b) \in \mathrm{xreg}(I)$. By Lemma 5.3, we get that $(I : h)_{\geq(a,b)} = I_{\geq(a,b)}$, so that Eq. (5.7) and Eq. (5.8) hold. Consider the short exact sequence which is induced by the multiplication by h,

$$0 \to (I:h)(-(1,0)) \to I \oplus (h) \to (I,h) \to 0. \tag{5.9}$$

For every $i \geq 1$, taking the graded components of the corresponding long exact sequence of local cohomology at degrees $(a',b') \geq (a-(i-1),b)$ yields

$$\cdots \to H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a',b')} \to H^i_{\mathfrak{m}_{\mathbf{x}}}(I,h)_{(a',b')} \to H^{i+1}_{\mathfrak{m}_{\mathbf{x}}}(I:h)_{(a'-1,b')} \to \cdots \tag{5.10}$$

We notice that the last term in (5.10) can be replaced by $H^{i+1}_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a'-1,b')}$ as $i \geq 1$, using Eq. (5.8). Moreover, as we are assuming that $(a,b) \in \mathrm{xreg}(I)$, the two graded components of the local cohomology of I in Eq. (5.10) vanish, and so $H^i_{\mathfrak{m}_{\mathbf{x}}}(I,h)_{(a',b')} = 0$ for all $(a',b') \geq (a-(i-1),b)$.

In order to prove that condition $ii)$ implies condition $i)$, we consider $(a,b) \in \mathrm{xreg}(I,h)$ such that $(I:h)_{(a',b')} = I_{(a',b')}$ for every $(a',b') \geq (a,b)$. By Lemma 5.1, we have that $H^1_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a',b')} = 0$ for $(a',b') \geq (a,b)$. As we did above, we consider the long exact sequence associated to Eq. (5.9) at the graded pieces given by $(a',b') \geq (a-(i-2),b)$,

$$\cdots \to H^{i-1}_{\mathfrak{m}_{\mathbf{x}}}(I,h)_{(a',b')} \to H^i_{\mathfrak{m}_{\mathbf{x}}}(I:h)_{(a'-1,b')}$$

$$\xrightarrow{\delta_i} H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a',b')} \to H^i_{\mathfrak{m}_{\mathbf{x}}}(I,h)_{(a',b')} \to \cdots .$$

As $(a,b) \in \mathrm{xreg}(I,h)$, the graded pieces of local cohomology modules associated to (I,h) vanish, and so the map δ_i is an isomorphism. By Lemma 5.2, there exists λ sufficiently big such that

$$H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a'+\lambda,b')} = 0 \quad (a',b') \geq (a-(i-1),b). \tag{5.11}$$

As we are assuming that $(I:h)_{\geq(a,b)} = I_{\geq(a,b)}$, Eq. (5.8) holds and, together with Eq. (5.11) and the isomorphism δ_i, we have that for every $i > 1$ and every $(a',b') \geq (a-(i-1),b)$,

$$H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a'+\lambda-1,b')} \cong H^i_{\mathfrak{m}_{\mathbf{x}}}(I:h)_{(a'+\lambda-1,b')} \underset{\delta_i}{\cong} H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a'+\lambda,b')} = 0. \tag{5.12}$$

where the first isomorphism follows from Eq. (5.8), the second isomorphism is δ_i, and the last equality to zero follows from Eq. (5.11). If we apply the above repeatedly, starting from big enough λ, we conclude that for every $i \geq 1$, $H^i_{\mathfrak{m}_{\mathbf{x}}}(I)$ vanishes at every degree bigger or equal to $(a-(i-1),b)$, and therefore $(a,b) \in \mathrm{xreg}(I)$. $\square$

The following theorem, which aims to characterize $\mathrm{xreg}(I)$, can be seen as a partial extension of the criterion of Bayer and Stillman to compute the Castelnuovo-Mumford regularity in the single graded case [BS87a, Theorem 1.10] to the setting of bigraded ideals.

Theorem 5.4. Consider a bihomogeneous ideal $I \subset S$. Then, for $(a,b) \in \mathbb{Z}^2_{\gtrsim 0}$, the following are equivalent:

i) $(a,b) \in \mathrm{xreg}(I)$.

ii) There exists a non-negative integer $k_0 \leq n$ such that for all $k = 0,\dots,k_0$ and $(a',b') \geq (a,b)$, we have:

$$(J_{k-1} : h_k)_{(a',b')} = (J_{k-1})_{(a',b')}$$

where $J_{k-1} = (I, h_0, \dots, h_{k-1})$ (with the convention $J_{-1} = I$), h_k are generic linear x-forms and $J_{k_0} \supset \mathfrak{m}_{\mathrm{x}}$.

Proof. We will proceed by induction in the minimal number k_0 such that $J_{k_0} \supset \mathfrak{m}_{\mathrm{x}}$. Consider I such that $k_0 = -1$. As $I \supset \mathfrak{m}_{\mathrm{x}}$, we have $I^{\mathsf{sat,x}} = S$. Moreover, as in the classical setting (see [BS08, Corollary 2.1.7]) the higher local cohomology modules of I are the same as those of the saturation, and so

$$H^i_{\mathfrak{m}_{\mathrm{x}}}(I) = H^i_{\mathfrak{m}_{\mathrm{x}}}(I^{\mathsf{sat,x}}) = H^i_{\mathfrak{m}_{\mathrm{x}}}(S) \text{ for every } i \geq 2.$$

By Remark 2.8, the previous cohomology $H^i_{\mathfrak{m}_{\mathrm{x}}}(S)_{(a,b)}$ vanish, unless $i = n+1, a \leq -n-1$ and $b \geq 0$. Therefore,

$$H^{n+1}_{\mathfrak{m}_{\mathrm{x}}}(I)_{(a',b')} = 0 \quad \text{for } (a',b') \geq (a-n,b).$$

Moreover, as $a > 0$ and $I \supset \mathfrak{m}_{\mathrm{x}}$, we have $I^{\mathsf{sat,x}}_{(a,b)} = I_{(a,b)} = S_{(a,b)}$. This last condition is equivalent to the fact that $H^1_{\mathfrak{m}_{\mathrm{x}}}(I)_{(a,b)} = 0$ for $a > 0$, and so $\mathrm{xreg}(I) \supset \mathbb{Z}^2_{\gtrsim 0}$. This shows that, for $k_0 = -1$ condition $i)$ is also always satisfied.

For the inductive step, assume that the theorem holds for every ideal such that its associated k_0 is at most t. Consider an ideal I such that its associated k_0 is $t+1$. Then, for a generic x-form h, the k_0 associated to (I,h) is t. Hence, we can apply our inductive hypothesis to (I,h). The proof follows straightforwardly from Lemma 5.4. ☐

Our next goal is to prove that the region $\mathrm{xreg}(I)$ and the region $\mathrm{xreg}(\mathrm{bigin}(I))$ coincide. This will only happen if we consider the $\mathrm{bigin}(I)$ with respect to the DRL monomial order in (5.4). In the next lemma, we analyze the behavior of ideals under change of coordinates.

Lemma 5.5. Let $I \subset S$ be a bihomogeneous ideal and $u \in \mathrm{GL}(n+1) \times \mathrm{GL}(m+1)$. Then, the following hold:

i) $u \circ (I^{\mathsf{sat,x}}) = (u \circ I)^{\mathsf{sat,x}}$.

ii) Let $h_0, \dots, h_n$ be linear forms satisfying that $x_k = u \circ h_k$ for $k = 0, \dots, n$ and $(a, b) \in \mathbb{Z}^2$. Then,

$$(I, h_0, \dots, h_{k-1} : h_k)_{(a,b)} = (I, h_0, \dots, h_{k-1})_{(a,b)} \iff \\ [(u \circ I), x_0, \dots, x_{k-1} : x_k]_{(a,b)} = [(u \circ I), x_0, \dots, x_{k-1}]_{(a,b)}.$$

for all $k = 0, \dots, n$.

Proof. In order to prove the part $i)$, we note that $\mathfrak{m}_{\mathrm{x}}$ is invariant under the action of $\mathrm{GL}(n+1) \times \mathrm{GL}(m+1)$. Therefore, if $f \in u \circ I^{\mathrm{sat,x}}$, then there is $g \in I^{\mathrm{sat,x}}$ such that $f = u \circ g$ and there exists t with $g\mathfrak{m}_x^t \subset I$. This is equivalent to the fact that $f\mathfrak{m}_x^t \subset u \circ I$. Similarly, the proof of part $ii)$ follows from the fact that the colon ideal commutes with the change of coordinates. □

The following lemma shows that we can verify the equality between I and its colon ideal with respect to a variable by looking at the initial ideal. This is a classical property of the DRL monomial order defined in Eq. (5.4); see Remark 5.3 for more details.

Lemma 5.6 ([BS87a, Lemma 2.2])**.** Let $I \subset S$ be a bihomogeneous ideal and let $(a, b) \in \mathbb{Z}^2_{\gtrsim 0}$. For $k = 0, \dots, n$, we have the following:

i) $\mathrm{in}(I, x_0, \dots, x_k) = (\mathrm{in}(I), x_0, \dots, x_k)$.

ii) Suppose that $x_0, \dots, x_{k-1} \in I$ and that we are using the DRL monomial order in Eq. (5.4), then:

$$(I : x_k)_{(a,b)} = I_{(a,b)} \iff (\mathrm{in}(I) : x_k)_{(a,b)} = \mathrm{in}(I)_{(a,b)}.$$

The following lemma generalizes Lemma 5.1 to the bigeneric initial ideal.

Lemma 5.7. Let $I \subset S$ be a bihomogeneous ideal. For every $k = 0, \dots, n$, let $J_{k-1} := (\mathrm{bigin}(I), x_0, \dots, x_{k-1})$ (with the convention $J_{-1} = I$). If $(J_{k-1})^{\mathrm{sat,x}} \neq S$, then x_k is a non-zero divisor in $S/(J_{k-1})^{\mathrm{sat,x}}$.

Proof. We first prove the case $k = 0$. Following the same argument as in [CDNG13, Lemma 2.1], we note that the associated primes of a bi-Borel fixed ideal are of the form:

$$\mathcal{P}_t = (x_{t_x}, \dots, x_n, y_{t_y}, \dots, y_m)$$

for some $t_x, t_y \in \mathbb{Z}^2$ such that $0 \le t_x \le n$ and $0 \le t_y \le m$. If $\mathrm{bigin}(I)^{\mathrm{sat,x}} \neq S$, the associated primes of $\mathrm{bigin}(I)^{\mathrm{sat,x}}$ must satisfy $t_x > 0$. Therefore, x_0 cannot be contained in the union of the associated primes of $\mathrm{bigin}(I)^{\mathrm{sat,x}}$.

In the case where $k > 0$, we note that $\mathrm{bigin}(I) \cap \mathbf{k}[x_k, \dots, x_n, y_0, \dots, y_m]$ is also bi-Borel fixed. Therefore, the associated primes of J_{k-1} which contain x_k are not associated primes of $(J_{k-1})^{\mathrm{sat,x}}$. The proof follows by the same argument as above. □

Using the above lemmas, we obtain the following theorem.

Theorem 5.5. Let $I \subset S$ be a bihomogeneous ideal and $h_0, \ldots, h_k$ are generic linear x-forms. Then, for every $(a,b) \in \mathbb{Z}^2_{\gtrsim 0}$ and $k = 0, \ldots, n$

$$(I, h_0, \ldots, h_{k-1} : h_k)_{(a,b)} = (I, h_0, \ldots, h_{k-1})_{(a,b)} \iff$$
$$[(\text{bigin}(I), x_0, \ldots, x_{k-1}) : x_k]_{(a,b)} = [(\text{bigin}(I), x_0, \ldots, x_{k-1})]_{(a,b)}. \quad (5.13)$$

In particular, $\text{xreg}(I) \cap \mathbb{Z}^2_{\gtrsim 0} = \text{xreg}(\text{bigin}(I)) \cap \mathbb{Z}^2_{\gtrsim 0}$.

Proof. The first part of the proof follows straightforwardly from Lemma 5.3 and Lemma 5.6. For the second part, we note that Lemma 5.7 implies that the proof of Theorem 5.4 can be reproduced for $\text{bigin}(I)$ using the variables $x_0, \ldots, x_n$ instead of generic linear x-forms $h_0, \ldots, h_n$. Namely, $(a,b) \in \text{xreg}(\text{bigin}(I)) \cap \mathbb{Z}^2_{\gtrsim 0}$, if and only if, there is $k_0 \in \mathbb{Z}_{\geq 0}$ such that for all $k = 0, \ldots, k_0$ and $(a', b') \geq (a, b)$, we have:

$$(J_{k-1} : x_k)_{(a',b')} = (J_{k-1})_{(a',b')}$$

where $J_{k-1} = (\text{bigin}(I), x_0, \ldots, x_{k-1})$ and $J_{k_0} \supset \mathfrak{m}_{\mathbf{x}}$. Therefore, the proof follows straightforwardly from the first part, i.e. from Eq. (5.13). □

Remark 5.3. In Lemma 5.6, the proof of the fact that for any $k = 0, \ldots, n$ and for any I such that if $x_0, \ldots, x_{k-1} \in I$, we have

$$(\text{in}(I) : x_k)_{(a,b)} = \text{in}(I)_{(a,b)} \implies (I : x_k)_{(a,b)} = I_{(a,b)} \quad (5.14)$$

does not require that the monomial order $<$ is degree reverse lexicographical. Therefore, Eq. (5.14) also holds for any other monomial order. On the other hand, in [Loh16], Loh proved that for any monomial order different than DRL, it is possible to find an ideal I such that the converse implication to Eq. (5.14) does not hold, regardless of the bigraded context. This motivates our choice of using the DRL monomial order in our study of the generalization of the Bayer-Stillman criterion to the bigraded setting.

As we noticed in Example 5.1, the relative order of the variables of different blocks will change the bidegrees of the generators of $\text{bigin}(I)$. Theorem 5.5 relies on the specific choice of Eq. (5.4). While the criterion in Theorem 5.4 would also hold symmetrically for $\text{yreg}(I)$, this region does not remain invariant under $\text{bigin}(I)$ unless we change the relative order of the blocks of variables.

Example 5.5. We continue with Example 5.1 and draw the regions $\text{yreg}(I)$ and $\text{yreg}(\text{bigin}(I))$. We note that, using the monomial order Eq. (5.4), they are different.

3. The partial regularity region and the minimal generators of $\text{bigin}(I)$

In the previous section, we provided the definition and main properties of the partial regularity region $\text{xreg}(I)$, including a criterion which generalizes the classical

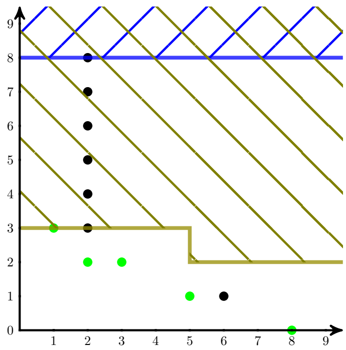


Figure 5.5: In olive, the region $\text{yreg}(I)$. In blue, the region $\text{yreg}(\text{bigin}(I))$.

result of Bayer and Stillman to the setting of regions of bidegrees that we are studying. In this section, we exploit $\text{xreg}(I)$ to prove the absence of minimal generators of $\text{bigin}(I)$ at some bidegrees (see Theorem 5.6) and to certify that there are generators near the border of the region $\text{xreg}(I)$ (see Theorem 5.7). Moreover, we also provide relations between $\text{reg}(I)$, $\text{xreg}(I)$ and the Betti numbers of I by relying on results by Chardin and Holanda [CH22].

The following lemma, which is the bigraded analogue of [BS87a, Lemma 2.2 iii)], provides sufficient conditions for the absence of minimal generators of bidegree (a, b).

Lemma 5.8. Consider a bihomogeneous ideal $I \subset S$ and $k \in \{0, \dots, n\}$ such that $x_0, \dots, x_{k-1} \in I$. Let $(a, b) \in \mathbb{Z}^2_{\gtrsim 0}$ with $a > 1$. Assume that there is no minimal generator of $\text{in}(I, x_k)$ of bidegree $(a, b) \in \mathbb{Z}^2_{\gtrsim 0}$ and that

$$(\text{in}(I) : x_k)_{(a-1,b)} = (\text{in}(I) + \mathfrak{m}_y(\text{in}(I) : x_k))_{(a-1,b)}. \tag{5.15}$$

Then, there is no minimal generator of $\text{in}(I)$ of bidegree (a, b).

Proof. Consider an element $f \in I_{(a,b)}$. If $x_0, \dots, x_{k-2}$, or x_{k-1} divides $\text{in}(f)$, then f cannot be a minimal generator of $\text{in}(I)$. Thus, up to substracting multiples of $x_0, \dots, x_{k-1}$, we may assume that $f \in \mathbf{k}[x_k, \dots, x_n, y_0, \dots, y_m]$. If x_k divides $\text{in}(f)$, then $\text{in}(f) = x_k\ \text{in}(\overline{f})$ for some

$$\text{in}(\overline{f}) \in (\text{in}(I) : x_k)_{(a-1,b)} = (\text{in}(I) + \mathfrak{m}_y(\text{in}(I) : x_k))_{(a-1,b)}.$$

Hence, there is a non-constant $x^\alpha y^\beta \in (x_k, \mathfrak{m}_y)$ and $l \in I$ of bidegree strictly smaller than (a, b) such that $\text{in}(f) = x^\alpha y^\beta\ \text{in}(l)$. Therefore, $\text{in}(f)$ cannot be a minimal generator.

Suppose now that x_k does not divide $\text{in}(f)$. As there is no generator of $\text{in}(I, x_k)$ of bidegree (a, b) and $\text{in}(f) \in \text{in}(I, x_k)$, then we can write $\text{in}(f) = x^{\alpha'} y^{\beta'}\ \text{in}(g)$ with $x^{\alpha'} y^{\beta'} \neq 1$ and $\text{in}(g) \in \text{in}(I, x_k)$. Write g as $g = g_1 + x_k g_2$ for $g_1 \in I$. Since $\text{in}(g) > \text{in}(x_k g_2)$, we have that $\text{in}(f) = x^{\alpha'} y^{\beta'}\ \text{in}(g_1)$ with $g_1 \in I$ an element of [strictly lower bidegree] than (a, b). Hence, $\text{in}(f)$ is not a generator of $\text{in}(I)$ $\square$

Applying repeatedly Lemma 5.8, we get sufficient conditions for the absence of minimal generators of $\mathrm{in}(I)$ of bidegree $(a,b) \in \mathbb{Z}^2_{\gtrsim 0}$.

Corollary 5.2. Let $I \subset S$ be a bihomogeneous ideal. Let $(a,b) \in \mathbb{Z}^2_{\gtrsim 0}$ with $a > 1$, and assume that:

$$(\mathrm{in}(J_{k-1}) : x_k)_{(a,b)} = \big(\mathrm{in}(J_{k-1}) + \mathfrak{m}_{\mathrm{y}}(\mathrm{in}(J_{k-1}) : x_k)\big)_{(a,b)}, \tag{5.16}$$

for all $k = 0, \ldots, n$ and $J_k = (I, x_0, \ldots, x_k)$ (with the convention $J_{-1} = I$). Then, there is no generator of $\mathrm{in}(I)$ of bidegree (a,b).

Proof. Note that there are no minimal generators of $\mathrm{in}(I, x_0, \ldots, x_n)$ of any bidegree $(a,b) \in \mathbb{Z}^2_{\gtrsim 0}$ as each of them must be divided by some x_i. By Lemma 5.6 and the hypothesis, this implies that there is no generator of $\mathrm{in}(I, x_0, \ldots, x_{n-1})$ of bidegree (a,b). Applying Lemma 5.6 $iii)$ recursively, we get that there is no generator of $\mathrm{in}(I)$ of bidegree $(a,b) \in \mathbb{Z}^2_{\gtrsim 0}$. $\square$

Applying Theorem 5.4, we derive the following result.

Theorem 5.6. Let $I \subset S$ be a bihomogeneous ideal. Let $(a,b) \in \mathrm{xreg}(I) \cap \mathbb{Z}^2_{\gtrsim 0}$. If $(a',b') \geq (a+1,b)$, then there is no minimal generator of $\mathrm{bigin}(I)$ of bidegree (a',b').

Proof. Note that for every $(a',b') \in \mathbb{Z}^2_{\gtrsim 0}$, the equality

$$(J_{k-1} : x_k)_{(a',b')} = (J_{k-1})_{(a',b')}$$

implies that

$$(J_{k-1} : x_k)_{(a',b')} = [J_{k-1} + \mathfrak{m}_{\mathrm{y}}(J_{k-1} : x_k)]_{(a',b')}$$

for every $J_{k-1} = (\mathrm{bigin}(I), x_0, \ldots, x_{k-1})$ with $k = 0, \ldots, n$. Therefore, applying Corollary 5.2 to $\mathrm{bigin}(I)$ and Theorem 5.4, we deduce that if $(a,b) \in \mathrm{xreg}(I) \cap \mathbb{Z}^2_{\gtrsim 0}$, then $(J_{k-1} : x_k)_{(a',b')} = (J_{k-1})_{(a',b')}$ for all $(a',b') \geq (a,b)$ and $k = 0, \ldots, n$ $\square$

In addition, we can use Lemma 2.2 to attest the presence of generators of some bidegrees, using the same criterion as in Theorem 5.4.

Theorem 5.7. Let $(a,b) \in \mathbb{Z}^2_{\gtrsim 0}$ with $a > 1$ such that $(a,b) \in \mathrm{xreg}(I)$, but $(a-1,b) \notin \mathrm{xreg}(I)$. Then, there exists $b' \leq b$ such that there is a minimal generator of $\mathrm{bigin}(I)$ of bidegree (a,b').

Proof. If $(a-1,b) \notin \mathrm{xreg}(I)$, then by Theorem 5.4 and Eq. (5.13) (not an equation), we can derive that, there is $0 \leq k \leq n$ such that we have $(J_{k-1} : x_k)_{(a-1,b)} \neq (J_{k-1})_{(a-1,b)}$ for $J_{k-1} = (\mathrm{bigin}(I), x_0, \ldots, x_{k-1})$. This result implies that there is a monomial $x^\alpha y^\beta \in S_{(a,b)}$ such that

$$x_k x^\alpha y^\beta \in (\mathrm{bigin}(I), x_0, \ldots, x_{k-1})_{(a,b)} \text{ but}$$
$$x^\alpha y^\beta \notin (\mathrm{bigin}(I), x_0, \ldots, x_{k-1})_{(a-1,b)}. \tag{5.17}$$

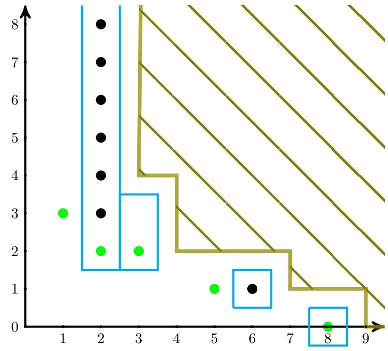


Figure 5.6: In olive, the region $\text{xreg}(I) + (1, 0)$. In blue, columns and squares where there are generators of $\text{bigin}(I)$.

Therefore, none of the variables $x_0, \ldots, x_{k-1}$ divides the monomial $x^\alpha y^\beta$. If $x_k x^\alpha y^\beta$ is a minimal generator of $\text{bigin}(I)$, we are done. Otherwise, write $x_k x^\alpha y^\beta = \overline{z} z^\gamma$ where z^γ is a minimal generator of $\text{bigin}(I)$. We need to show that the bidegree of z^γ is (a, b') for some $b' \leq b$. If this is not true, then there is some $k' \geq k$ such that $x_{k'}$ divides $\overline{z}$. At this point, we have two cases:

- If $k' = k$ then $x_k x^\alpha y^\beta = x_k \frac{\overline{z}}{x_k} z^\gamma$, which implies that $x^\alpha y^\beta = \frac{\overline{z}}{x_k} z^\gamma \in \text{bigin}(I)$, in contradiction with (5.17).

- If $k' > k$, then x_k divides z^γ and $x_{k'}$ divides $\overline{z}$. In this case, we write $z^\gamma = x_k z^{\gamma'}$ and $\overline{z} = x_{k'} \overline{z}'$. Using the property of $\text{bigin}(I)$ in Lemma 2.2, we get $x_{k'} z^{\gamma'} \in \text{bigin}(I)$ and so $x^\alpha y^\beta = x_{k'} \overline{z}' z^{\gamma'} \in \text{bigin}(I)$ getting a contradiction with Eq. (5.17).

Therefore, z^γ has bidegree (a, b') for some $b' \leq b$. □

Example 5.6. Consider the ideal I in Example 5.4. In Figure 5.6, one shows the region $\text{xreg}(I) + (1, 0)$ where there cannot be any generators of $\text{bigin}(I)$ (using Theorem 5.6). Moreover, we mark the columns and squares in which Theorem 5.7 guarantees that there must be minimal generators of $\text{bigin}(I)$ of such bidegrees. Due to the vanishing of $H^i_{\mathfrak{m}_x}(I)_{(a,b)}$ for $a \gg 0$, we note that the region $\text{xreg}(I)$ always provides a tight bound for the degrees of the generators of $\text{bigin}(I)$ with respect to the x's. The bound of provided in Theorem 5.2 is thus recovered.

In what follows, we study the relation between $\text{xreg}(I)$, the multigraded Castelnuovo-Mumford regularity of I and the bidegrees of the generators of $\text{bigin}(I)$.

Theorem 5.8. Let $I \subset S$ be a bihomogeneous ideal. Then, there is $0 \leq s \leq \text{cd}_{\mathfrak{m}_x}(I) - 1$, such that $\text{reg}(I) + (s, 0) \subset \text{xreg}(I)$.

Proof. If $(a, b) \in \text{reg}(I)$, then we have $H^i_{\mathfrak{b}}(I)_{(a',b')} = 0$ for all $i \geq 1$ and $(a', b') \geq (a - \lambda_x, b - \lambda_y)$ with $(\lambda_x, \lambda_y) \in \mathbb{Z}^2_{\geq 0}$ such that $\lambda_x + \lambda_y = i - 1$. In particular, $H^i_{\mathfrak{b}}(I)_{(a',b')} = 0$

for all $(a',b') \geq (a,b)$. This implies that $(a,b) \notin \mathrm{Supp}_{\mathbb{Z}^2}(H^{\bullet}_{\mathfrak{b}}(I))^{\star}$. Hence, by Theorem 2.14, we get $(a,b) \notin \mathrm{Supp}_{\mathbb{Z}^2}(H^{\bullet}_{\mathfrak{m}_{\mathbf{x}}}(I))^{\star}$. It follows that

$$H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a',b')} = 0 \text{ for all } (a',b') \geq (a,b) \text{ and } i \geq 1.$$

Therefore, there is some $0 \leq s \leq \mathrm{cd}_{\mathfrak{m}_{\mathbf{x}}}(I) - 1$ such that for $i \geq 1$, we have $H^i_{\mathfrak{m}_{\mathbf{x}}}(I)_{(a',b')} = 0$ for all $(a',b') \geq (a+s-(i-1), b)$. This implies that $(a+s,b) \in \mathrm{xreg}(I)$ and so does every $(a',b') \geq (a+s,b)$. □

Remark 5.4. Remark **??** implies that for every ideal, the integer s appearing in the above theorem is bounded by n. In many cases, we can also bound the cohomological dimension using the dimension of I, as a module over $\mathbf{k}[x_0,\ldots,x_n]$; see [Gro57].

As a consequence of Theorem 5.8, we derive a relation between $\mathrm{reg}(I)$ and the minimal generators of $\mathrm{bigin}(I)$.

Corollary 5.3. Let $I \subset S$ be a bihomogeneous ideal and $(a,b) \in \mathrm{reg}(I) \cap \mathbb{Z}^2_{\gtrsim 0}$. Then, there is $1 \leq s \leq \mathrm{cd}_{\mathfrak{m}_{\mathbf{x}}}(I)$ such that for every $(a',b') \geq (a+s,b)$, there is no minimal generator of $\mathrm{bigin}(I)$ of bidegree (a',b').

Proof. The proof follows from applying Theorem 5.6 and Theorem 5.8. □

Using Theorem 2.16, we can also relate $\mathrm{xreg}(I)$ with the Betti numbers of I.

Theorem 5.9. Let $I \subset S$ be any bihomogeneous ideal and let $(a,b) \in \mathrm{xreg}(I) \cap \mathbb{Z}^2_{\gtrsim 0}$. Then, $(a+n+1, b+m+1) \notin \beta_i(I)$ for all $i \geq 1$.

Proof. If $(a,b) \in \mathrm{xreg}(I)$, then $(a-i+1,b) \notin \mathrm{Supp}_{\mathbb{Z}^2}(H^i_{\mathfrak{m}_{\mathbf{x}}}(I))^{\star}$ for all $i \geq 1$. In particular, $(a,b) \notin \mathrm{Supp}_{\mathbb{Z}^2}(H^{\bullet}_{\mathfrak{m}_{\mathbf{x}}}(I))^{\star}$. Using Theorem 2.16, we derive that $(a+n+1, b+m+1) \notin \cup_i \beta_i(I)^{\star}$, concluding the proof. □

Corollary 5.4. Let $I \subset S$ be a bihomogeneous ideal and let $(a,b) \in \mathrm{xreg}(I) \cap \mathbb{Z}^2_{\gtrsim 0}$. Then, there is $0 \leq s \leq \mathrm{cd}_{\mathfrak{m}_{\mathbf{x}}}(I) - 1$, such that $\beta_{i,(a',b')} = 0$ for all $i \geq 1$ and $(a',b') \geq (a+n+s+1, b+m+1)$.

Proof. Apply Proposition 5.9 and Theorem 5.8. □

We refer to [BC17, Corollary 3.8] for a finer version of Corollary 5.4.

Example 5.7. We continue with Example 5.2. In Figure 5.7, we illustrate the region $\mathrm{xreg}(I) + (3,2)$ and the Betti numbers, i.e., the bidegrees (a,b) such that there is $i \geq 1$ with $\beta_{i,(a,b)}(I) \neq 0$ in the minimal free resolution of I. Proposition 5.9 guarantees that there is no Betti number in the region $\mathrm{xreg}(I) + (3,2)$.

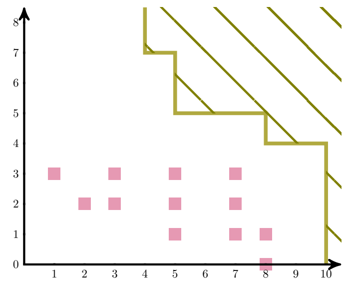


Figure 5.7: We illustrate in olive the region $\mathrm{xreg}(I) + (3, 2)$ and in pink squares, the Betti numbers, i.e. the bidegrees (a, b) such that there is $i \geq 1$ with $\beta_{i,(a,b)}(I) \neq 0$.

4. Bounds on the cohomological dimension

In Theorem 5.8, we have seen that the multigraded Castelnuovo-Mumford regularity $\mathrm{reg}(I)$ is contained in the partial regularity region $\mathrm{xreg}(I)$, up to a shift by $(s, 0)$, where s is bounded above by the cohomological dimension of I with respect to $\mathfrak{m}_x$. In general, the cohomological dimension is bounded above by the minimal number of generators of $\mathfrak{m}_x$, which is $n + 1$.

However, we can clearly see in Example 5.6 that the bound by $n + 1$ can be refined. In fact, we have already used in Lemma 4.6 of Chapter 4, the classical result of Grothendieck [Gro57, Theorem 3.6.5] that says that the cohomology modules can be bounded above in terms of the dimension of the module.

Remark 5.5. In the paper, we describe the generators of $\mathrm{bigin}(I)$ in terms the cohomology of I. However, the dimension discussions we have next depend on the local cohomology with respect to S/I, which can be more standard. However, we can always consider the short exact sequence:

$$0 \to I \to S \to S/I \to 0$$

As $H^i_{\mathfrak{m}_x}(S)_{(a,b)} = 0$ for all $i \in \mathbb{Z}_{\geq 0}$ and $(a, b) \in \mathbb{Z}^2_{\gtrsim 0}$. As a consequence, we have $H^{i+1}_{\mathfrak{m}_x}(I)_{(a,b)} = H^i_{\mathfrak{m}_x}(S/I)_{(a,b)}$ for $(a, b) \in \mathbb{Z}^2_{\gtrsim 0}$. Indeed, as we are studying the degrees in $\mathbb{Z}^2_{\gtrsim 0}$, as study the cohomological dimension in this subset, denoted as $\mathrm{cd}^{\mathbb{Z}^2_{\gtrsim 0}}_{\mathfrak{m}_x}(I)$.

Remark 5.6. A different approach to the bound that we tried to can be given by the Mayer-Vietoris spectral sequence which, as it is the case of the results of Chardin and Holanda that we used in the previous sections; see [CHN23; MBZ18].

Our way to give a bound on the cohomological dimension is considering Noether normalization; see Theorem 2.3, which implies that there are algebraically independent elements $y_1, \ldots, y_r$ such that S/I is a finite module over $\mathbf{k}[y_1, \ldots, y_r]$. Moreover, if $\mathbf{k}$ is an infinite field, then $y_1, \ldots, y_r$ can be chosen to be linear forms in $x_0, \ldots, x_n$; see [BH98, Theorem 1.5.17]. We recall that the Čech complex with respect to $(y_1, \ldots, y_r)$ is bounded above by r. Therefore, $H^i_{(y_1,\cdots,y_r)}(M) = 0$ for $i > r$.

Notation 5.4. For $\nu \in \mathbb{Z}$, we denote by $M_{(*,\nu)}$ the module $\oplus_{\mu\in\mathbb{Z}} M_{(\mu,\nu)}$.

Lemma 5.9. If $y_1, \ldots, y_r$ is a Noether normalization of the $\mathbf{k}[x_0, \ldots, x_n]$-module $(S/I)_{(*,0)}$, then

$$\sqrt{(y_1, \ldots, y_r)} = \sqrt{(x_0, \ldots, x_n)}$$

as ideals in $(S/I)_{(*,0)}$. Therefore, $H^i_{\mathfrak{m}_x}(S/I_{(*,0)}) = H^i_{(y_1,\ldots,y_r)}(S/I_{(*,0)})$.

Proof. The inclusion $\sqrt{(y_1, \ldots, y_r)} \subset \sqrt{(x_0, \ldots, x_n)}$ is trivial as $y_1, \ldots, y_r$ are linear forms in $x_0, \ldots, x_n$. On the other hand, if $f \in \sqrt{(x_0, \ldots, x_n)}$, then there is $r \in \mathbb{Z}_{\geq 0}$ such that $f^r \in (x_0, \ldots, x_n)^M$ for some $M \in \mathbb{Z}_{\geq 0}$. The finiteness of $(S/I)_{(*,0)}$ over $\mathbf{k}[y_1, \ldots, y_r]$ implies that there is a set of generators $x^{b_1}, \ldots, x^{b_s} \in (S/I)_{(*,0)}$ as a $\mathbf{k}[y_1, \ldots, y_r]$-module. Let $M' \in \mathbb{Z}_{\geq 0}$ be an integer such that these generators are of degree $< M'$. Therefore, every monomial x^a of degree M' in $x_0, \ldots, x_n$ can be written as:

$$x^a = \sum_{i=1}^{s} P_{i,a}(y_1, \ldots, y_r) x^{b_i} \text{ modulo } I$$

for some polynomial $P_{i,a}$ which must be of degree ≥ 1 in the variables $y_1, \ldots, y_r$. Therefore, if we consider r' such that $rr' \geq M'$, we must have:

$$(f^r)^{r'} \in (y_1, \ldots, y_r).$$

Thus, $f \in \sqrt{(y_1, \ldots, y_r)}$. The second part follows from [Bus06, Proposition 3.5] □

As local cohomology can be defined as the homology of the Cech complex using Remark 5.5, we get:

$$\mathrm{cd}_{\mathfrak{m}_x}(I_{(*,0)}) \leq r + 1. \tag{5.18}$$

Moreover, for any $\nu \in \mathbb{Z}_{\geq 0}$, the $(S/I)_{(*,0)}$-module $(S/I)_{(*,\nu)}$ is finite. Thus, these modules are also finite over $\mathbf{k}[y_1, \ldots, y_r]$ for all $\nu \in \mathbb{Z}$, getting the same result for $(S/I)_{(*,\nu)}$.

Lemma 5.10. [CH22, Lemma 3.7] If $\nu \in \mathbb{Z}$, we have $H^i_{\mathfrak{m}_x}(M_{(*,\nu)}) = H^i_{\mathfrak{m}_x}(M)_{(*,\nu)}$.

Theorem 5.10. Let $I \subset S$ be a bihomogeneous ideal. Then,

$$\mathrm{cd}^{\mathbb{Z}^2_{\gtrsim 0}}_{\mathfrak{m}_x}(I) \leq \dim_{\mathbb{P}^n}((S/I)_{(*,0)}) + 2, \tag{5.19}$$

where $\dim_{\mathbb{P}^n}((S/I)_{(*,0)})$ is the dimension of $\mathrm{Proj}((S/I)_{(*,0)})$. Geometrically, if I is saturated with respect to $\mathfrak{m}_x$, then $\dim_{\mathbb{P}^n}((S/I)_{(*,0)})$ corresponds to $\pi(V_{\mathbb{P}^n\times\mathbb{P}^m}(I))$ where π is the natural projection:

$$\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n.$$

Proof. The theorem of Noether normalization indicates that r is the Krull dimension of $(S/I)_{(*,0)}$. Using (2.1), we can see that this corresponds to the dimension of $\mathrm{Proj}((S/I)_{(*,0)})$ plus one, where Proj is considered with respect to $\mathbb{P}^n$, i.e. all homogeneous ideals P in the ring $(S/I)_{(*,0)}$ such that $P \not\supset \mathfrak{m}_x$.

As a consequence, using Lemma 5.10 and (5.18), we get that for $i > r+1$:

$$H^i_{\mathfrak{m}_x}(S/I) = \oplus_{\nu\in\mathbb{Z}} H^i_{\mathfrak{m}_x}(S/I)_{(*,\nu)} = \oplus_{\nu\in\mathbb{Z}} H^i_{\mathfrak{m}_x}(S/I_{(*,\nu)}) = 0,$$

which provides the proof of (5.19). The Main Theorem of elimination theory [BCP23, Chapter 3, Theorem 3.14] implies that the ideal:

$$I^{\mathsf{sat},x} \cap \mathbf{k}[x_0, \dots, x_n]$$

Therefore, if $I^{\mathsf{sat},x} = I$, then the dimension of $(S/I)_{(*,0)}$ corresponds to the dimension of $\pi(V_{\mathbb{P}^n\times\mathbb{P}^m}(I))$, finalizing the proof of Theorem 5.10. $\square$

Chapter 6

Applications and computations

In this final chapter, we provide examples of the use of resultants in concrete applications such as *geometric modelling* or *computer vision*.

- *Geometric modelling:* A classical problem where polynomials appear in computer-aided design is the implicitization of curves or surfaces. Namely, given a polynomial or rational map which models a curve or surface, one aims to find the implicit equation of that surface by manipulating the entries of the map. For instance, in the case of surfaces, consider the following polynomial map

  $$\phi : \mathbb{R}^2 \to \mathcal{X} \subset \mathbb{R}^3, \quad (s,t) \to (\phi_1(s,t), \phi_2(s,t), \phi_3(s,t)). \tag{6.1}$$

  The polynomials ϕ typically correspond to piecewise information coming from the representation of an object, for instance B-splines; see [Sha+06]. Representing $\mathcal{X}$ in its implicit form has the advantage of being able to check more easily whether a given point $p \in \mathbb{R}^3$ belongs to $\mathcal{X}$ or finding the intersection of $\mathcal{X}$ with some other surface. Moreover, matrix representations of the implicit equation also exhibit some advantages; see [BLY19; Bus14].

  In particular, we would like to find a polynomial equation in three variables $P(X,Y,Z)$ that represents the surface, i.e.

  $$\mathcal{X} = \{P(X,Y,Z) = 0\}.$$

  Therefore, we are obliged to eliminate the variables s, t from the polynomial system:

  $$X - \phi_1(s,t), Y - \phi_2(s,t), Z - \phi_3(s,t).$$

  A plausible way to eliminate these variables is computing the resultant, which can be done by exploiting the monomial structure of the polynomials with the methods that we explained in the previous sections. Other methods include Gröbner bases; see [Big16], or approximation complexes; see [Bot11; Cha06]. The setting of overdetermied polynomial systems which is explored in this thesis is naturally attached to this problem.

All in all, we can use the examples provided in this thesis to give representations of the matrices in the computation of the resultant.

Example 6.1. Consider the problem of finding the implicit representation a surface $\mathcal{S}$ given by the following polynomials:

$$\phi_1(s,t) = a_0 + a_1 s + a_2 s^2 + a_3 t + a_4 st$$
$$\phi_1(s,t) = b_0 + b_1 s + b_2 s^2 + b_3 t + b_4 st \quad \phi_3 = c_0 + c_1 s + c_2 t \tag{6.2}$$

Under generic assumptions in the coefficients, the system $\phi_1 = \phi_2 = \phi_3 = 0$ has no solutions. Under these assumptions, resultants can be used to eliminate the variables s and t. In particular, the following matrix

$$\mathbb{M}_{(2,1)}(X,Y,Z) = \begin{pmatrix} a_0 - X & a_1 & a_2 & a_3 & a_4 \\ b_0 - Y & b_1 & b_2 & b_3 & b_4 \\ c_0 - Z & c_1 & 0 & c_3 & 0 \\ 0 & c_0 & c_1 & 0 & c_3 \\ [013] & [023]+[014] & 0 & [024] & 0 \end{pmatrix}.$$

has a rank drop after evaluating (X,Y,Z) at a point $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, if and only if, $p \in \mathcal{S}$.

- *Computer vision:* A variety of polynomial systems arising in vision consists of matching problems between snapshots captured by cameras. In this type of problems, thousands of polynomial systems will have to be simultaneously solved [Duf+18; Kuk13; BKH19] so small differences in the computations will be helpful in the final result. As one thinks of the cameras as linear projections, interesting algebraic objects such as Chow forms [OT19] or distorion varieties [Kil+16] also arise.

  A typical problem consists in computing the displacement of a calibrated camera between two positions in a static environment. Namely, we would like to find the displacement of a rigid body between two snapshots taken by a stationary camera. The identifiable features of the body include only points.

  Usually, a minimum number of 5 point matches is available. The algebraic problem reduces to a well-constrained system of polynomial equations and we are able to give a closed-form solution. Typically, computer vision applications use at least 8 points in order to reduce the number of possible solutions to one, in generic coordinates. In addition, computing the displacement reduces to a linear problem and the effects of noise in the input can be diminished [LH81].

  Let $a_i \in (\mathbb{R}^3)$ for $i = 1, \ldots, 5$ be the 5 points in the first snapshot and $a'_i \in (\mathbb{R}^3)$ for $i = 1, \ldots, 5$ be the points in the second snapshot. A quaternion formulation of this problem was proposed in [Hor91]. This quaternion formulation reduces the problem to solving the polynomial system given by the following equations in the variables $q \in \mathbb{R}^3$ (representing a rotation) and $d \in \mathbb{R}^3$ (representing a translation)

$$(a_i^T q)(d^T a_i') + a_i^T a_i' + (a_i \times q)^T a_i' + (a_i \times q)^T (d \times a_i') + a_i^T (d \times a_i') = 0, \quad i = 1, \dots, 5,$$
$$1 - d^T q = 0, \quad (6.3)$$

where $\times$ represents the usual exterior product. The first five equations represents each of the 5 displacements while the last one represents a normalization between the vectors q, d. This system is bilinear in the two groups of variables. We can solve it by building the u-resultant. Namely, we introduce a new linear equation $P_u = u_0 + u_1 d_1 + u_2 d_2 + u_3 d_3 + u_4 q_1 + u_5 q_2 + u_6 q_3$. Once we consider the resultant of this system, we get a polynomial that factors into linear forms, whose coefficients are the values of the solutions (they are a finite number in this case); see also [Emi94] for a similar approach.

1. Some JULIA code for resultants and elimination matrices

The Canny-Emiris formula In [CE22], we included an JULIA implementation of the Canny-Emiris formula for the cases of n-zonotopes and multihomogeneous systems. Instead of applying the formula to polynomials of any Newton polytope (which has already been done in other implementations), our goal was to provide the rows of the Canny-Emiris formula by simply providing the type functions of each of the lattice points that are used after the greedy implementation, as we described in Chapter 3. The package can be found in the URL

https://github.com/carleschecanualart/CannyEmiris.

As input, one can introduce the vectors generating the n-zonotope (the matrix H below) and the $a_{i,j}$ appearing in (3.1) (the matrix A below).

```
julia> using CannyEmiris

julia> A = [[1,1] [1,1] [1,1]]
2×3 Matrix{Int64}:
 1  1  1
 1  1  1

julia> H = [[1,0] [0,1]]
2×2 Matrix{Int64}:
 1  0
 0  1
```

The command CannyEmiris.Zonotopes considers the setting given by A and H and produces the matrix $\mathcal{H}$ in the Canny-Emiris formula. In particular, the command specifies which are the exponent vectors in the greedy subset $\mathcal{G}$ and their corresponding polynomials providing the rows of $\mathcal{H}$ (the matrix CE below) and the principal submatrix $\mathcal{E}$ (the matrix PM below).

```
julia> CE, PM = CannyEmiris.Zonotopes(A,H,true)
The rows of the Canny-Emiris matrix x^{b-a(b)}F_{i(b)} are:
[0, 1]-> x^[0, 1]*F_2
[0, 2]-> x^[0, 1]*F_1
[1, 0]-> x^[1, 0]*F_2
[1, 1]-> x^[1, 1]*F_2
[1, 2]-> x^[0, 1]*F_0
[2, 0]-> x^[1, 0]*F_1
[2, 1]-> x^[1, 0]*F_0
[2, 2]-> x^[1, 1]*F_1

The size of the greedy Canny-Emiris matrix is: 8
The degree of the resultant is: 6

The sparse resultant is the ratio of the determinants of the returned matrices to the power 1.0
```

```
julia> CE
8×8 Matrix{SymPy.Sym}:
 (u_{2, [0, 0]})  (u_{2, [0, 1]})                0  (u_{2, [1, 0]})  (u_{2,
 (u_{1, [0, 0]})  (u_{1, [0, 1]})                0  (u_{1, [1, 0]})  (u_{1,
               0                0  (u_{2, [0, 0]})  (u_{2, [0, 1]})
               0                0                0  (u_{2, [0, 0]})  (u_{2,
 (u_{0, [0, 0]})  (u_{0, [0, 1]})                0  (u_{0, [1, 0]})  (u_{0,
               0                0  (u_{1, [0, 0]})  (u_{1, [0, 1]})
               0                0  (u_{0, [0, 0]})  (u_{0, [0, 1]})
               0                0                0  (u_{1, [0, 0]})  (u_{1,
```

In the case of multihomogeneous systems, the command `CannyEmiris.Multihomogeneous` takes the list of the exponents of the projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, i.e. $(n_1, \ldots, n_r)$ (the vector N below). Moreover, it also considers the matrix of multi-degrees of the polynomials (the matrix D below) and provides the matrices $\mathcal{H}$ and $\mathcal{E}$ as in the case of zonotopes.

```
julia> N = [2]
1-element Vector{Int64}:
 2

julia> D = [2 2 1]
1×3 Matrix{Int64}:
 2  2  1

julia> CE,PM = CannyEmiris.Multihomogeneous(D,N,true)
The rows of the Canny-Emiris matrix x^{b-a(b)}F_{i(b)} are:
[2, 1]-> x^[2, 1]*F_2
[3, 1]-> x^[3, 1]*F_2
[1, 2]-> x^[1, 2]*F_2
[2, 2]-> x^[2, 2]*F_2
[1, 3]-> x^[1, 3]*F_2
[4, 1]-> x^[2, 1]*F_1
[3, 2]-> x^[1, 2]*F_1
[2, 3]-> x^[2, 1]*F_0
[1, 4]-> x^[1, 2]*F_0

The size of the greedy Canny-Emiris matrix is: 9
The degree of the resultant is: 8
```

We also included the implementation of the resultant matrix for the equations of the 5-point problem in (6.3) (the two sets of 5 random points in $\mathbb{R}^3$ are the matrices $A1$ and $A2$ below).

```
julia> A1 = rand(3,5); A2 = rand(3,5)
3×5 Matrix{Float64}:
 0.606563  0.137791  0.347151  0.677246   0.790655
 0.157243  0.116453  0.897549  0.369342   0.182918
 0.189332  0.325438  0.793318  0.0780656  0.934121

julia> CE, a = CannyEmiris.MatrixOfTheFivePointsLinearForm(A1,A2)
```

Sylvester forms The package regarding Sylvester forms was not included in [BC22]. However, we included it here as part of showing the potential computational interest of Sylvester forms in the framework of elimination matrices. In this package, we only develop the construction of the matrix. However, this construction can be included in other packages for solving polynomial systems such as `AlgebraicSolvers.jl` or `EigenvalueSolver.jl`. It is relevant to note that we give the construction for affine polynomials and so the cokernel of the matrix will also contain the solutions at infinity. The package can be found in the URL

https://github.com/carleschecanualart/Sylvester.

For the case of dense polynomial systems, we initialize the packages and use `DynamicPolynomials` to manage the variables.

```
julia> using DynamicPolynomials

julia> @polyvar x[1:2]
(PolyVar{true}[x₁, x₂],)

julia> using Sylvester
```

Assuming that we manage with polynomials of a certain list of degrees (the vector *ds* below) which we use to generate random polynomials of those degrees (the list *f* below). One can also introduce any list of given polynomials.

```
julia> ds = [2; 2; 3]
3-element Vector{Int64}:
 2
 2
 3

julia> f = EigenvalueSolver.getRandomSystem_dense(x,ds)
3-element Vector{Polynomial{true, Float64}}:
 -1.6923041859862062x₁² - 1.5323841354928134x₁x₂ + 1.163862279944845x₂² + 2
- 0.08793758894274863
 -0.47345770172033036x₁² - 0.7805926804371628x₁x₂ + 0.8670343715925974x₂² +
x₂ - 1.5577804668722373
 0.797566920072599x₁³ + 2.155253114690668x₁²x₂ - 1.475642808383832x₁x₂² - 0
x₁² - 0.48623192983748753x₁x₂ - 0.09694876902940987x₂² + 0.4813182319424892
29137
```

The command `Sylvester.getResDense` outputs the elimination matrix $\mathcal{M}_\nu$ (written as *res* below) at the smallest possible degree ν, which is given by

$$\sum_{i=0}^{n} d_i - n - \min_i d_i \tag{6.4}$$

if $d_0, \ldots, d_n$ are the degrees of the system. The command also outputs the list of monomials of degree ν (the list *S* below), which label the columns of the matrix.

```
julia> res, S = Sylvester.getResDense(f, x)
(ComplexF64[-1.6923041859862062 + 0.0im -1.5323841354928134 + 0.0im … 0.0
859862062 + 0.0im … -0.08793758894274863 + 0.0im 0.0 + 0.0im; … ; 0.0 + 0
2407659645 + 0.0im -2.0376139336777808 + 0.0im; 0.0 + 0.0im -1.3852118449
4.605474537430942 + 0.0im], Monomial{true}[x₁³, x₁²x₂, x₁x₂², x₂³, x₁², x

julia> res
10×10 Matrix{ComplexF64}:
   -1.6923+0.0im   -1.53238+0.0im    1.16386+0.0im  …  -0.0879376+0.0im
       0.0+0.0im    -1.6923+0.0im   -1.53238+0.0im            0.0+0.0im
       0.0+0.0im        0.0+0.0im        0.0+0.0im        2.07407+0.0im
 -0.473458+0.0im  -0.780593+0.0im   0.867034+0.0im       -1.55778+0.0im
       0.0+0.0im  -0.473458+0.0im  -0.780593+0.0im            0.0+0.0im
       0.0+0.0im        0.0+0.0im        0.0+0.0im  …    0.319124+0.0im
  0.797567+0.0im    2.15525+0.0im   -1.47564+0.0im       0.481318+0.0im
  0.901239+0.0im    1.94951+0.0im   -2.90787+0.0im        4.50515+0.0im
       0.0+0.0im  -0.256917+0.0im  -0.308325+0.0im       0.135512+0.0im
       0.0+0.0im   -1.38521+0.0im   -2.76035+0.0im        4.61975+0.0im
```

In the case of multihomogeneous polynomial systems, the groups of variables can be assigned using the below *vargroups* vector and specifying the number of variables of each group with the *varsize* vector. Moreover, one can specify the multi-degrees of the polynomials (with the matrix *ds* below) and provide a polynomial system with $n+1$ polynomials.

```
julia> vargroups = [[x[1:1]];[x[2:2]]]
2-element Vector{Vector{PolyVar{true}}}:
 [x₁]
 [x₂]

julia> varsize = [1;1]
2-element Vector{Int64}:
 1
 1

julia> ds = [1 1; 1 1; 1 1]
3×2 Matrix{Int64}:
 1  1
 1  1
 1  1

julia> f = [x[1]*x[2] - 1; x[1] - 1; x[2] - 1]
3-element Vector{Polynomial{true, Int64}}:
 x₁x₂ - 1
 x₁ - 1
 x₂ - 1
```

The command `Sylvester.getMultiResDense` outputs the elimination matrix $\mathcal{M}_\nu$ (written as *res* below) of the smallest bi-degree which as before is given by the multigraded analogue of (6.4). In this case, one can also specify more than $n+1$ polynomials but, following the construction of Section 4.9 in Chapter 4, one has to specify a set S of polynomials satisfying the hypotheses of Theorem 4.8.

```
julia> res, S = Sylvester.getResMultiDense(f,x,vargroups,varsize,ds,[1:3;])
(ComplexF64[1.0 + 0.0im 0.0 + 0.0im 0.0 + 0.0im -1.0 + 0.0im; 0.0 + 0.0im 1.0
 0.0im 0.0 + 0.0im 1.0 + 0.0im -1.0 + 0.0im; 0.0 + 0.0im 0.0 + 0.0im 1.0 + 0.0
x₁, x₂, 1])

julia> res
4×4 Matrix{ComplexF64}:
 1.0+0.0im  0.0+0.0im  0.0+0.0im  -1.0+0.0im
 0.0+0.0im  1.0+0.0im  0.0+0.0im  -1.0+0.0im
 0.0+0.0im  0.0+0.0im  1.0+0.0im  -1.0+0.0im
 0.0+0.0im  0.0+0.0im  1.0+0.0im  -1.0+0.0im
```

In the most general sparse case, one has to specify the vectors generating the fan of the toric variety (see the matrix U) and, after this, the integer vector $(a_{i,n+j})$

defining the inequalities of each Newton polytope (see the matrix a). Once these are clarified, the `Julia` interface of the famous package `Polymake` [KLT20] is used to generate the lattice points and the command `Sylvester.randomSparsePoly` generates random polynomials with these Newton polytopes. Note that the matrix U that we specify below is the negative of the matrix in (2.9).

```
julia> U = [-1 0; 0 -1; 1 1; 0 1]
4×2 Matrix{Int64}:
 -1   0
  0  -1
  1   1
  0   1

julia> a = [0 0 2 1; 0 0 2 1; 0 0 1 1]
3×4 Matrix{Int64}:
 0  0  2  1
 0  0  2  1
 0  0  1  1

julia> f = Sylvester.randomSparsePoly(U,a,x)
3-element Vector{Any}:
 -0.642851404855405x₁² - 0.9055582043989475x₁x₂ - 0.5439562378001662x₁ - 1.067363
 -0.698419385047925x₁² - 0.4165717279595798x₁x₂ - 2.8163569091729745x₁ + 0.9693908
 -1.4195710741863383x₁ + 0.016049160747017312x₂ - 0.9864555867154953
```

Once these are specified, the command `Sylvester.getResSparse` outputs the elimination matrix $\mathcal{M}_\nu$ (written as *res* below) of the smallest bi-degree which as before is given by the sparse analogue of (6.4).

```
julia> res, S = Sylvester.getResSparse(f,x,U,a,[1:3;])
(ComplexF64[-1.037314088651869 + 0.0im -1.0673631220309368 + 0.0im … -0.905558204398947
+ 0.0im; 1.4309016382099484 + 0.0im 0.9693908532865807 + 0.0im … -0.4165717279595798 + 
im; -0.9864555867154953 + 0.0im 0.016049160747017312 + 0.0im … 0.0 + 0.0im 0.0 + 0.0im;
2292048735672 + 0.0im … 0.0 + 0.0im 0.0 + 0.0im], Monomial{true}[1, x₂, x₁, x₁x₂, x₁²])

julia> res
4×5 Matrix{ComplexF64}:
  -1.03731+0.0im   -1.06736+0.0im  -0.543956+0.0im  -0.905558+0.0im  -0.642851+0.0im
    1.4309+0.0im   0.969391+0.0im   -2.81636+0.0im  -0.416572+0.0im  -0.698419+0.0im
 -0.986456+0.0im  0.0160492+0.0im   -1.41957+0.0im        0.0+0.0im        0.0+0.0im
   4.28553+0.0im    1.33229+0.0im    1.37649+0.0im        0.0+0.0im        0.0+0.0im
```

Chapter 7

Open problems

All the contributions in the previous chapters leave many open discussions and problems, which the author aims to dedicate in his future research. In the following, we can list some of these problems and highlight their interest.

Resultants and sparse elimination

A conjecture on the greedy Canny-Emiris formula. A very natural question for sparse polynomial systems is which are the matrices of smallest size that one can build to represent the sparse resultant. In [CE23], we stated a conjecture on the case of using the Canny-Emiris formula that can be described as follows.

Assume that we are working with coefficients in the field of complex numbers $\mathbb{C}$. Consider $\mathcal{A}_0, \ldots, \mathcal{A}_n$ be a family of supports corresponding to a multihomogeneous system. Assume, also, that each of the $\mathcal{A}_i$ can be associated to a multidegree in $\mathbf{d}_i \in \mathbb{Z}^d$. The generic Hilbert function is defined as:

$$\mathrm{HF}(\mathbf{d}) = \dim(S/I)_{\mathbf{d}} \quad \mathbf{d} \in \mathbb{Z}^d$$

where I is the ideal in $\mathbb{C}[M]$ after specializing the $u_{i,a}$ to generic values in $\mathbb{C}$. This generic Hilbert function exists as we cannot have two different generic behaviours for coefficients in $\mathbb{C}$. Using the correspondence between polytopes and multi-degrees that we described in the preeliminaries, we can associate some of the subsets $\mathcal{G} \subset \mathcal{B}$ to multi-degrees.

In the case of generic coefficients (still in the homogeneous case), this coincides with the degree at which one can build resultant matrices: the Macaulay bound (1). There is a multihomogeneous analogue of the Macaulay bound [Ben19, Proposition 8.2.2] but, in general, it is not tight for the resultant construction [ACG05]. We can relate these bounds to the mixed subdivisions that we considered in the previous sections. For instance, the whole set of lattice points $\mathcal{B}$ can be naturally associated to the multi-homogeneous Macaulay bound.

Conjecture 7.1. Assume that $\mathcal{G} \subset \mathcal{B}$ is a set of lattice points in the translated cells that corresponds to a multi-degree $\mathbf{d} \in \mathbb{Z}^d$. If $\mathrm{HF}(\mathbf{d}) = 0$, then there is a lifting function $\omega \in \prod_{i=0}^n \mathbb{R}^{\mathcal{A}_i}$ such that $\mathcal{G}$ is the greedy subset of such system. Moreover, if $\mathrm{HF}(\mathbf{d}') \neq 0$ for $\mathbf{d}' \lneq \mathbf{d}$, $\mathcal{G}$ contains no greedy subset.

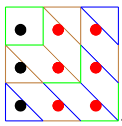
This idea can easily be extended to the sparse case by considering generic values of the Hilbert function associated to degrees in the Cox ring of a toric variety. Following the use that we made of the degree reverse lexicographical monomial order in Chapter 5, it is natural to think that this lifting function must be related to the degree reverse lexicograpical order. Namely, that for two monomials $x^A, x^B \in k[M]$, we have:

$$x^A <_{\mathrm{DRL}} x^B \iff \omega(A) \leq \omega(B).$$

Here $\omega(A)$ refers to evaluating the exponents of x^A in the inf-convolution of ω as in Definition 3.1.

In [CE22], we only considered affine lifting functions, for the sake of simplicity on the combinatorics of the greedy algorithm. However, the results are known to be not optimal, in the sense that there exist other lifting functions that provide smaller resultant matrices. Therefore, a natural question is to ask which subsets $\mathcal{G} \subset \mathcal{B}$ can be obtained using the greedy algorithm for some lifting function and which of them are minimal.

Example 7.1. Consider the same bilinear system as in Example 3.1. Another possible non-affine mixed subdivision $S(\rho)$ is the following:

.

The red dots indicate the greedy subset that one obtains by starting the algorithm at the lattice points in mixed cells. A possible lifting function giving this mixed subdivision is $\omega_0 = (0, 1, 1, 3), \omega_1 = (0, 2, 2, 5), \omega_2 = (0, 3, 3, 7)$, which is not affine. This lifting function satisfies this degree reverse lexicographical condition. Moreover, the subset $\mathcal{B}$ obtained by considering all the lattice points in translated cells can be related to the bi-degree $(2, 2)$. However, the greedy subset $\mathcal{G}$ that we have found corresponds to the bi-degree $(2, 1)$. In particular, this bi-degree corresponds to some existing exact resultant formulas [DE03].

Where should we perform elimination? Throughout the whole thesis, it is assumed that if we are given a polynomial system, $F_0 = \cdots = F_n = 0$ with Newton polytopes $\Delta_0, \ldots, \Delta_n$, then it is a good idea to exploit and understand this structure for a better design of the algorithms of algebraic elimination and thus, X_Δ (for $\Delta = \sum_{i=0}^n \Delta_i$) is the toric variety in which we should work.

However, it might not always be the case that the best representations for resultants and elimination matrices come from exploiting this sparse setting. For intstance, it might be the case that there are polytopes $\overline{\Delta}_i \supset \Delta_i$ such that computing the resultant in this setting is sufficient for performing elimination we have simpler representations for the resultant. Namely, we have that:

$$\mathrm{Res}_{\overline{\Delta}_0,\ldots,\overline{\Delta}_n}(F_0,\ldots,F_n) \neq 0.$$

In fact, D'Andrea, Jerónimo and Sombra provided necessary and sufficient conditions for that to happen in terms of mixed integrals [DJS22, Theorem 3.19].

On the other hand, if we consider the greedy algorithm and the setting of Conjecture 7.1, it is possible that the resulting polytope formed by the lattice points in $\mathcal{G}$ does not correspond to a divisor in the same toric variety given by the polytopes $\Delta_0,\ldots,\Delta_n$. For instance, in Example 3.5, the resulting polytope does not correspond to a nef divisor in $\mathbb{P}^n$, even if the starting point were polytopes corresponding to divisors in this variety. Thus, even in the dense case, it is possible to consider elimination matrices and resultants which are built from polytopes which do not correspond to divisors in the toric variety defined by $\Delta_0,\ldots,\Delta_n$.

As the rows of the Macaulay matrix for the resultant are related to the Castelnuovo-Mumford regularity, it is clear that in the resulting polytope, there will be lattice points associated to the degree of regularity, but this does not mean that all the lattice points of that degree belong to the Newton polytope of $\mathcal{G}$.

All in all, it is possible to state the following big question: *given a polynomial system which is the best toric variety in which we can work for performing algebraic elimination using resultants and elimination matrices?*

Type functions. As we showed in the computational section, providing a description of the lattice points that are required after the greedy algorithm in terms of type functions can simplify the implementation of the Canny-Emiris formula. *Can we give this type of description in a wider context that the cases of n-zonotopes and multihomogeneous systems?*

Generic dimension. The tecnique that D'Andrea, Jerónimo and Sombra used in [DJS22] to derive that the Canny-Emiris matrix does not vanish is very interesting. They used the fact that the lifting functions provide a tropical degeneration of the polynomials and that the initial part of the determinant of the matrix with respect to that degeneration is nonzero. This idea can be very interesting to use this tecnique to derive the non-vanishing of Macaulay matrices for computing the generic dimension depending on parameters. This can be very interesting both in applications [FHPE23] or in the toric setting [BS24].

Sylvester forms

The σ-positive property. The use of the σ-positive property follows the intuition coming from the dense and multihomogeneous case that if a polytope is described in terms of a degree which has a negative entry, then there are no lattice points in that polytope; see Remark 2.3. As noted in Conjecture 4.1, we can expect that a broad range of toric varieties (specially smooth) satisfy this property. On the other hand, we can notice that some relevant counterexamples do not satisfy the σ-positive property. For instance, the counterexample given by Maclagan and Smith in [MS04, Example 6.11] of a toric variety such that its Cox ring is not 0-regular does not satisfy this property.

Computational aspects of Sylvester forms. Further work is needed to analyze if some toric Sylvester forms can be avoided or combined to gain in efficiency. A more practical approach for future improvements would be to add Sylvester forms step by step (similarly to the "degree-by-degree" approach developed in [BT21]) until the expected corank is achieved, or some other criterion needed to solve the polynomial systems is satisfied (see e.g. [BT21, Definition 2.1]). An interestic topic related to further study of Sylvester forms is how they might be included in computer algebra systems. The fact that they are defined only in the case of $n+1$ polynomials in n (affine) variables is a big restriction for this use. However, in some cases, a part of the variables could be considered as the parameters of the system and the Sylvester forms can be computed with respect to the rest. With this, one can try to compute the saturation with respect to some of the variables, which can be contained in the saturation with respect to the irrelevant, which contains the interesting geometric information of the polynomial system.

Hybrid resultant formulas. In Example 4.7, we mentioned that the case $n = 2$ and $\alpha = \delta$ is the only case for which a method for choosing a minor of the hybrid elimination matrices, extending in the Canny-Emiris formula; see [DE01]. Finding a more general method for choosing a minor of the hybrid elimination matrices could help generalizing the Canny-Emiris formula to the matrices that use toric Sylvester forms could be interesting for finding further compact representations of the sparse resultant.

Multigraded regularity and generic initial ideals

Complete the picture and relation with other objects. The definition of $\mathrm{xreg}(I)$ is pivotal to the contribution of this paper, as it allowed us to generalize the criterion of Bayer and Stillman in [BS87a] to the case of the bi-generic initial ideal $\mathrm{bigin}(I)$, using the DRL monomial order (5.4). This region is described in terms of the local cohomology with respect to $\mathfrak{m}_x$. The study of the local cohomology modules by Chardin and Holanda in [CH22] allows us to relate $\mathrm{xreg}(I)$ with the Castelnuovo-Mumford regularity $\mathrm{reg}(I)$ [MS04]. On the other hand, this invariant preserves some of the properties of the degree of regularity $\mathfrak{R}_{\mathrm{x}}(I)$ in [ACDN00; Röm01], but allows us to describe the bi-degrees of the generators of $\text{bigin}(I)$ in terms of a region which, in general, is larger than the one described by $\mathfrak{R}_{\mathrm{x}}(I)$. All these results follow identically in the multigraded case.

As we see Figure **??**, there can be unbounded regions that do not intersect $\text{xreg}(I)$. Therefore, we cannot use $\text{xreg}(I)$ to give a complete characterization of the bi-degrees of the generators of $\text{bigin}(I)$. The idea of the proof of Theorem 5.7 can be used in further generality to derive that

$$(J_{k-1} : x_k)_{(a-1,b)} = (J_{k-1} + \mathfrak{m}_{\mathrm{y}}(J_{k-1} : x_k))_{(a-1,b)} \quad \forall k = 0, \dots, n$$
$$\iff \text{There is no generator of } \text{bigin}(I) \text{ of degree } (a, b). \quad (7.1)$$

for $J_{k-1} = (\text{bigin}(I), x_0, \dots, x_{k-1})$. This last result is the closest we can get to the complete characterization of de bi-degrees of the generators of $\text{bigin}(I)$, generalizing (1.7) to the bigraded setting. However, we were not able to characterize the left hand side of (7.1) in terms of the algebra of I (local cohomology, Betti numbers...), generalizing $\text{xreg}(I)$.

A sparse generic initial ideal. Knowing for which toric varieties there is an analogue of the generic initial ideal can also be very helpful for extending the theory we described for multihomogeneous systems to other toric varieties.

How to structure Gröbner-based computations with multigraded polynomials? In our work on multigraded regularity, we motivated the use of the DRL monomial order (see Remark **??**) and we assumed a fixed relative order of the variables of different multi-degree. However, we do not derive that this monomial order is necessarily going to provide the best approach towards the computation of a Gröbner. Proving that requires an extra study, considering also the relative orders in which the variables of each block can be intermixed.

$\text{xreg}(I)$ **and effective computations.** In Theorem 5.4, we showed that the regularity region $\text{xreg}(I)$ extends the criterion of Bayer and Stillman to the setting of multidegrees. This criterion has been applied in other contexts, for instance in the use of Macaulay matrices for the construction of normal forms; see [TMVB17]. More concretely, as $\text{xreg}(I)$ only depends on the structure of the multigraded ideal with respect to one group of variables, we believe that our criterion can be applied in the case that we want to recover the geometry of the projection of the solution set to the projective space corresponding to that group of variables. For instance, in the case that the projected solution set is formed by a finite number of points, we could try to recover those points by posing an eigenvalue problem, as we explained in the introduction. This idea could be very interesting in the context of systems that depend on parameters.

Other applications. During the thesis, we focused on sparsity as a general and widely used structure to exploit. However, the systems that usually appear in applications have further structure which cannot always be seen from this paradigm. For instance, the polynomial systems modeling the steady states of chemical reaction networks. They sometimes exhibit toric structure [Cra+09], but their structure is much more particular than the general type of polynomial systems we studied in this thesis.

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